REVIEW OF LAST LECTURE

Joint work with

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Fix $m, n \geq 1$.

$c, C, C' \ldots = \left(\begin{matrix}\text{CONST'S DEPENDING} \\ \text{ONLY on } m, n\end{matrix}\right)$

$X \sim Y$ means

$$cX \leq Y \leq CX$$

GIVEN m, n, E, f, σ

$E \subset \mathbb{R}^n$ finite $(\#E = N)$

$f : E \to \mathbb{R}$

$\sigma : E \to [0, \infty)$

WANT TO COMPUTE $\begin{Bmatrix} F \in C^m(\mathbb{R}^n) \\ M > 0 \end{Bmatrix}$

S.T.

$$\boxed{\begin{array}{l} \bullet \ \|F\| \leq M \\ \bullet \ |F(x) - f(x)| \leq M\sigma(x) \ \text{(all } x \in E\text{)} \end{array}}$$

$$\|f\|_{(E,\sigma)} := \inf \begin{Bmatrix} \text{all } M \text{ s.t.} \\ \text{for some } F \end{Bmatrix}$$

PROBLEMS:

- COMPUTE ORDER of MAG. of $\|f\|_{(E,\sigma)}$
- COMPUTE $F \in C^m(\mathbb{R}^n)$, $M > 0$, S.T.

$$\boxed{\begin{array}{l} \|F\| \leq M \\ |F(x) - f(x)| \leq M\sigma(x) \ (\text{all } x \in E) \end{array}}$$

WITH $M \leq C\|f\|_{(E,\sigma)}$

$J_x(F) = \left\{ \begin{array}{l} (m-1)^{\text{rst}} \text{ ORDER TAYLOR POLY.} \\ \text{OF } F \text{ at } x \end{array} \right.$

$\Gamma(x_0, M) = $ SET OF ALL $J_{x_0}(F)$
for $F \in C^m(\mathbb{R}^n)$ s.t.

$$\boxed{\begin{array}{c} \|F\| \leq M, \text{ and} \\ |F(x) - f(x)| \leq M\sigma(x) \\ \text{for all } x \in E \end{array}}$$

PROBLEM:

Compute the approximate size & shape of $\Gamma(x_0, M)$ for $\left\{ \begin{array}{l} x_0 \in E \\ \text{all } x_0 \in \mathbb{R}^n \end{array} \right\}$

A **BLOB** IN A VECTOR SPACE V

is a family

$$\mathcal{K} = (K_M)_{M>0}$$

of (possibly empty) convex subsets of V, s.t.

$$M \leq M' \Longrightarrow K_M \subseteq K_{M'}.$$

Two BLOBS

$$\mathcal{K} = (K_M)_{M>0} \quad \& \quad \mathcal{K}' = (K'_M)_{M>0}$$

are

C-EQUIVALENT

if & only if

$$\boxed{\begin{array}{l} K_M \subseteq K'_{CM} \quad \& \quad K'_M \subseteq K_{CM} \\ \text{for all } M > 0. \end{array}}$$

$\Gamma(x_0) := (\Gamma(x_0, M))_{M>0}$

IS A BLOB IN THE VECTOR SPACE

$\mathcal{P} = \left\{ \begin{array}{l} (m-1)^{\underline{\text{RST}}} \text{ DEGREE} \\ \text{POLYS ON } \mathbb{R}^n \end{array} \right\}$

PROBLEM:

COMPUTE $\Gamma(x_0)$ up to C-EQUIV.

RECALL $\Gamma(x_0, \ell, M) \subset \mathcal{P}$

- DEFINED FOR

$$\left(\begin{array}{l} x_0 \in E \\ \ell \geq 0 \text{ integer} \\ M > 0 \text{ real} \end{array} \right)$$

- DEFINED BY INDUCTION ON ℓ

$\underline{\ell = 0} \Longrightarrow$

$\Gamma(x_0, 0, M) =$

SET OF ALL $P \in \mathcal{P}$ s.t.

$$\boxed{\begin{aligned} &|\partial^\alpha P(x_0)| \leq M \quad (\text{all } \alpha) \\ &|P(x_0) - f(x_0)| \leq M\,\sigma(x_0) \end{aligned}}$$

"

INDUCTION STEP

- Fix $l \geq 0$.
- Assume $\Gamma(x_0, l, M)$ have been defined for all $x_0 \in E$, $M > 0$.
- Will define $\Gamma(x_0, l+1, M)$ for all $x_0 \in E$, $M > 0$.

$\Gamma(x_0, \ell+1, M) =$

SET OF ALL $P \in \mathcal{P}$ such that

$$\boxed{\begin{array}{l} \forall y_0 \in E \quad \exists P' \in \Gamma(y_0, \ell, M) \\ \text{s.t.} \\ |\partial^\alpha (P - P')(x_0)| \leq \\ \qquad M |x_0 - y_0|^{m-|\alpha|} \quad (\text{all } \alpha) \end{array}}$$

$$\Gamma(x_0, \ell) := \left(\Gamma(x_0, \ell, M) \right)_{M>0}$$

is a blob in $\mathcal{P}$.

TRIVIAL INDUCTION ON $\ell \Longrightarrow$

$$\Gamma(x_0, M) \subseteq \Gamma(x_0, \ell, M)$$

(all $\ell \geq 0$).

MAIN THM: (1rst Version)

For an integer const. ℓ_* depending only on m & n, the blobs $\Gamma(x_0, \ell_*)$ and $\Gamma(x_0)$ are C-equivalent, for each $x_0 \in E$.

Roughly speaking,

- Proof of MAIN THM is constructive.
- Proofs of our other Thms. arise from implementing proof of Main Thm by efficient algorithms.

However, ...

- Need cheaper Γ's

- Make strong use of ideas from COMPUTER SCIENCE, both to make the cheaper Γ's, & to implement **EFFICIENTLY** the proof of the MAIN THM.

PLAN

- Define convex sets $\sigma(x_0, l) \subset P$
- State the basic properties of the Γ's & σ's
- Generalize Main Thm
- Sketch Proof of Main Thm
- Making cheap Γ's & σ's.

DEFINING $\sigma(x_0, \ell)$

Fix m, n.

$\boxed{\text{RECALL: Given } E, f, \sigma, \text{ obtain } \Gamma(x_0, M), \Gamma(x_0, \ell, M).}$

Now: Change f to 0.

Obtain $\Gamma_0(x_0, M)$, $\Gamma_0(x_0, \ell, M)$.

$\Gamma_0(x_0, M)$ consists of all $J_{x_0}(F)$

for $F \in C^m(\mathbb{R}^n)$ s.t.

$$\boxed{\begin{array}{l} \|F\| \leq M \\ |F(x)| \leq M\sigma(x) \text{ for all } x \in E. \end{array}}$$

<u>Note</u>: $\Gamma_0(x_0, M)$ has trivial

M-dependence !

$$\Gamma_0(x_0, M) = M \cdot \underline{\sigma}(x_0)$$

for a convex symmetric set

$\underline{\sigma}(x_0) \subset \mathcal{P}$.

Similarly, $\Gamma_0(x_0, \ell, M)$ has trivial M-dependence:

$$\Gamma_0(x_0, \ell, M) = M \cdot \sigma(x_0, \ell)$$

for $\sigma(x_0, \ell) \subset \mathcal{P}$ convex, symmetric.

BASIC PROPERTIES OF THE

Γ's and σ's

PROPERTY 0 :

$$\Gamma(x_0, M) \subset \Gamma(x_0, \ell, CM)$$

for any $\ell \geq 0$.

(FOLLOWS BY a trivial induction on ℓ.)

PROPERTY 1:

Given $x_0, y_0 \in E$, $P \in \Gamma(x_0, \ell, M)$.

Then there exists $P' \in \Gamma(y_0, \ell-1, M)$

s.t.

$$\boxed{|\partial^\alpha (P - P')(x_0)| \leq M |x_0 - y_0|^{m-|\alpha|} \quad (\text{all } \alpha)}$$

Similarly for the $\sigma(x_0, \ell)$.

PROPERTY 2 :

- $\Gamma(x_0, \ell, M) + M\sigma(x_0, \ell)$
  $\subseteq \Gamma(x_0, \ell, CM)$

- $\Gamma(x_0, \ell, M) - \Gamma(x_0, \ell, M)$
  $\subseteq CM\sigma(x_0, \ell)$

$$\Gamma(x_0, M) + M\underline{\sigma}(x_0) \subseteq \Gamma(x_0, CM)$$

because

$$\boxed{\|F\| \le M, \quad |F(x) - f(x)| \le M\sigma(x) \text{ (all } x \in E)}$$

$+$

$$\boxed{\|G\| \le 1, \quad |G(x)| \le \sigma(x) \text{ (all } x \in E)}$$

$\Downarrow$

$$\boxed{\begin{array}{l} \|F + MG\| \le 2M, \quad \& \\ |(F+MG)(x) - f(x)| \le 2M\sigma(x) \\ \qquad \text{(all } x \in E) \end{array}}$$

Similarly,

$$\Gamma(x_0, M) - \Gamma(x_0, M) \leq 2M\underline{\sigma}(x_0).$$

Analogous properties of $\Gamma(x_0, \ell, M)$ and $\sigma(x_0, \ell)$ follow from trivial induction on ℓ.

PROPERTY 3:

Given $P \in \sigma(x_0, \ell)$, $Q \in \mathcal{P}$, $\delta > 0$.

Assume

$$\left\{ \begin{array}{l} |\partial^\alpha P(x_0)| \leq \delta^{m-|\alpha|} \\ |\partial^\alpha Q(x_0)| \leq \delta^{-|\alpha|} \end{array} \right\} \text{(all } \alpha\text{)}$$

Then $P \odot Q \in C\sigma(x_0, \ell)$

(PRODUCT AS TAYLOR POLY'S AT x_0)

- "Whitney convexity" of $\sigma(x_0, \ell)$
- Lets us use partitions of unity at any lengthscale δ.
- Proven by an (almost) trivial induction on ℓ.

PROPERTY 4 :

$$\Gamma(x_0, 0, M) =$$

$$\Big\{ P \in \mathcal{P} : |\partial^\alpha P(x_0)| \leq M \quad (\text{all } \alpha)$$

$$\& \quad |P(x_0) - f(x_0)| \leq M\sigma(x_0) \Big\}$$

& SIMILARLY FOR $\sigma(x_0, 0)$.

BY CLEVER USE OF THE

"WELL-SEPARATED PAIRS DECOMPOSITION"

from COMPUTER SCIENCE,

can construct <u>cheaply</u> <u>OTHER</u>

$\Gamma(x_0, \ell), \sigma(x_0, \ell)$ satisfying

PROPERTIES 0, 1, 2, 3, 4.

MAIN THM (2$^{\text{ND}}$ VERSION):

Suppose we are given blobs $\Gamma(x_0, \ell)$ and convex sets $\sigma(x_0, \ell)$, for $x_0 \in E$, $\ell \geq 0$.

Assume the Γ's & σ's satisfy PROPERTIES 0, 1, 2, 3, 4.

Then, for large enough ℓ_*, dep. only on m & n, the blobs $\Gamma(x_0, \ell_*)$ and $\Gamma(x_0)$ are C-EQUIVALENT (all $x_0 \in E$).

Proof of the MAIN THM is constructive.

All our results follow by implementing proof of the MAIN THM as an algorithm.

To get efficient algorithms, use ideas from Computer Science

(WELL-SEPARATED PAIRS DECOMP.
+
BALANCED BOX DECOMP. TREE)

WSPD due to Callahan & Kosaraju

BBD Tree due to Arya, Mount, Netanyahu, Silverman & Wu

See also

HAR-PELED

&

MENDEL

PLAN OF THE PROOF OF THE MAIN THM

(MORE OR $\boxed{\text{LESS}}$ ACCURATE)

"OUR

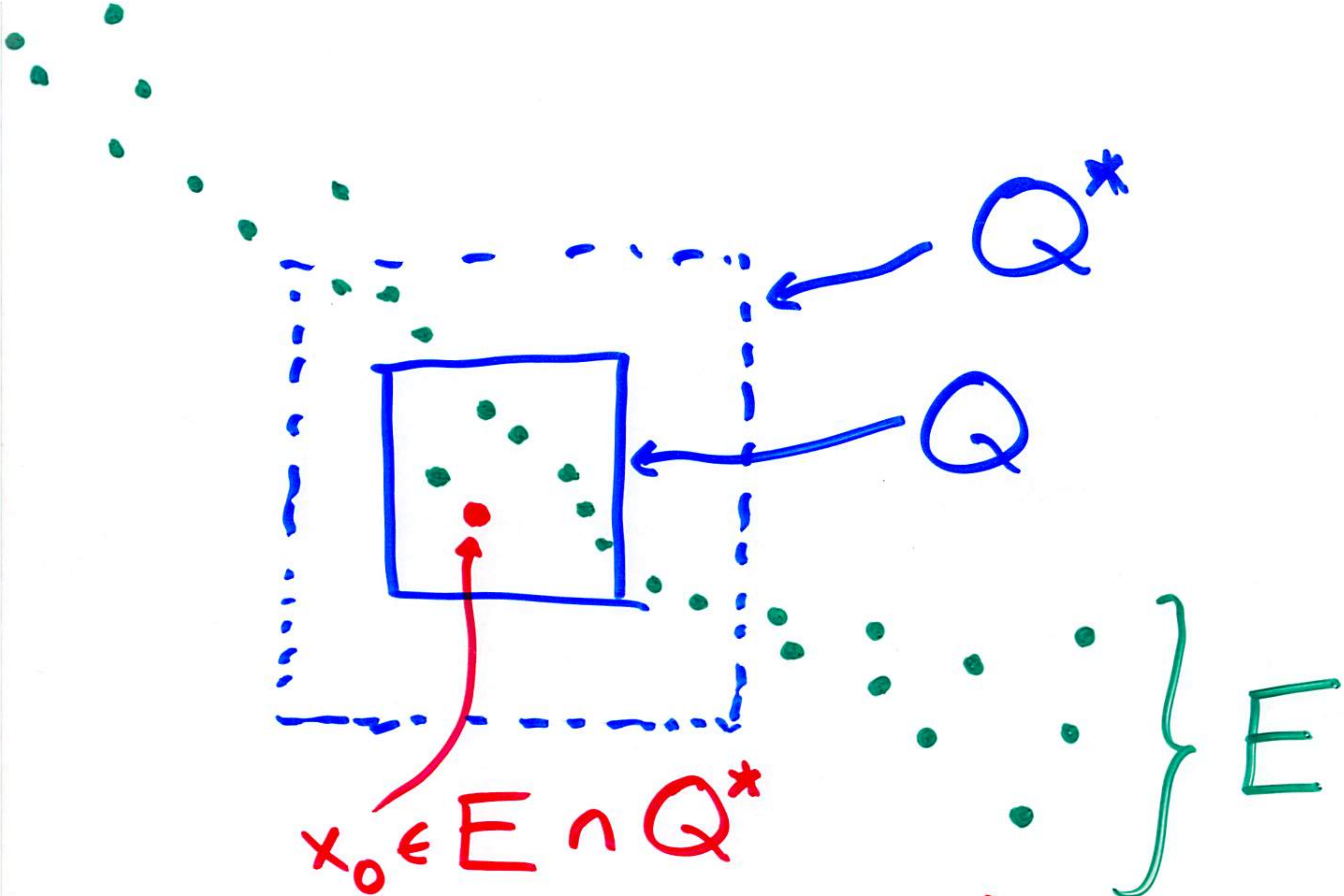
PROBLEM"

Given m, n, E, f, σ.

Given $\left[\begin{array}{l} Q \subset \mathbb{R}^n \text{ cube} \\ x_0 \in E \cap Q^* \\ P_0 \in \Gamma(x_0, \ell, M). \end{array}\right]$

<u>PROBLEM</u>: Find $F \in C^m(\mathbb{R}^n)$ s.t.

- $\|F\| \leq CM$
- $|F(x) - f(x)| \leq CM\sigma(x) \;\; \forall x \in E \cap Q^*$
- $J_{x_0}(F) = P_0$



$x_0 \in E \cap Q^*$

$$P_0 \in \underbrace{\Gamma(x_0, \ell, M)}_{\text{should be BLUE}}$$

PROVING THE MAIN THM AMOUNTS TO SOLVING THE ABOVE PROBLEM FOR $\underline{\underline{m, n, E, f, \sigma, Q, x_0, P_0}}$

with Q = UNIT CUBE.

Fix m, n, E, f, σ.

For any $Q \subset \mathbb{R}^n$, will attach a "LABEL" to Q, to tell us how hard our problem is.

- LABELS arise from looking at the size & shape of $\sigma(x, \ell)$ for $x \in E \cap Q^*$.

- LABELS are <u>ordered</u>.

  If $\mathcal{A} < \mathcal{B}$, then a problem with label $\mathcal{A}$ is EASIER than a problem with label $\mathcal{B}$.

- For the <u>easiest</u> label (minimal under $<$), our problem is trivial. We can just set

$$F = P_0 \,,$$

& it works.

- Solving our problem for the <u>HARDEST</u> label (MAXIMAL under $<$) amounts to proving the MAIN THM in the general case.

- The total no. of LABELS is a constant depending only on m & n.
- WE SOLVE OUR PROBLEM BY INDUCTION ON THE LABEL (with resp. to $<$)

BASE CASE OF THE INDUCTION

For the minimal label (under $<$),
we have already promised
that our problem is trivial,
because we can just use

$$F := P_0$$

INDUCTION STEP

- Fix a label a_0 (not the easiest)
- Suppose we know how to solve our problem whenever the label is easier than a_0 (i.e., $< a_0$).
- Will solve our problem whenever the label is a_0.

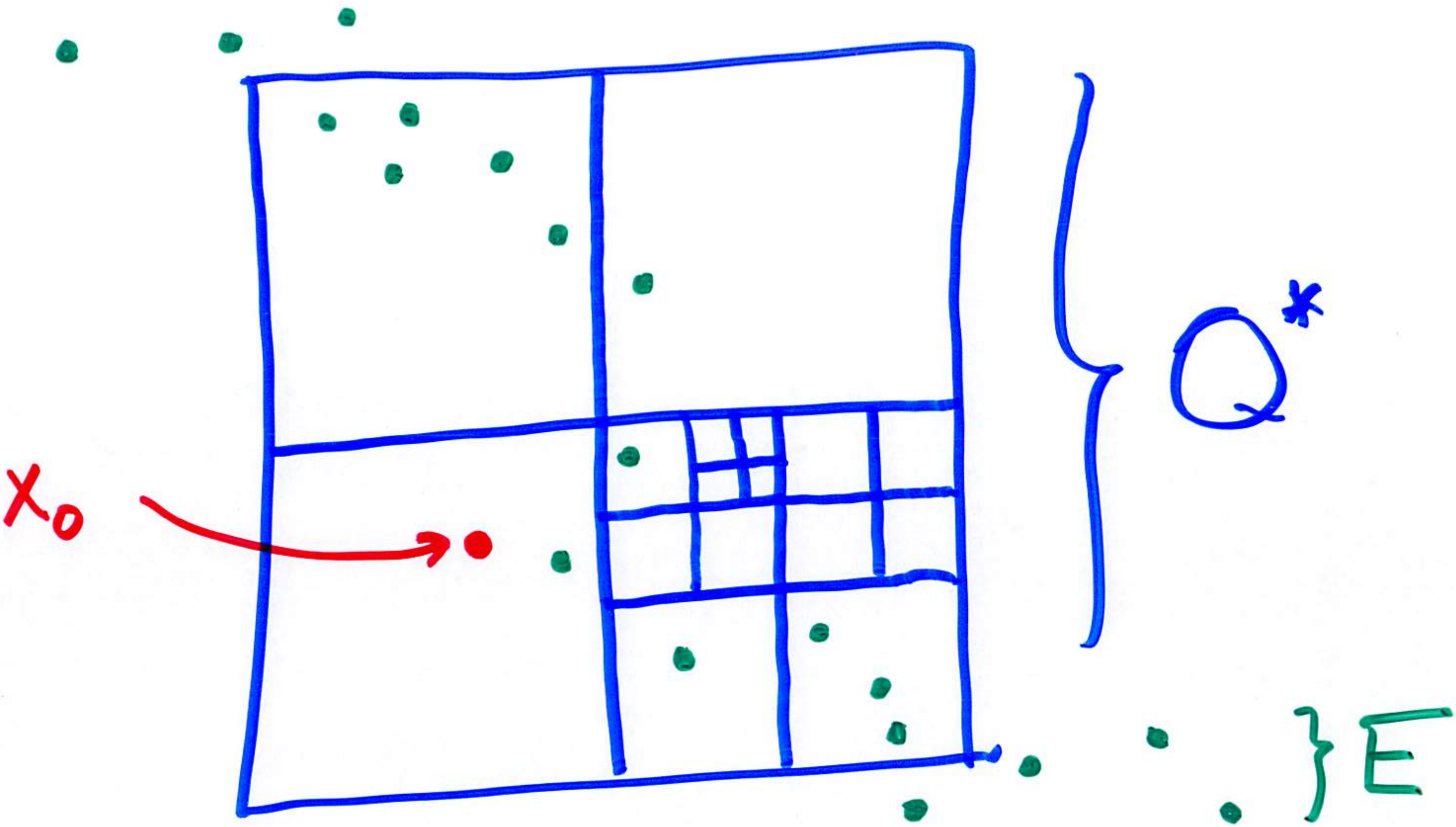
RECALL, WE ARE GIVEN

$\underline{m, n, E, f, \sigma}$, and

- $Q \subset \mathbb{R}^n$ cube
- $x_0 \in E \cap Q^*$
- $P_0 \in \Gamma(x_0, \ell, M)$

WE HAVE TO FIND $F \in C^m(\mathbb{R}^n)$

SUCH THAT ....



MAKE A C-Z DECOMP. OF Q^* INTO $\{Q_\nu\}$

THE C-Z RULE

Stop cutting Q_ν

as soon as

EITHER

- Q_ν carries a label easier than Q_0

OR

- $\#(E \cap Q_\nu^*) \leq 1$.

- In each Q_ν we pick a

<u>REPRESENTATIVE</u>

$$x_\nu \in E \cap Q_\nu^*$$

(Ignore here the case

$$E \cap Q_\nu^* = \emptyset .)$$

- For each ν, find

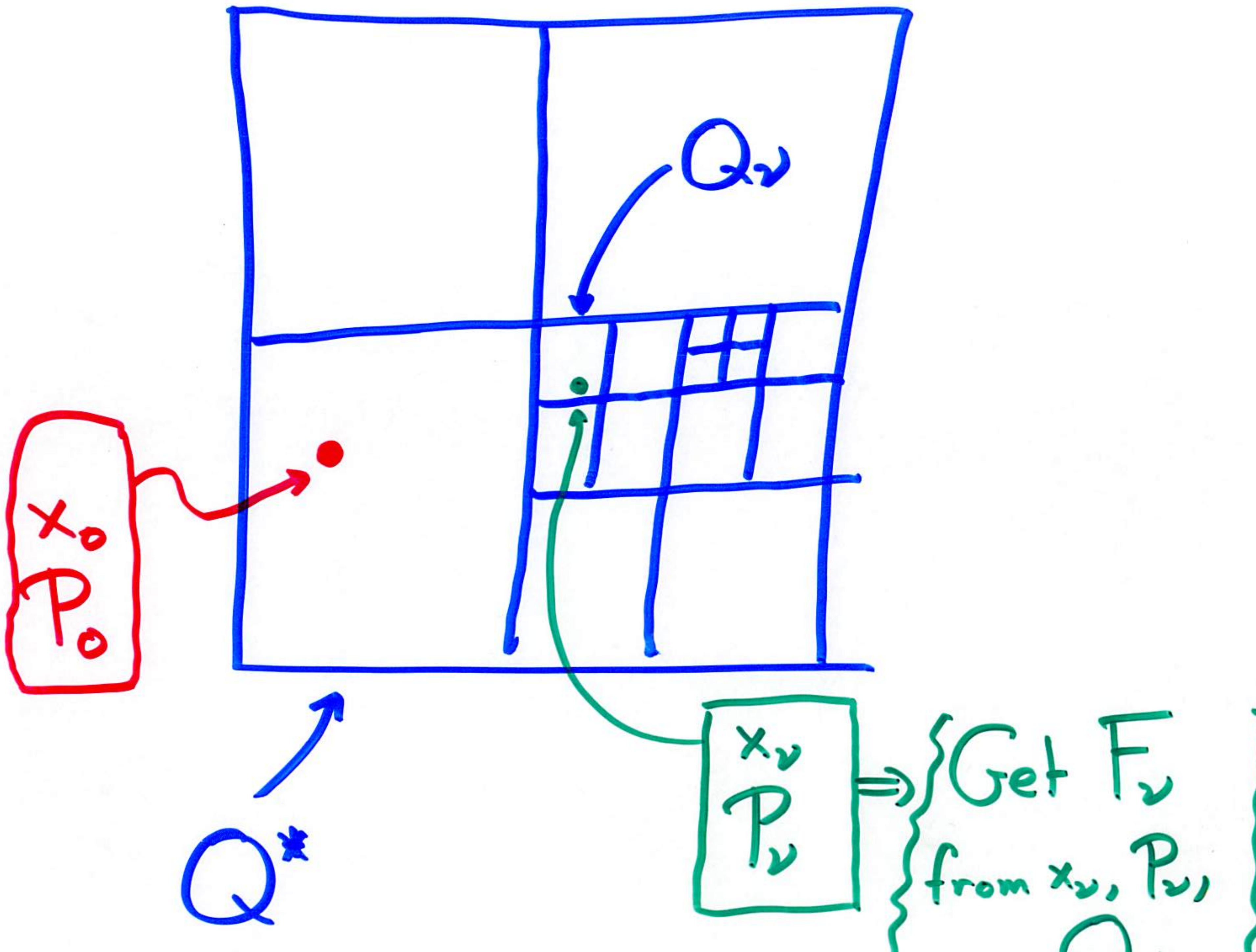
$P_\nu \in \Gamma(x_\nu, \ell-1, CM)$

s.t.

$$\left|\partial^\alpha (P_\nu - P_0)(x_\nu)\right| \leq CM \left|x_0 - x_\nu\right|^{m-|\alpha|}$$

(all α).

(RECALL BASIC PROP's OF Γ's.)


Q_ν
x_0
P_0
Q^*
x_ν
P_ν
Get F_ν
from x_ν, P_ν,
Q_ν.

- For each ν, find $F_\nu \in C^m(\mathbb{R}^n)$ s.t.

$$\boxed{\begin{array}{l} \|F_\nu\| \leq CM \\ |F_\nu(x) - f(x)| \leq CM\sigma(x) \ \forall x \in E \cap Q_\nu^* \\ J_{x_\nu}(F_\nu) = P_\nu. \end{array}}$$

We can do that, <u>either</u> because Q_ν^* carries a label easier than $\mathcal{A}_0$ <u>or</u> because $\#(E \cap Q_\nu^*) \leq 1$.

- COMBINE THE "LOCAL SOLUTIONS" F_ν INTO A "GLOBAL SOLUTION"

$$F = \sum_\nu \theta_\nu F_\nu$$

USING A PARTITION OF UNITY

$$1 = \sum_\nu \theta_\nu$$

ADAPTED TO THE CZ DECOMP.

IT WORKS,

So we have solved "OUR PROB",

proven the MAIN THM,

& therefore also proven

Thms 1, 2, 3, 4

from previous lecture.

NO TIME FOR

- The balanced box decomp. tree
- Def. of the LABELS & their order
- Any more details of the proof.
- Generalizations & further results.

THERE MUST BE
AN EASIER
PROOF.