Percolation on transitive graphs

Mark Sapir

A short survey plus Iva Kozáková's work

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For any realization $\omega \in \Omega$, open edges form a random subgraph of $\mathcal{G}$. Connected components of that subgraph are called *clusters*.The *Percolation function* $\theta(p)$ is defined to be the probability that the origin is contained in an infinite cluster.

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Therefore if the graph is transitive, then either with probability 1 there are no infinite clusters (that is $p < p_c$), or with probability 1, there exists a unique infinite cluster or with probability 1 there are infinitely many of them.

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- Free products of transitive graphs; p_c can be expressed in terms of the expected cluster sizes at the origin of the free factors. (Kozakova)

Other critical characteristics of percolation

The uniqueness phase p_u - the infimum of all p such that P_p-a.s. the infinite cluster is unique. By Häggström and Peres, the infinite cluster is unique a.s. for all $p > p_u$.

It is known that $p_c \leq p_u$ for every group.

For amenable groups, $p_c = p_u$.

For the Cayley graph of F_2, $p_c = \frac{1}{3}, p_u = 1$. For every transitive graph with infinitely many ends, $p_u = 1$.

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Pak and Smirnova-Nagnibeda:For every non-amenable group, there exists a generating set, possibly with repetitions, for which $p_c < p_u$.

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Theorem (Gaboreau, Lyons) For any finitely generated non-amenable group G, there is $n \in \mathbb{N}$ and a non-empty interval (p_1, p_2) of parameters p for which there is an ergodic essentially free action of F_2 on $\Pi_1^n(\{0,1\}^G, \mu_p)$ such that almost every G-orbit of the diagonal Bernoulli shift decomposes into F_2-orbits.

Schonmann's critical value

Schonmann's critical value of p:

$$p_{\rm exp} = \mathsf{sup}\{p : \exists_{C,\gamma>0} \forall_{x,y\in V} \mathrm{P}_p(x \leftrightarrow y) \leq Ce^{-\gamma \mathrm{dist}(x,y)}\}$$

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$p_c \leq p_{\exp} \leq p_u$.

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$$\theta(p) \approx (p - p_c)^\beta \quad \text{as } p \searrow p_c,$$
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For the triangular lattice in $\mathbb{R}^2$ we have $\beta = 5/36$, $\gamma = -43/18$ as proved by Smirnov and Werner.

The main problems about percolation

Problem (Benjamini-Schramm)

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Find p_c for the cubic lattice in $\mathbb{R}^3$?

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Problem (Smirnova-Nagnibeda)

Find p_c of known groups with "standard" generating sets.

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What is the possible expected distortion of an open cluster in a Cayley graph of a group? Same question for hyperbolic groups is also open.

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Schonmann: the critical exponents take their mean-field values for all non-amenable planar graphs with one end, and for unimodular graphs with infinitely many ends (in particular, for all Cayley graphs of groups with infinitely many ends).

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Pete+Nekrashevich+S: $Z_2 wr Z$.

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Problem

Is there a Cayley graph with $p_c > .6$?

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- If all the border sets are finite then the branching process has finitely many types, and the first moment matrix is of finite size.
- If, in addition, the pieces are finite, then the entries of the first moment matrix are algebraic functions in p. In this case p_c is an algebraic number, which is the smallest value of p such that the spectral radius of the first moment matrix is 1. Moreover there exists an algorithm, that, given the pieces of the tree-like structure, produces a finite extension K of the field $\mathbb{Q}(x)$, and an algebraic function $f(x)$ such that p_c is the smallest positive root of $f(x)$.

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This is the first quasi-isometry complete class of Cayley graphs of groups such that there exists an algorithm to find the p_c of any graph in the class (except the class of virtually cyclic groups where p_c is always 1).