JOINT WORK

WITH

BO'AZ KLARTAG

ALSO, CREDIT IS DUE TO:

- H. WHITNEY,
- G. GLAESER,
- Y. BRUDNYI & P. SHVARTSMAN
- E. BIERSTONE, P. MILMAN & W. PAWŁUCKI
- N. ZOBIN

CALLAHAN-
KOSARAJU

HAR-PELED &
MENDEL

ARYA-MOUNT-NETANY-
-SILVERMAN-WU

Fix $m, n \geq 1$.

$C^m(\mathbb{R}^n)$ = space of real-valued fns. on $\mathbb{R}^n$, with (bounded) continuous deriv's up to order m.

$$\|F\| = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)| .$$

WHITNEY'S PROBLEM

GIVEN: $m, n \geq 1$, $f: E \to \mathbb{R}$,

$E \subset \mathbb{R}^n$ compact.

DECIDE: DOES THERE EXIST

$F \in C^m(\mathbb{R}^n)$ such that

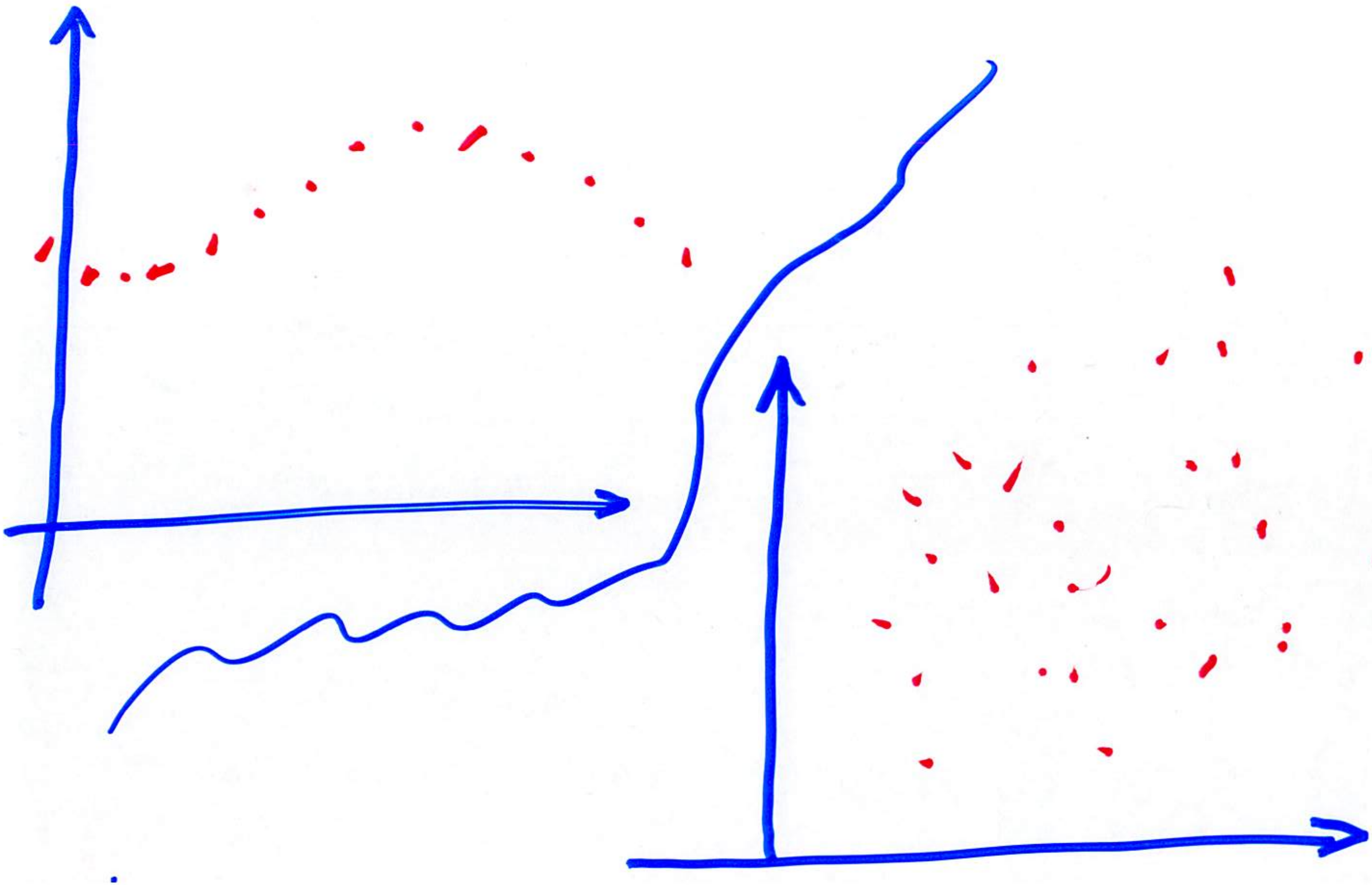
$F = f$ on E ?

ALSO,

- If F exists, how small can we take its C^m-norm ?
- What can we say about the Taylor poly. of F at a given point ?
- Can we take F to depend linearly on f ?

EFFECTIVE, FINITE VERSION

- Fix $m, n \geq 1$.
- Given points $z_1, \cdots, z_N \in \mathbb{R}^{n+1}$.
- FIND $F \in C^m(\mathbb{R}^n)$, whose graph passes $\left\{\begin{matrix}\text{through} \\ \text{close to}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{all} \\ \text{most}\end{matrix}\right\}$ of the points $z_1, \cdots, z_N$, with the order of magnitude of $\|F\|$ as small as possible.



$\left[\begin{array}{l} X, Y \geq 0 \text{ determined by} \\ m, n, \text{ other data.} \end{array}\right] \Longrightarrow$

$X \sim Y$ means $cX \leq Y \leq CX$

with c, C depending only on m, n.

(c, C, C', etc **ALWAYS** dep. only on m, n)

To COMPUTE THE ORDER OF MAGNITUDE of X

is to compute some Y

such that

$$X \sim Y.$$

COMPUTING A FUNCTION F

- WE ENTER THE DATA.
- THE COMPUTER PERFORMS "ONE-TIME WORK", & then says "READY!"

WE THEN "QUERY" the computer.

A <u>QUERY</u> consists of a pt.

$\underline{x} \in \mathbb{R}^n$.

The computer responds by computing $\partial^\alpha F(\underline{x})$ for all $|\alpha| \leq m$.

COMPUTER RESOURCES

- ONE-TIME WORK
- WORK TO ANSWER A QUERY
- STORAGE

A "COMPUTER"

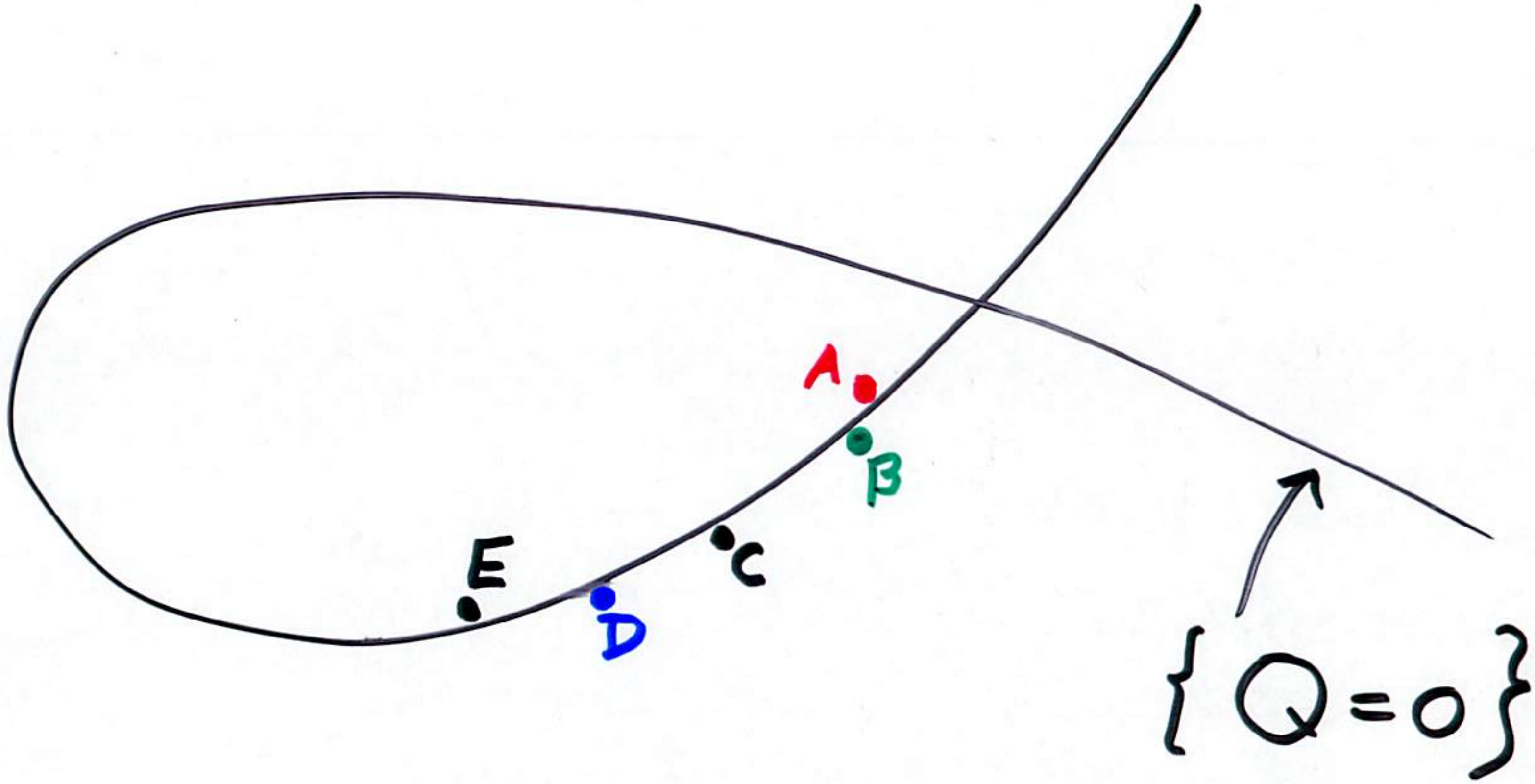
Standard Von Neumann

architecture.

Manipulates (EXACT)
real numbers.

(also: FINITE PRECISION)

WANT ALGORITHMS THAT *ALWAYS* WORK



Do we use F or $F+Q$?

THE PROBLEMS

Fix $m, n \geq 1$.

GIVEN:

- $E \subset \mathbb{R}^n$ finite $(\#E = N)$
- $f : E \to \mathbb{R}$
- $\sigma : E \to [0, \infty)$

WANT TO FIND

$F \in C^m(\mathbb{R}^n)$ and $M > 0$

SUCH THAT

$$\boxed{\begin{array}{l} \|F\| \le M, \quad \text{and} \\ |F(x) - f(x)| \le M\sigma(x) \text{ for all } x \in E \end{array}}$$

DEFINE $\|f\|_{(E,\sigma)} =$

$$\inf\{M > 0 : \exists F \in C^m(\mathbb{R}^n) \text{ s.t. } \}$$

PROBLEMS: GIVEN m, n, E, f, σ,

(1) Compute the order of magnitude of $\|f\|_{(E,\sigma)}$.

---

(2) Compute a fn. $F \in C^m(\mathbb{R}^n)$ s.t.

$$\boxed{\|F\| \le M, \quad |F(x) - f(x)| \le M\sigma(x) \quad (\text{all } x \in E)}$$

with $M \le C\|f\|_{(E,\sigma)}$.

(3) Given m, n, E, f, σ (with $\#E = N$).

Given $Z \leq N$.

PROBLEM:

Find $S \subset E$, with $\# S \leq Z$

and $\| f \|_{(E \setminus S, \sigma)}$

as small as possible.

How small can we make it?

Thm 1: Given E, f, σ, m, n $\quad (\#E = N)$

We can compute the order of magnitude of $\|f\|_{(E,\sigma)}$

using $\left\{ \begin{array}{l} CN \log N \text{ operations} \\ CN \text{ storage} \end{array} \right\}$

Thm 2: Given m, n, E, f, σ with $\# E = N$,

we compute $F \in C^m(\mathbb{R}^n)$ such that

- $\|F\| \leq M$
- $|F(x) - f(x)| \leq M\sigma(x)$ (all $x \in E$)
- $M \leq C\|f\|_{(E,\sigma)}$.

One-time work $\leq CN \log N$

Work to answer QUERY $\leq C \log N$

Storage $\leq CN$

Thm 3: Given m, n, E, f, σ $(\#E = N)$,

& GIVEN $1 \leq Z \leq N$.

We compute $S^* \subset E$ such that

- $\#S^* \leq CZ$
- For any $S \subset E$ with $\#S \leq Z$, we have $\|f\|_{(E \setminus S^*, \sigma)} \leq C \|f\|_{(E \setminus S, \sigma)}$

Work $\leq CZN \log N$, Storage $\leq CN$.

Thm 4: Given m, n, E, σ $(\#E = N)$,

there exist $S_1, S_2, \dots, S_L \subset E$ s.t.

- $\#S_\ell \leq C$ for each ℓ
- $L \leq CN$
- $\|f\|_{(E,\sigma)} \sim \max_{\ell} \|f\|_{(S_\ell, \sigma)}$

for any $f : E \longrightarrow \mathbb{R}$.

Moreover, we can compute the list $S_1, S_2, \ldots, S_L$ with

$$\text{Work} \leq C N \log N,$$

$$\text{Storage} \leq C N.$$

REMOVE_OUTLIERS (m, n, E, σ, f, Z)

{ While ($E \neq \emptyset$ and $Z \neq 0$)

{ • Compute $S_1, S_2, \cdots, S_L$ as in Thm 4.

• For $\ell = 1, \cdots, L$
compute $X_\ell \sim \|f\|_{(S_\ell, \sigma)}$.

• Pick ℓ_* to maximize X_{ℓ_*}.

• Remove S_{ℓ_*} from E.

• Replace Z by $Z-1$.

} }

WHY IT WORKS

- Work $\leq CZN\log N$, Storage $\leq CN$.

Let $S^* = \{$all the points removed$\}$.

- **MUST SHOW:**
- $\#S^* \leq CZ$
-
- If $S \subset E$ with $\#S \leq Z$, then

$$\|f\|_{(E \setminus S^*, \sigma)} \leq C\|f\|_{(E \setminus S, \sigma)}.$$

WHY $\# S^* \le CZ$

- Each time we execute the body of the loop, we remove S_{ℓ_*}.
- We have $\# S_{\ell_*} \le C$ by Thm 4.
- We execute the body of the loop at most Z times.

Comparing $\|f\|_{(E \setminus S^*; \sigma)}$ with $\|f\|_{(E \setminus S, \sigma)}$

Suppose $S \subset E$, $\# S \leq Z$.

We don't care about S unless

(1) $\|f\|_{(E \setminus S, \sigma)} \ll \|f\|_{(E, \sigma)}$.

OK, let's assume (1).

Then S must have a point in common with S_{ℓ_*} (as computed the first time).

Otherwise, $S_{\ell_*} \subset E \setminus S$, hence

$$\|f\|_{(E,\sigma)} \sim \max_{\ell} \|f\|_{(S_\ell,\sigma)}$$

$$\sim \|f\|_{(S_{\ell_*},\sigma)} \leq \|f\|_{(E\setminus S,\sigma)},$$

Contradicting (1).

So, 1rst time we execute the body of the loop,

We remove from E at most C points, at least one of which must belong to S (any viable competitor).

REPEAT

SOME IDEAS FROM PROOFS OF THMS 1, 2, 4

For $F \in C^m(\mathbb{R}^n)$, $x \in \mathbb{R}^n$,

define

$$J_x(F) = \left[\begin{array}{l} \text{the } (m-1)^{\underline{\text{rst}}} \text{ degree} \\ \text{Taylor poly.} \\ \text{of } F \text{ at } x \end{array} \right]$$

So $J_x(F)$ belongs to

$$\mathcal{P} = \left[\begin{array}{l} \text{VECTOR SPACE OF (REAL)} \\ (m-1)^{\text{rst}} \text{ DEGREE POLY'S} \\ \text{ON } \mathbb{R}^n \end{array} \right]$$

RING STRUCTURE ON $\mathcal{P}$

$$J_x(F) \odot_x J_x(G) = J_x(FG)$$

KEY QUESTION:

Given m, n, E, f, σ.

Given $x_0 \in \mathbb{R}^n$, $M > 0$.

Suppose $\exists\, F \in C^m(\mathbb{R}^n)$, with

$$\boxed{\begin{array}{l} \|F\| \leq M \\ |F(x) - f(x)| \leq M\sigma(x) \text{ (all } x \in E). \end{array}}$$

WHAT CAN WE SAY ABOUT

$J_{x_0}(F)$?

DEFINE A CONVEX SET

$$\Gamma(x_0, M) =$$

$$\left\{ J_{x_0}(F) : F \in C^m(\mathbb{R}^n), \text{ s.t. } \boxed{\begin{array}{c} \|F\| \leq M, \text{ and} \\ |F(x) - f(x)| \leq M\sigma(x) \\ (\text{all } x \in E). \end{array}} \right\}.$$

$$\Gamma(x_0, M) \subset \mathcal{P}$$

(ALWAYS CONVEX
MAYBE EMPTY)

$$\boxed{\Gamma(x_0, M) \neq \emptyset}$$

$$\Updownarrow$$

$\exists\, F \in C^m(\mathbb{R}^n)$, such that

$\|F\| \leq M$, &

$|F(x) - f(x)| \leq M\,\sigma(x)$, all $x \in E$.

PROBLEM :

Compute the approximate size and shape of

$\Gamma(x_0, M)$

Thm 5: Given m, n, E, f, σ $(\#E = N)$.

With work $\leq C N \log N$, storage $\leq C N$,

we can exhibit convex sets

$\hat{\Gamma}(x_0, M)$ (all $x_0 \in E$, $M > 0$)

s. t.

$$\boxed{\begin{array}{l} \Gamma(x_0, M) \subset \hat{\Gamma}(x_0, CM) \\ \hat{\Gamma}(x_0, M) \subset \Gamma(x_0, CM) \end{array}}$$

Then, can answer QUERIES

QUERY: ANY $x_0 \in \mathbb{R}^n$.

Response: FAMILY $\left(\hat{\Gamma}(M)\right)_{M>0}$

s.t.

$$\boxed{\begin{array}{l} \Gamma(x_0, M) \subset \hat{\Gamma}(CM) \\ \hat{\Gamma}(M) \subset \Gamma(x_0, CM) \end{array}}$$

WORK TO ANSWER QUERY:

$C \log N$

CONSTRUCTING $\hat{\Gamma}(x_0, M)$

I. By induction on $\ell \geq 0$, construct $\Gamma(x_0, \ell, M)$ s.t. $\Gamma(x_0, M) \subset \Gamma(x_0, \ell, M)$.

II. For large enough $\ell = \ell(m, n)$, have $\Gamma(x_0, \ell, M) \subset \Gamma(x_0, CM)$.

BASE CASE $\ell = 0$: For $x_0 \in E$, $M > 0$,

$\Gamma(x_0, 0, M)$ CONSISTS OF ALL $P \in \mathcal{P}$ SUCH THAT

$$\boxed{\begin{aligned} &|P(x_0) - f(x_0)| \leq M\sigma(x_0), \quad \& \\ &|\partial^\alpha P(x_0)| \leq M \quad (\text{all } |\alpha| \leq m-1) \end{aligned}}.$$

OBVIOUSLY, $\Gamma(x_0, M) \subset \Gamma(x_0, 0, M)$.

PREPARING FOR INDUCTION STEP

SUPPOSE

$$\|F\| \le M, \quad P = J_{x_0}(F), \quad P' = J_{y_0}(F).$$

THEN

$$\left|\partial^\alpha (P - P')(x_0)\right| \le M |x_0 - y_0|^{m-|\alpha|}$$

(all α)

(by Taylor's Thm.)

INDUCTION STEP

- Fix $\ell \geq 0$.
- Suppose we have already defined $\Gamma(x_0, \ell, M)$ for all $x_0 \in E$, $M > 0$.

(and suppose $\Gamma(x_0, M) \subset \Gamma(x_0, \ell, M)$).

- Then we will define the

$\Gamma(x_0, \ell+1, M)$

for all $x_0 \in E$, $M > 0$

$\Gamma(x_0, \ell+1, M)$

consists of all $P \in \mathcal{P}$ s.t.

$$\boxed{\begin{array}{l} \forall y_0 \in E \quad \exists P' \in \Gamma(y_0, \ell, M) \\ \text{satisfying} \\ |\partial^\alpha (P - P')(x_0)| \leq M |x_0 - y_0|^{m-|\alpha|} \\ \qquad\qquad\qquad (\text{all } \alpha). \end{array}}$$

NOTE: $\Gamma(x_0, M) \subset \Gamma(x_0, \ell+1, M)$

So, we have constructed

$\Gamma(x_0, \ell, M)$

for $x_0 \in E$, $\ell \geq 0$, $M > 0$,

such that

$$\Gamma(x_0, M) \subset \Gamma(x_0, \ell, M).$$

Thm 6: For large enough $\ell = \ell(m,n)$, we have

$$\Gamma(x_0, \ell, M) \subset \Gamma(x_0, CM).$$

[TRIVIALLY,

$$\Gamma(x_0, M) \subset \Gamma(x_0, \ell, M).]$$

ACTUALLY, WE CONSTRUCT DIFFERENT $\Gamma(x_0, \ell, M)$.

[The above Γ's are too expensive.

WORK $\sim N^2$]

NEED A

GENERALIZATION

OF

THM. 6

The COMPUTER SCIENCE ideas

PROBLEM :

Given $\left\{ \begin{array}{l} E \subset \mathbb{R}^n, \quad \#(E) = N. \\ f : E \to \mathbb{R} \end{array} \right\}$

COMPUTE the Lip(1) - norm of f.

NAÏVE METHOD $\Rightarrow$

WORK $\sim N^2$

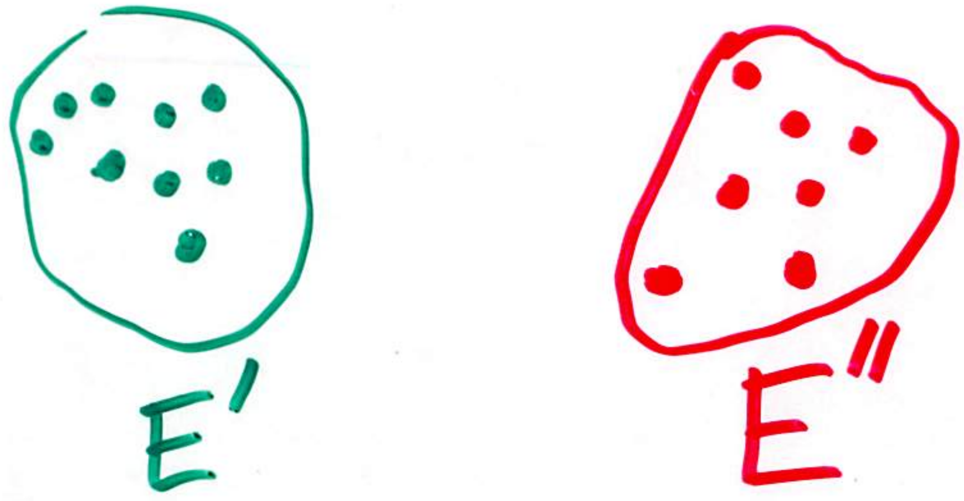
WSPD $\Rightarrow$

Work $\sim C\, N \log N$

(DEP. only on n)

(Computes Lip(1) norm up to factor of 2)

(or 1.00001)



"WELL-SEPARATED"

Assume

$$\boxed{\begin{aligned} \mathrm{dist}(E', E'') &\geq 2\,\mathrm{diam}(E') \\ \mathrm{dist}(E', E'') &\geq 2\,\mathrm{diam}(E'') \end{aligned}}$$

Then we can compute

$$\max_{x \in E',\, y \in E''} |f(x) - f(y)| / |x - y| \quad \text{"EFFICIENTLY"}$$

THE WSPD:

Given $E \subset \mathbb{R}^n$, $\#E = N$,

we can partition

$E \times E \setminus \text{Diagonal}$

into Cartesian products

$E'_\nu \times E''_\nu$, $\nu = 1, \ldots, \nu_{max}$ s.t.

- E'_ν & E''_ν WELL-SEPARATED (each ν)
- $\nu_{max} \leq CN$, C dep only on n.

- MOREOVER, the E'_ν, E''_ν $(\nu = 1, \dots, \nu_{max})$ can be COMPUTED with work $C N \log N$ & storage $C N$. (C dep. only on N.)

P.S. To compute Lip(1) norms from a WSPD, it's enough to examine

$$|f(x_\nu) - f(y_\nu)| / |x_\nu - y_\nu|$$

for just ONE pair $(x_\nu, y_\nu) \in E'_\nu \times E''_\nu$ for each ν.

OPEN PROBLEMS

1. PROOFS ARE HARD.
[SIMPLIFY THEM!]

2. CONSTANTS C DEPEND ONLY ON m, n but are huge (?) [REDUCE THEM!]

3. HOW MUCH WORK DOES IT REALLY TAKE TO REMOVE OUTLIERS?

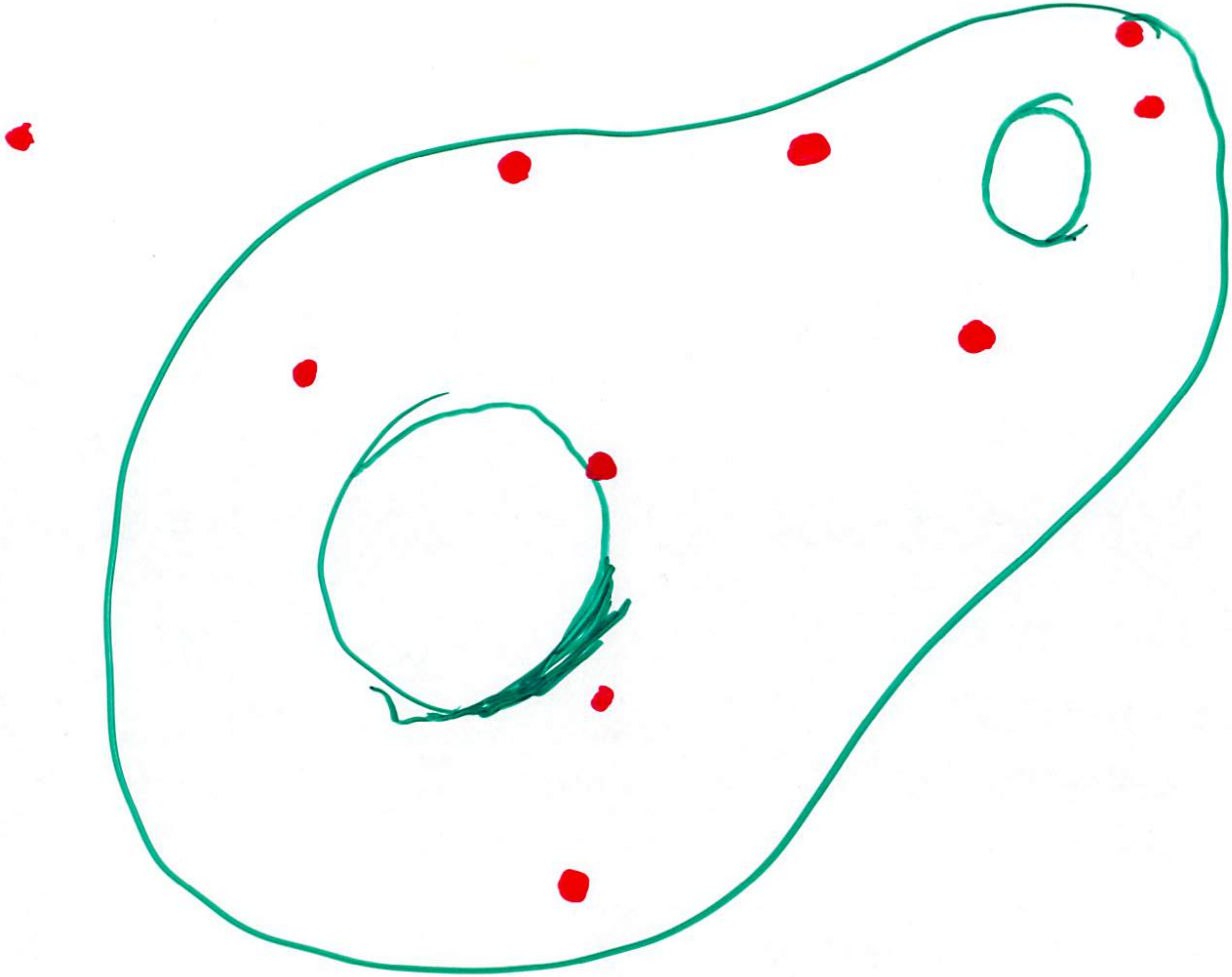
Can we improve on $2N \log N$?

4. OTHER NORMS in place of $C^m(\mathbb{R}^n)$,

e.g.,

SOBOLEV NORMS

5. HOW TO FIT N GIVEN POINTS IN $\mathbb{R}^n$ to a C^m-HYPERSURFACE THAT NEEDN'T BE A GRAPH?



6. ARE THERE ANY PRACTICAL APPLICATIONS OF ANY OF THE IDEAS ?

FURTHER
PROGRESS

GIVEN:

- POINTS $x_1, \cdots, x_N \in \mathbb{R}^n$
- POLY'S $P_1, \cdots, P_N \in \mathcal{P}$
- $\varepsilon > 0$.

FIND: $F \in C^m(\mathbb{R}^n)$ s.t.

- $J_{x_i}(F) = P_i$ (each i),
- $\|F\|_{C^m(\mathbb{R}^n)}$ within ϵ percent of "min".

GIVEN:

- POINTS $x_1, \cdots, x_N \in \mathbb{R}^n$
- REAL NUMBERS $y_1, \cdots, y_N$.
- $\varepsilon > 0$

FIND: $F \in C^m(\mathbb{R}^n)$ s.t.

- $F(x_i) = y_i$ (each i),
- $\|F\|_{C^m(\mathbb{R}^n)}$ within ε percent of "min"

WHICH C^m-NORM ?

$$\|F\| = \max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |\partial^\alpha F(x)|$$

$$\|F\| = \sup_{x \in \mathbb{R}^n} \left(\sum_{|\alpha| \leq m} |\partial^\alpha F(x)|^2 \right)^{1/2}$$

MANY OTHERS.

ASSUME :

For each $x \in \mathbb{R}^n$, we are given a norm $|\cdot|_x$ on $\mathcal{P}$.

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |J_x(F)|_x$$

EXAMPLES

$$|P|_x = \max_{|\alpha| \le m} |\partial^\alpha P(x)|$$

$$|P|_x = \left(\sum_{|\alpha| \le m} |\partial^\alpha P(x)|^2 \right)^{1/2}$$

ASSUME OUR COMPUTER

CAN LEARN $|P|_x$

(P, x given)

FROM AN "ORACLE"

THM 1: CAN SOLVE THE "$J_{x_i}(F) = P_i$" PROB.

- ONE-TIME WORK
  $\exp(C/\varepsilon)\, N \log N$
- QUERY WORK
  $C \log(N/\varepsilon)$
- STORAGE $\exp(C/\varepsilon) N$

THM 2: CAN SOLVE THE "$F(x_i) = y_i$" PROB.

- ONE-TIME WORK
  $$\exp(C/\varepsilon)\, N^5 (\log N)^2$$
- QUERY WORK
  $$C \log(N/\varepsilon)$$
- STORAGE
  $$\exp(C/\varepsilon)\, N^2$$

PROOF OF THM 1
BASED ON A
"FINITENESS PRINCIPLE"

ANALOGOUS FINITENESS PRINCIPLE FOR THM 2 IS FALSE.

(cf & Bo'az Klartag)

SHARP ALGORITHM NEEDS NEW IDEAS.