Composition Operators on Bergman Spaces

John H. Clifford **Dechao Zheng***

Abstract

We obtain function theoretic characterizations of the compactness on the standard weighted Bergman spaces of the two operators formed by multiplying a composition operator with the adjoint of another composition operator.

1 Introduction

Let $\varphi : D \to D$ be a holomorphic self-map of the unit disk $D = \{z : |z| < 1\}$. The *composition operator* C_φ induced by φ is the linear map on the space of all holomorphic functions on the unit disc defined by $C_\varphi(f) = f \circ \varphi$. By the Littlewood Subordination Theorem [7] the composition operator C_φ is bounded on the standard weighted Bergman spaces $L^2_a(dA_\alpha)$. In this paper we consider the compactness of $C_\varphi C^*_\psi$ or $C^*_\psi C_\varphi$ where C^*_ψ is the adjoint of C_ψ on $L^2_a(dA_\alpha)$.

For $\alpha > -1$, let dA_α denote the normalized measure on D defined by

$$dA_\alpha(z) = (-\log|z|^2)^\alpha dA(z)/\Gamma(\alpha+1).$$

The *standard weighted Bergman space* $L^2_a(dA_\alpha)$ is the Hilbert space of holomorphic functions on D that are also in $L^2(dA_\alpha)$ with inner product given by

$$\langle f, g\rangle_\alpha = \int_D f(z)\overline{g(z)}dA_\alpha(z).$$

As pointed out in [11], in defining the space $L^2_a(dA_\alpha)$, the measure $dA_\alpha(z)$ is frequently replaced by $(1-|z|^2)^\alpha dA(z)/\Gamma(\alpha+1)$ resulting in the same space and an equivalent norm. We let $\alpha = -1$ denote the classical Hardy space H^2. Since weighted Bergman spaces are Hilbert spaces, the adjoint C^*_ψ is a bounded operator on $L^2_a(dA_\alpha)$.

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The main goal of this paper is to provide a function theoretic characterization of the inducing maps φ and ψ for which the operators $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ are compact on $L_a^2(dA_\alpha)$.

For univalent inducing maps, the compactness of $C_\varphi C_\psi^*$ and $C_\psi^* C_\varphi$ has been characterized on the Hardy space in [3]. In this paper the same questions are addressed on the weighted Bergman spaces. Similar methods of proof yield even more complete results than those obtained on the Hardy space.

In order to outline our main results we start with some background material. The *general counting function* $N_{\varphi,\alpha+2}$ defined for $\alpha \geq -1$ is

$$N_{\varphi,\alpha+2}(w) = \sum_{z\in\varphi^{-1}(w)} (-\log|z|)^{\alpha+2}, \qquad w \in \varphi(D) \setminus \{\varphi(0)\},$$

and $N_{\varphi,\alpha+2}(w) = 0$ if w is not in $\varphi(D)$. The points of the inverse image φ^{-1} are regarded as being repeated according to their φ-multiplicity. Note that when $\alpha = -1$ we recover the Nevanlinna counting function for the Hardy space and when $\alpha = 0$ we have the counting function for the classical Bergman space.

We say the *angular derivative* of φ exists at a point $\zeta \in \partial D$ if there exists $\omega \in \partial D$ such that the difference quotient $(\varphi(z) - \omega)/(z - \zeta)$ has a (finite) limit as z tends non-tangentially to ζ through the unit disc. B. D. MacCluer and J. H. Shapiro's [9] showed that C_φ is compact on $L_a^2(dA_\alpha)$ for $\alpha > -1$ if and only if φ does not have a finite angular derivative. This angular derivative criterion completely characterizes the compactness of a composition operator on the weighted Bergman spaces. They also show that the angular derivative criterion fails on the Hardy space. The seminal results on compact composition operators in [11] completely characterizes compactness of a composition operator on the Bergman and the Hardy spaces by proving a composition operator, C_φ, is compact on $L_a^2(dA_\alpha)$ for $\alpha \geq -1$ if and only if

$$\lim_{|w|\to 1^-} \left\{ N_{\varphi,\alpha+2}(w) / (-\log|w|)^{\alpha+2} \right\} = 0.$$

In [3] the compactness of $C_\varphi C_\psi^*$, for general inducing maps, is not completely characterized on H^2.

Our main result completely characterizes the compactness of $C_\varphi C_\psi^*$ on the Bergman spaces $L_a^2(dA_\alpha)$ in terms of both the angular derivative and general counting functions of the inducing maps:

Theorem 1.1 *Suppose that φ and ψ are holomorphic self-maps of D. Then, for $\alpha > -1$, the following three conditions are equivalent.*

(a) $C_\varphi C_\psi^*$ *is compact on* $L_a^2(dA_\alpha)$.

(b) $\lim_{|w|\to 1^-} \left\{N_{\varphi,\alpha+2}(w)N_{\psi,\alpha+2}(w)/ (\log|w|)^{2\alpha+4}\right\} = 0.$

(c) *There does not exist points* ζ_1 *and* ζ_2 *on the unit circle such that* φ *has a finite angular derivative at* ζ_1, ψ *has a finite angular derivative at* ζ_2, *and* $\varphi(\zeta_1) = \psi(\zeta_2)$.

Our main result for the operator $C_\psi^* C_\varphi$ on the weighted Bergman spaces is a sharp upper bound on the essential norm. In [3], for general inducing maps, only a sufficient condition for compactness on H^2 is proved.

Theorem 1.2 *Suppose that* φ *and* ψ *are holomorphic self-maps of* D. *Then, for* $\alpha \geq -1$,

$$\|C_\psi^* C_\varphi\|_{e,\alpha}^2 \leq \limsup \left\{\frac{N_{\varphi,\alpha+2}(\varphi(z))N_{\psi,\alpha+2}(\psi(z))}{(\log|\varphi(z)|\log|\psi(z)|)^{\alpha+2}}\right\}$$

as $|\varphi(z)| \to 1^-$ *or* $|\psi(z)| \to 1^-$.

An immediate corollary of Theorem 1.2 is that if $\alpha \geq -1$

$$\lim \left\{\frac{N_{\varphi,\alpha+2}(\varphi(z))N_{\psi,\alpha+2}(\psi(z))}{(\log|\varphi(z)|\log|\psi(z)|)^{\alpha+2}}\right\} = 0 \quad \text{as} \quad |\varphi(z)| \to 1^- \text{or } |\psi(z)| \to 1^-$$

then $C_\psi^* C_\varphi$ is compact on $L_a^2(dA_\alpha)$.

Finally, if φ and ψ are univalent functions we completely characterize the compactness of $C_\psi^* C_\varphi$ on the Bergman spaces $L_a^2(dA_\alpha)$.

Theorem 1.3 *Suppose* φ *and* ψ *are univalent self-maps of* D. *Then, for* $\alpha > -1$, *the following three conditions are equivalent.*

(a) $C_\psi^* C_\varphi$ *is compact on* $L_a^2(dA_\alpha)$ *for* $\alpha > -1$.

(b) $\lim_{|\varphi(z)|\to 1^- \text{ or } |\psi(z)|\to 1^-} \left\{\frac{N_{\varphi,\alpha+2}(\varphi(z))N_{\psi,\alpha+2}(\psi(z))}{(\log|\varphi(z)|\log|\psi(z)|)^{\alpha+2}}\right\} = 0.$

(c) $\lim_{|z|\to 1^-} \frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} = 0.$

(d) *For each ζ on the unit circle, either φ or ψ cannot have finite angular derivative at ζ.*

The paper is organized as follows. In sections 2 and 3 we develop the results used to prove the three main theorems. Specifically, in section 2 we develop a connection between the operator $C_\varphi C_\psi^*$ on $L_a^2(dA_\alpha)$ and the product of Toeplitz operators on $L_a^2(dA_{\alpha+2})$. While in section 3 we derive two connections between the angular derivatives of φ and ψ and an asymptotic limit of the their generalized counting functions. In section 4 we prove Theorem 1.1 and in section 5 we proof Theorems 1.2 and 1.3.

2 Composition operators via Toeplitz operators.

For functions $f(z) = \sum \hat{f}(n)z^n$ belonging to $L_a^2(dA_\alpha)$, $\alpha \geq -1$ it is well known that the norm of f has the series representation

$$(1) \qquad \|f\|_\alpha^2 = \int_D |f(z)|^2 dA_\alpha = \sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{(n+1)^{\alpha+1}}.$$

As pointed out in [8], there is a natural connection between $L_a^2(dA_\alpha)$ and $L_a^2(dA_{\alpha+2})$ given by

$$(2) \qquad \|f\|_\alpha^2 = \|(zf)'\|_{\alpha+2}^2 \quad \text{for} \quad f \in L_a^2(A_\alpha).$$

Let P_α denote the orthogonal projection from $L^2(dA_\alpha)$ onto $L_a^2(dA_\alpha)$. For a function u in $L^2(dA_\alpha)$, the Toeplitz operator T_u with symbol u is the operator on $L_a^2(dA_\alpha)$ defined by $T_u f = P_\alpha(uf)$, for $f \in L_a^2(dA_\alpha)$.

For each $w \in D$ let $k_w(z)$ be the normalized reproducing kernel of $L_a^2(dA_\alpha)$. For simplicity of notation we use $k_w(z)$ for different normalized reproducing kernels $k_w^\alpha(z)$ of $L_a^2(dA_\alpha)$. Normalized reproducing kernels play a crucial role in the study of compact operators in that $k_w(z)$ converges weakly to zero as $|w| \to 1^-$. In addition, it follows from [11] that C_φ is compact on $L_a^2(dA_\alpha)$ if and only it $\|C_\varphi k_w\| \to 0$ as $w \to 1^-$.

The *Berezin transform* $B_\alpha(f)(w)$ is defined by

$$B_\alpha(f)(w) = \langle f k_w, k_w \rangle_\alpha, \quad \text{for} \quad f \in L_a^2(A_\alpha).$$

The Berezin transform is useful in studying compact Toeplitz operators on the Bergman space [1]. As in [3] and [13], using the inner product formula and local estimates of Toeplitz operators on the Bergman space $L_a^2(dA_\alpha)$ we obtain the following result, which we state here without proof.

Theorem 2.1 *Suppose that f and g are bounded on $D \setminus rD$ for some $0 < r < 1$. If*
$$\lim_{|z|\to 1^-} B_\alpha(f)(z)B_\alpha(g)(z) = 0$$
then $T_f T_g$ is compact on $L^2_a(dA_\alpha)$.

In [14] it is shown that if $\varphi(0) = 0$ then there is a unitary operator $U : zL^2_a(dA_\alpha) \to L^2_a(dA_{\alpha+2})$ defined by $Uf(z) = f'(z)$ such that

$$UC_\varphi U^* = D_\varphi$$

where D_φ is the weighted composition operator on $L^2_a(dA_{\alpha+2})$ defined by

$$D_\varphi f(z) = f(\varphi(z))\varphi'(z).$$

Moreover $D^*_\varphi D_\varphi$ is the Toeplitz $T_{\tau_{\varphi,\alpha+2}}$, where

$$\tau_{\varphi,\alpha}(w) = N_{\varphi,\alpha}(w)/\left(\log 1/|w|\right)^\alpha.$$

Proposition 6.3, in [11], shows that $\tau_{\varphi,\alpha}(w)$ is bounded on $D \setminus rD$ for some $0 < r < 1$.

Following the approach in [14], we decompose $L^2_a(dA_\alpha)$ into $\mathfrak{C} \oplus zL^2_a(dA_\alpha)$ where $\mathfrak{C}$ consists of constants, define the operator $U : L^2_a(dA_\alpha) \to \mathfrak{C} \oplus L^2_a(dA_{\alpha+2})$ by
$$U(c \oplus f) = c \oplus f'.$$
It is easy to check that U is an unitary operator. For $z \in D$, let $\mathfrak{P}_z$ denote the operator on $L^2_a(dA_\alpha)$ given by

$$\mathfrak{P}_z f = f(z).$$

Since $\mathfrak{P}_z$ is a rank-one operator we will view it as a compact operator from $L^2_a(dA_\alpha)$ to $\mathfrak{C}$. The proof of the following lemma follows directly from the the proof of Lemma 5.5 in [3].

Lemma 2.2 *Suppose φ is a holomorphic self-map of D. Then*

$$UC_\varphi U^* = \begin{bmatrix} 1 & \mathfrak{P}_{\varphi(0)} \\ 0 & D_\varphi \end{bmatrix},$$

where D_φ is the weighted composition operator on $L^2_a(dA_{\alpha+2})$ given by

$$D_\varphi f(z) = f(\varphi(z))\varphi'(z).$$

We now state the connection between the compactness of the operator $T_{\tau_{\varphi,\alpha+2}}T_{\tau_{\psi,\alpha+2}}$ on $L^2_a(dA_{\alpha+2})$ and compactness of the operator $C_\varphi C_\psi^*$ on $L^2_a(dA_\alpha)$, without proof as it follows from the proof of Theorem 5.6 in [3].

Lemma 2.3 *If* $T_{\tau_{\varphi,\alpha+2}}T_{\tau_{\psi,\alpha+2}}$ *is compact on* $L^2_a(dA_{\alpha+2})$ *then* $C_\varphi C_\psi^*$ *is compact on* $L^2_a(dA_\alpha)$.

3 Generalized counting function and angular derivatives.

The Julia-Carathéodory Theorem [4, 11, 12] states that a holomorphic self-map of the disc φ has a finite angular derivative at $\zeta \in \partial D$ if and only if

$$\liminf_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|) < \infty$$

where z is allowed to tend unrestrictedly to ζ through the unit disk.

Lemma 3.1 *Suppose that* φ *and* ψ *are univalent self-maps of* D. *For* $\alpha > -1$ *the following are equivalent.*

(a) $$\limsup_{|\varphi(z)|\to 1^- \text{ or } |\psi(z)|\to 1^-}\left\{\frac{N_{\varphi,\alpha+2}(\varphi(z))N_{\psi,\alpha+2}(\psi(z))}{(\log|\varphi(z)|\log|\psi(z)|)^{\alpha+2}}\right\} > 0.$$

(b) $$\limsup_{|z|\to 1^-}\frac{(1-|z|)^2}{(1-|\varphi(z)|)(1-|\psi(z)|)} > 0.$$

(c) There exists a point ζ *on the unit circle such that* φ *and* ψ *have finite angular derivatives at* ζ.

The proof of Lemma 3.1 follows from the univalence of the the inducing maps and the Julia-Carathéodory theorem. By Theorem 1.3 the statements in this lemma are equivalent to the univalently induced operator $C_\varphi C_\psi^*$ not being compact on $L^2_a(dA_\alpha)$.

For a general inducing map the key to characterizing the compactness of a composition operator on $L^2_a(dA_\alpha)$ in terms of the inducing maps' angular derivative are the following two results from [11].

Lemma 3.2 *Suppose* $w \in D\setminus\{0,\varphi(0)\}$ *and* $z \in \varphi^{-1}(w)$ *is of minimum modulus. Then*

$$\frac{N_{\varphi,\alpha+2}(w)}{(\log 1/|w|)^{\alpha+2}} \le \frac{N_{\varphi,1}(w)}{\log 1/|w|}\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\alpha+1}.$$

Lemma 3.3 *Suppose $w \in D \setminus \{0, \varphi(0)\}$ and $z \in \varphi^{-1}(w)$. Then there exists a positive constant m, depending only on α, such that*

$$m\left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{\alpha+2} \le \frac{N_{\varphi,\alpha+2}(w)}{(\log 1/|w|)^{\alpha+2}}.$$

In the definition of the Berezin transform $B_\alpha(f)$, replace the measure $dA_\alpha(z)$ by the measure $(\alpha+1)(1-|z|^2)^\alpha dA(z)$. Then the Berezin transform becomes

$$B_\alpha(f)(z) = (\alpha+1)\int_D \frac{(1-|z|^2)^{\alpha+2}(1-|w|^2)^\alpha}{|1-z\overline{w}|^{4+2\alpha}} f(w)dA(z).$$

By a change of variables, we have

$$B_\alpha(f)(z) = \int_D f \circ \varphi_z(w) dA_\alpha(w),$$

where $\varphi_z(w)$ is the Möbius transformation $\varphi_z(w) = \frac{z-w}{1-\overline{z}w}$.

For $z, w \in D$, define the pseudohyperbolic metric

$$\rho(z,w) = |\varphi_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|.$$

For any $z \in D$ and $0 < r < 1$ define the pseudohyperbolic disk

$$D(z,r) = \{w \in D : \rho(z,w) < r\}.$$

It is well-known that $D(z,r)$ is actually a (Euclidean) disk with center $[(1-r^2)/(1-r^2|z|^2)]z$ and radius $r[(1-|z|^2)/(1-r^2|z|^2)]$. So when r is fixed, if z converges to a point η in the unit circle, then the entire pseudohyperbolic disc $D(z,r)$ converges to η. Moreover, for fixed r,

$$A(D(z,r)) \approx (1-|z|^2)^{-2},$$

and

$$(3) \qquad \left|\frac{1}{(1-\overline{z}w)^2}\right| \approx (1-|z|^2)^{-2}, \quad \text{for } w \in D(z,r).$$

Here the symbol $\approx$ indicates that either quantity is bounded by a constant multiple of the other as z and w vary. For more details about the Berezin transform and the pseudohyperbolic disc , see [5].

We will use the following lemma to connect an asymptotic limit of the product of the Berezin transforms of $\tau_{\varphi,\alpha}$ and $\tau_{\psi,\alpha}$ with the existence of the angular derivatives of φ and ψ.

Lemma 3.4 *Suppose φ is a holomorphic self-map of D and $\alpha > -1$. If there exists ω on the unit circle such that*

$$\lim_{w \to \omega} B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w) > 0$$

then there exists $\zeta \in \varphi^{-1}(\omega)$ such that $\varphi'(\zeta)$ exists and is finite.

PROOF: Let $\delta > 0$. Let $\{w_n\} \subset D$ be a sequence converging to $\omega \in \partial D$ such that

$$B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w_n) \geq \frac{1}{\delta}, \tag{4}$$

for some n sufficiently large.

By Littlewood's Inequality $|\tau_{\varphi,\alpha+2}(w)|$ is bounded near the boundary of the unit disc, so let $0 < s < 1$ and M be such that $|\tau_{\varphi,\alpha+2}(w)|$ is bounded by M for $s < |w| < 1$. Fix r sufficiently close to 1 such that

$$M|D \setminus rD| \leq \frac{1}{4\delta}. \tag{5}$$

Since the integral of $\tau_{\varphi,\alpha}(w)(1-|w_n|^2)^{\alpha+2}/|1-\overline{w}_n w|^{2\alpha+6}$ over sD is bounded by

$$\frac{(1-|w_n|^2)^{\alpha+2}}{|1-\overline{w}_n s|^{2\alpha+6}} \int_{sD} \tau_{\varphi,\alpha+2}(w) dA_{\alpha+2}(w),$$

which tends to zero as $|w_n| \to 1^-$, we choose n sufficiently large so that

$$\int_{sD} \tau_{\varphi,\alpha+2}(w) \frac{(1-|w_n|^2)^{\alpha+2}}{|1-\overline{w}_n w|^{2\alpha+6}} dA_{\alpha+2}(w) \leq \frac{1}{4\delta}. \tag{6}$$

Now split the Berezin transform,

$$B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w_n) = \int_D \tau_{\varphi,\alpha+2} \circ \varphi_{w_n}(\lambda) dA_{\alpha+2}(\lambda),$$

into the sum of three integrals over the regions rD, $(D \setminus rD) \cap \varphi_{w_n}(D \setminus sD)$, and $(D \setminus rD) \cap \varphi_{w_n}(sD)$. Then solving for the integral over rD and using inequalities (4), (5) and (6) we obtain,

$$\int_{rD} \tau_{\varphi,\alpha+2} \circ \varphi_{w_n}(\lambda) dA_{\alpha+2}(\lambda) \geq \frac{1}{2\delta}.$$

Thus,

$$\frac{\int_{rD} N_{\varphi,\alpha+2}\circ\varphi_{w_n}(\lambda)dA_{\alpha+2}(\lambda)}{(1-|w_n|^2)^{\alpha+2}} \geq \frac{1}{2\delta}. \tag{7}$$

By Lemma 3.2 there is a point w_n' in the pseudohyperbolic disc $\overline{D(w_n,r)}$ such that

$$\max_{w\in\overline{D(w_n,r)}} N_{\varphi,\alpha+2}(w) \leq C(1-|w_n'|^2)^{\alpha+2} \sup_{z\in\varphi^{-1}(w_n')}\left(\frac{1-|z|}{1-|w_n'|}\right)^{\alpha+1}.$$

Therefore inequality (7) simplifies to

$$\sup_{z\in\varphi^{-1}(w_n')} C\left(\frac{1-|z|}{1-|w_n'|}\right)^{\alpha+1} \geq \frac{1}{2\delta}.$$

Since w_n' is in $\overline{D(w_n,r)}$, w_n' also converges to ω. Choose $z_n'\in\varphi^{-1}(w_n')$ such that

$$C\left(\frac{1-|z_n'|}{1-|w_n'|}\right)^{\alpha+1} \geq \sup_{z\in\varphi^{-1}(w_n')} C\left(\frac{1-|z|}{1-|w_n'|}\right)^{\alpha+1} - \frac{1}{4\delta} \geq \frac{1}{4\delta}$$

and using the fact that $-\log|z|\leq 1-|z|$ for $|z|>1/2$, the above inequality becomes

$$\frac{1-|\varphi(z_n')|}{1-|z_n'|} \leq M_r\delta.$$

Since w_n' converges to ω and $w_n'=\varphi(z_n')$ we may assume that z_n' converges to some point ζ in the closure of the unit disk D. Thus ζ is in $\varphi^{-1}(\omega)$ and by the Julia-Carathéodory theorem φ does have an angular derivative at ζ.

This completes the proof of the lemma.

The following lemma connects an asymptotic limit of the product of the generalized counting functions of φ and ψ, with the existence of the angular derivatives of φ and ψ, and with the asymptotic limit of the product of the Berezin transforms of $\tau_{\varphi,\alpha+2}$ and $\tau_{\psi,\alpha+2}$. Note that as a consequence of Theorem 1.1 the lemma characterizes when the operator $C_\psi^*C_\varphi$ is not compact.

Lemma 3.5 *Suppose that φ and ψ are holomorphic self-maps of D. For $\alpha>-1$ the following three conditions are equivalent.*

(a) $\limsup \left\{ \dfrac{N_{\varphi,\alpha+2}(w) N_{\psi,\alpha+2}(w)}{(\log|w|)^{2\alpha+4}} \right\} > 0$ *as* $|w| \to 1^-$.

(b) *There exists points* ζ_1 *and* ζ_2 *on the unit circle such that* φ *has a finite angular derivative at* ζ_1, ψ *has a finite angular derivative at* ζ_2, *and* $\varphi(\zeta_1) = \psi(\zeta_2)$.

(c) $\limsup \{ B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w) B_{\alpha+2}(\tau_{\psi,\alpha+2})(w) \} > 0$ *as* $|w| \to 1^-$.

PROOF: We will prove the equivalence of (a) and (b), then $(c) \Rightarrow (b)$, and finally $(a) \Rightarrow (c)$.

We will start by showing $(a) \Rightarrow (b)$. Let $\{w_n\}$ be a sequence in D such that $|w_n| \to 1^-$ and

$$\limsup_{n\to\infty} \left\{ \frac{N_{\varphi,\alpha+2}(w_n) N_{\psi,\alpha+2}(w_n)}{(\log|w_n|)^{2\alpha+4}} \right\} > 0. \tag{8}$$

Choose $z_n = z(w_n) \in \varphi^{-1}(w_n)$ and $z'_n = z'(w_n) \in \psi^{-1}(w_n)$ both of minimum modulus. Set $C(\varphi) = (1+|\varphi(0)|)/(1-|\varphi(0)|)$ and $C(\psi) = (1+|\psi(0)|)/(1-|\psi(0)|)$. Then by Lemma 3.2 and inequality (8)

$$C(\varphi)C(\psi) \limsup_{n\to\infty} \left(\frac{1-|z_n|}{1-|\varphi(z_n)|} \right)^{\alpha+1} \left(\frac{1-|z'_n|}{1-|\psi(z'_n)|} \right)^{\alpha+1} > 0.$$

Thus, $\limsup (1-|z_n|)/(1-|\varphi(z_n)|) > 0$ and $\limsup (1-|z'_n|)/(1-|\psi(z'_n)|) > 0$. Hence by the Julia-Carathéodory theorem there exists ζ_1 and ζ_2 such that φ and ψ have finite angular derivatives at ζ_1 and ζ_2 respectively. This proves (b).

We will now prove $(b) \Rightarrow (a)$. Set $\omega = \varphi(\zeta_1) = \psi(\zeta_2)$. Let $\{w_n\}$ be a sequence in $\varphi(D) \bigcap \psi(D)$ converging to ω. Then

$$\limsup_{|w|\to 1^-} \left\{ \frac{N_{\varphi,\alpha+2}(w) N_{\psi,\alpha+2}(w)}{(\log|w|)^{2\alpha+4}} \right\} \geq \limsup_{n\to\infty} \left\{ \frac{N_{\varphi,\alpha+2}(w_n) N_{\psi,\alpha+2}(w_n)}{(\log|w_n|)^{2\alpha+4}} \right\}. \tag{9}$$

Since $\varphi(\zeta_1) = \omega$ there exists $z_n = z(w_n) \in \varphi^{-1}(w_n)$ such that the sequence $\{z_n\}$ converges to ζ_1. Also since $\psi(\zeta_2) = \omega$ there exists $z'_n = z'(w_n) \in \psi^{-1}(w_n)$ such that the sequence $\{z'_n\}$ converges to ζ_2. Thus Lemma 3.3 and inequality (9)

$$\limsup_{|w|\to 1^-} \left\{ \frac{N_{\varphi,\alpha+2}(w) N_{\psi,\alpha+2}(w)}{(\log|w|)^{2\alpha+4}} \right\} \geq m \left(\frac{1-|z_n|}{1-|\varphi(z_n)|} \frac{1-|z'_n|}{1-|\psi(z'_n)|} \right)^{\alpha+2}.$$

Hence by the Julia-Carathéodory theorem we obtain our desired result

$$\limsup_{|w|\to 1^-} \left\{ \frac{N_{\varphi,\alpha+2}(w) N_{\psi,\alpha+2}(w)}{(\log |w|)^{2\alpha+4}} \right\} \geq m \left(|\varphi'(\zeta_1)\psi'(\zeta_2)| \right)^{-(\alpha+2)}. \tag{10}$$

This proves (a).

The implication $(c) \Rightarrow (b)$ is a direct consequence of Lemma 3.4.

In order to finish the proof we will show that $(a) \Rightarrow (c)$. By Corollary 6.7 in [11], $N_{\varphi,\alpha}(w)$ has the subharmonic mean value property. Thus we have

$$N_{\varphi,\alpha+2}(w) \leq A_r \frac{\int_{rD} N_{\varphi,\alpha+2}(\varphi_w(z)) dA_{\alpha+2}(z)}{\int_{rD} dA_{\alpha+2}},$$

for some positive constant A_r. So

$$\begin{aligned}
B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w) &= \int_D \tau_{\varphi,\alpha+2}(\lambda) |k_w(\lambda)|^2 dA_{\alpha+2}(\lambda) \\
&\geq \int_{\varphi_w(rD)} \tau_{\varphi,\alpha+2}(\lambda) |k_w(\lambda)|^2 dA_{\alpha+2}(\lambda) \\
&\geq \frac{C_r}{(1-|w|^2)^{\alpha+2}} \int_{rD} N_{\varphi,\alpha+2}(\varphi_w(z)) dA_{\alpha+2}(z) \\
&\geq \frac{C_r}{(1-|w|^2)^{\alpha+2}} \frac{\int_{rD} N_{\varphi,\alpha+2}(\varphi_w(z)) dA_{\alpha+2}(z)}{\int_{rD} dA_{\alpha+2}} \\
&\geq \frac{C_r N_{\varphi,\alpha+2}(w)}{A_r (1-|w|^2)^{\alpha+2}}
\end{aligned}$$

for some constant C_r. Here the second inequality follows from the change of variables $\lambda = \varphi_w(z)$ and the fact that $\log(1/|\lambda|)$ is equivalent to $\log(1/|w|)$ for $\lambda \in \varphi_w(rD)$, and $\log(1/|w|)$ is equivalent to $(1-|w|^2)$ for w near the unit circle. Thus, for w near the unit circle,

$$B_{\alpha+2}(\tau_{\varphi,\alpha+2})(w) B_{\alpha+2}(\tau_{\psi,\alpha+2})(w) \geq C \frac{N_{\varphi,\alpha+2}(w) N_{\psi,\alpha+2}(w)}{(\log |w|)^{2\alpha+4}}.$$

The above inequality shows that $(a) \Rightarrow (c)$, to complete the proof.

4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Clearly, the equivalence of (b) and (c) follows from Lemma 3.5. To finish the proof of this theorem we only need to show the equivalence of (a) and (c).

We start with $(c) \Rightarrow (a)$. Suppose that (c) holds. Then by the equivalence of the statements (b) and (c) in Lemma 3.5 we see

$$\lim_{|z|\to 1^-} B_\alpha(\tau_{\varphi,\alpha+2})(w) B_\alpha(\tau_{\psi,\alpha+2})(w) = 0.$$

By Theorem 2.1, we see that the product $T_{\tau_{\varphi,\alpha+2}} T_{\tau_{\psi,\alpha+2}}$ of Toeplitz operators is compact on $L^2_a(dA_{\alpha+2})$. It follows from Theorem 2.3 that $C_\varphi C^*_\psi$ is compact on $L^2_a(dA_\alpha)$.

Now we turn to the proof of $(a) \Rightarrow (c)$. To do so we need the following lemma from [11].

Lemma 4.1 *For $0 < r < 1$ there exists $\delta > 0$ such that,*

$$\|C_\varphi k_w\|^2_\alpha \geq m(1-r)^{2\alpha+2} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}} \quad \textit{for all} \quad 1-\delta \leq |w| \leq 1.$$

We will prove the contrapositive of $(a) \Rightarrow (c)$, which by Lemma 3.5 is: *If ζ_1 and ζ_2 are two points on the unit circle such that $\varphi'(\zeta_1)$ and $\psi'(\zeta_2)$ both exist and $\varphi(\zeta_1) = \psi(\zeta_2) = \omega$ then $C_\varphi C^*_\psi$ is not compact on $L^2_a(dA_\alpha)$ for $\alpha > -1$.*

We start by observing that we may assume $\zeta_1 = \zeta_2 = \omega$. Let ρ_1 and ρ_2 be rotations of the unit disc such that $\rho_i(\omega) = \zeta_i$ for $i = 1$ and 2. Since the composition operators induced ρ_i for $i = 1$ and 2 are invertible operators the compactness of $C_\varphi C^*_\psi$ is equivalent to $C_{\rho_1} C_\varphi C^*_\psi C^*_{\rho_2} = C_{\varphi\circ\rho_1} C^*_{\psi\circ\rho_2}$, and $\varphi \circ \rho_1$ and $\psi \circ \rho_2$ have the desired properties. Thus we will assume that $\varphi'(\omega)$ and $\psi'(\omega)$ exist and $\varphi(\omega) = \psi(\omega) = \omega$.

Let $k_w(z)$ be the normalized reproducing kernel at the point $w \in D$ of $L^2_a(dA_\alpha)$,

$$k_w(z) = K_w(z)/\|K_w\| = \left(1-|w|^2\right)^{1+\alpha/2} / (1-\overline{w}z)^{\alpha+2}, \qquad (w \in D).$$

Let $\{w_n\}$ be a sequence in D converging to ω. Since k_{w_n} converges weakly to zero as $n \to \infty$ it suffices to show that

$$\lim_{n\to\infty} \|C_\varphi C^*_\psi k_{w_n}\|_\alpha > 0.$$

Using the identity $C^*_\psi K_w = K_{\psi(w)}$ and normalizing $K_{\psi(w_n)}$ we obtain,

$$\|C_\varphi C^*_\psi k_{w_n}\|^2_\alpha = (1-|w_n|^2)^{\alpha+2} \|C_\varphi K_{\psi(w_n)}\|^2_\alpha = \left(\frac{1-|w_n|^2}{1-|\psi(w_n)|^2}\right)^{\alpha+2} \|C_\varphi k_{\psi(w_n)}\|^2_\alpha.$$

Now fix $0 < r < 1$ and by Lemma 4.1,

$$\|C_\varphi k_{\psi(a)}\|_\alpha^2 \geq \frac{N_{\varphi,\alpha+2}(\psi(w_n))}{(1-|\psi(w_n)|)^{\alpha+2}} c_r(w_n)$$

for $\psi(w_n)$ sufficiently close to ∂D. Thus, $\lim_{n\to\infty} \|C_\varphi C_\psi^* k_{w_n}\|_\alpha^2$ is bounded below by

$$\lim_{n\to\infty} c_r(w_n) \left(\frac{1-|w_n|}{1-|\psi(w_n)|}\right)^{\alpha+2} \frac{N_{\varphi,\alpha+2}(\psi(w_n))}{(1-|\psi(w_n)|)^{\alpha+2}}. \tag{11}$$

Let $\{w_n'\}$ be a sequence in D converging to ω such that $\varphi(w_n') = \psi(w_n)$. Thus

$$\lim_{n\to\infty} \frac{1-|\psi(w_n)|}{1-|w_n|} = |\psi'(\omega)|, \quad \text{and} \quad \lim_{n\to\infty} \frac{1-|\varphi(w_n')|}{1-|w_n'|} = |\varphi'(\omega)|.$$

By Lemma 3.3,

$$\frac{N_{\varphi,\alpha+2}(\psi(w_n))}{(1-|\psi(w_n)|)^{\alpha+2}} \geq C \left(\frac{1-|w_n'|}{1-|\varphi(w_n')|}\right)^{\alpha+2}.$$

we can conclude by inequality (11) that

$$\lim_{n\to\infty} \|C_\varphi C_\psi^* k_{w_n}\|_\alpha^2 \geq C \left(\frac{1}{|\psi'(\omega)\varphi'(\omega)|}\right)^{\alpha+2}.$$

This completes the proof.

5 Proof of Theorem 1.2 and 1.3

The proof of Theorem 1.2 is based on the approach Shapiro used in [11] to obtain an upper estimate of the essential norm of a composition operator on the Hardy and weighted Bergman spaces. We will obtain an upper estimate of the essential norm of $C_\psi^* C_\varphi$ in the more general setting following Kriete and MacCluer [6] and the presentation in [4]. We consider the Hilbert spaces $\mathcal{H}$ of holomorphic functions with inner product given by, or equivalent to,

$$\langle f,\ g\rangle_{\mathcal{H}} = f(0)\overline{g(0)} + \int_D f'(z)\overline{g'(z)} H(|z|)dA(z)$$

where $H(r)$ is non-negative, continuous on (0,1), and integrable on $[0,1)$. We will call such a Hilbert space a *weighted Dirichlet space*. The choice $H(r) =$

$|\log r^2|^{\alpha+2}$ or $(1-r^2)^{\alpha+2}$ gives the weighted Bergman spaces $L^2_a(dA_\alpha)$ for $\alpha > -1$, and $\alpha = -1$ gives the Hardy space $H^2(D)$. For more information on weighted Dirichlet spaces see [4], page 133. We will need the following generalized change of variables formula where $z_j(w)$ are the points of $\varphi^{-1}(w)$ repeated according to multplicity.

Proposition 1 *([4], Theorem 2.32, page 36) If g and W are non-negative measurable functions on D, then*

$$\int_D g(\varphi(z))|\varphi'(z)|^2 W(z)dA(z) = \int_{\varphi(D)} g(w)\left(\sum_{j\geq 1} W(z_j(w))\right) dA(w).$$

We also require the following estimates on functions in $z^n\mathcal{H}$. Let R_n is the orthogonal projection of $\mathcal{H}$ onto $z^n\mathcal{H}$.

Proposition 2 *([4], Proposition 3.15, page 133) Suppose $f \in \mathcal{H}$. Then for each $z \in D$:*

1. $|(R_n f)(z)| \leq \left(\sum_{k=n}^{\infty} \frac{|z|^{2k}}{\beta(k)^2}\right)^{1/2} \|f\|_{\mathcal{H}},$ *and*

2. $|(R_n f)'(z)| \leq \left(\sum_{k=n}^{\infty} \frac{k^2|z|^{2k-2}}{\beta(k)^2}\right)^{1/2} \|f\|_{\mathcal{H}}$

where $\beta(k) = \|z^k\|_{\mathcal{H}}$.

We will use the following general formula for the essential norm of a linear operator on a Hilbert space which we present in terms of the operator $C_\psi^* C_\varphi$ acting on the Hilbert space $\mathcal{H}$.

Proposition 3 *Suppose $C_\psi^* C_\varphi$ is a bounded operator on $\mathcal{H}$. Then*

$$\|C_\psi^* C_\varphi\|_{e,\mathcal{H}} = \lim_{n\to\infty} \|R_n C_\psi^* C_\varphi R_n\|_{\mathcal{H}}.$$

The proof of Proposition 3 follows directly from the proof of Proposition 5.1 in [11].

We now start our proof of the following upper estimate on $\|C_\psi^* C_\varphi\|_{e,\mathcal{H}}$ when $C_\psi^* C_\varphi$ is bounded on the general weighted Dirichlet space $\mathcal{H}$. We will show that

$$\|C_\psi^* C_\varphi\|_{e,\mathcal{H}}^2 \leq \limsup_{|\varphi(z)|\to 1 \text{or} |\psi(z)|\to 1} \frac{\sum H(|z_j(\varphi(z))|) \sum H(|w_j(|\psi(z)|)}{H(|\varphi(z)|)H(|\psi(z)|)}.$$

We start by applying Proposition 3 and representing the norm using the inner product:

$$\begin{aligned} \|C_\psi^* C_\varphi\|_{e,\mathcal{H}} &= \lim_{n\to\infty} \|R_n C_\psi^* C_\varphi R_n\|_{\mathcal{H}} \\ &= \lim_n \sup_{f,g\in(H)_1} |\langle C_\varphi R_n f,\ C_\psi R_n g\rangle_{\mathcal{H}}| \end{aligned}$$

where $(\mathcal{H})_1$ is the unit ball of $\mathcal{H}$. By fixing f and g in $(\mathcal{H})_1$, we see that $|\langle C_\varphi R_n f,\ C_\psi R_n g\rangle_{\mathcal{H}}|$ is bounded by

$$(12)\quad |R_n f(\varphi(0)) R_n g(\psi(0))| + \int_D |(R_n f\circ\varphi)'(z)(R_n g\circ\psi)'(z)| H(|z|) dA(z).$$

Since $R_n f$ and $R_n g$ are in $(\mathcal{H})_1$, Proposition 2 implies that

$$|R_n f(\varphi(0))| \leq \left(\sum_{k=n}^{\infty} \frac{|\varphi(0)|^{2k}}{\beta(k)^2}\right)^{1/2} \quad \text{and} \quad |R_n g(\psi(0))| \leq \left(\sum_{k=n}^{\infty} \frac{|\psi(0)|^{2k}}{\beta(k)^2}\right)^{1/2},$$

approach zero as n tends to infinity. Thus we need only concern ourselves with the integral in equation (12).

Now fix $0 < r < 1$ and split the integral in equation (12) into two parts: one over the set $D\setminus\{E_1\cup E_2\}$ where $E_1 = \varphi^{-1}(D\setminus rD)$ and $E_2 = \psi^{-1}(D\setminus rD)$ and the other over its complement. Let I represent the integral over $D\setminus\{E_1\cup E_2\}$.

First we will show that the integral I tends to zero as n tends to infinity. To estimate I we use successively the Cauchy-Schwartz inequality and Propositions 1 and 2 to obtain

$$\begin{aligned} I &= \int_{D\setminus\{E_1\cup E_2\}} |(R_n f\circ\varphi)'(z)(R_n g\circ\psi)'(z)| H(|z|)\, dA(z) \\ &\leq \left(\int_{D\setminus E_1} |(R_n f\circ\varphi)'(z)|^2 H(|z|)\, dA(z)\right)^{1/2} \times \\ &\qquad \left(\int_{D\setminus E_2} |(R_n g\circ\psi)'(z)|^2 H(|z|)\, dA(z)\right)^{1/2} \end{aligned}$$

$$= \left(\int_{\varphi(D\setminus E_1)} |(R_n f\circ\varphi)'(w)|^2 \left(\sum_{j\geq 1} H(|z_j(w)|)\right) dA(z)\right)^{1/2} \times$$

$$\left(\int_{\psi(D\setminus E_2)} |(R_n g\circ\psi)'(w)|^2 \left(\sum_{j\geq 1} H(|w_j(w)|)\right) dA(z)\right)^{1/2}$$

$$= \left(\int_{\varphi(D)\cap rD} |(R_n f\circ\varphi)'(w)|^2 \left(\sum_{j\geq 1} H(|z_j(w)|)\right) dA(z)\right)^{1/2} \times$$

$$\left(\int_{\psi(D)\cap rD} |(R_n g\circ\psi)'(w)|^2 \left(\sum_{j\geq 1} H(|w_j(w)|)\right) dA(z)\right)^{1/2}$$

$$\leq \sup_{|w|\leq r} |(R_n f\circ\varphi)'(w)| \sup_{|w|\leq r} |(R_n g\circ\psi)'(w)| \times$$

$$\left(\int_{\varphi(D)} \left(\sum_{j\geq 1} H(|z_j(w)|)\right) dA(z)\right)^{1/2} \left(\int_{\psi(D)} \left(\sum_{j\geq 1} H(|w_j(w)|)\right) dA(z)\right)^{1/2}.$$

Using Propositions 2, 1), and the fact that f and g are in $(\mathcal{H})_1$ we see the last expression is bounded by

$$\left(\sum_{k=n}^{\infty} \frac{k^2}{\beta(k)^2} r^{2k-2}\right) \left(\int_D |\varphi'(z)|^2 H(|z|)\, dA(z)\right)^{1/2} \left(\int_D |\psi'(z)|^2 H(|z|)\, dA(z)\right)^{1/2}$$

which in turn is bounded by a multiple of

$$\left(\sum_{k=n}^{\infty} \frac{k^2}{\beta(k)^2} r^{2k-2}\right) (\|\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} - |\varphi_n(0)||\psi_n(0)|)\,.$$

Now as n approaches infinity, this last expression tends to zero. Thus we have shown

$$\|C_\psi^* C_\varphi\|_{e,\mathcal{H}} \leq \lim_n \sup_{f,g\in(\mathcal{H})_1} \int_{E_1\cup E_2} |(R_n f\circ\varphi)'(z)(R_n g\circ\psi)'(z)| H(|z|) dA(z).$$

This is bounded by

$$\sup_{f,g\in(\mathcal{H})_1}\int_{E_1\cup E_2}|(f\circ\varphi)'(z)(g\circ\psi)'(z)|H(|z|)dA(z).$$

To finish the proof set

$$\gamma_r=\sup_{E_1\cup E_2}Q(z),\ \text{ where } Q(z)=\left(\frac{\sum H(|z_j(\varphi(z))|)\sum H(|w_j(\psi(z))|)}{H(|\varphi(z)|)H(|\psi(z)|)}\right)^{1/2}.$$

We have

$$\begin{aligned}
\|C_\psi^*C_\varphi\|_{e,\mathcal{H}} &\leq \sup\int_{E_1\cup E_2}|(f\circ\varphi)'(g\circ\psi)'|H(|z|)dA(z)\\
&\leq \gamma_r\sup\int_D|(f\circ\varphi)'(g\circ\psi)'|\frac{H(|z|)}{Q(z)}dA(z)\\
&\leq \gamma_r\sup\left(\int_D|(f\circ\varphi)'|^2\frac{H(|\varphi(z)|)}{\sum H(|z_j(\varphi(z))|)}H(|z|)dA(z)\right)^{1/2}\times\\
&\qquad\left(\int_D|(g\circ\psi)'|^2\frac{H(|\psi(z)|)}{\sum H(|w_j(\psi(z))|)}H(|z|)dA(z)\right)^{1/2}
\end{aligned}$$

where the last line follows from the Cauchy-Schwarz inequality. Now we will calculate the two integrals in the last expression above. Because the calculations are similar we will only explicitly compute the first integral. To calculate the first integral, use Proposition 1 and then recognize the result as the norm of f in $\mathcal{H}$,

$$\begin{aligned}
&\int_D|(f\circ\varphi)'(z)|^2\frac{H(|\varphi(z)|)}{\sum H(|z_j(\varphi(z))|)}H(|z|)dA(z)\\
&\quad= \int_D|f'(w)|^2\frac{H(|w|)}{\sum H(|z_j(w)|)}\sum H(|z_j(w)|)dA(z)\\
&\quad= \int_D|f'(w)|^2H(|w|)dA(w)\\
&\quad\leq \|f\|^2_{\mathcal{H}}.
\end{aligned}$$

Similarly,

$$\int_D|(g\circ\psi)'(z)|^2\frac{H(|\psi(z)|)}{\sum H(|w_j(\psi(z))|)}H(|z|)dA(z)\quad\leq\quad\|g\|^2_{\mathcal{H}}.$$

Since $\|f\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = 1$, we have arrived at the desired upper estimate on the essential norm,

$$\|C_\psi^* C_\varphi\|_{e,\mathcal{H}} \quad \leq \quad \lim_{r\to 1} \gamma_r.$$

In order to prove Theorem 1.2 consider the weight $H_{\alpha+2}(|z|) = (1 - |z|^2)^{\alpha+2}/(\alpha+2)$. Since $\|C_\psi^* C_\varphi\|_{e,\alpha} \leq \|C_\psi^* C_\varphi\|_{e,H_{\alpha+2}}$, a proof of which can be found in [10], we obtain our desired estimate

$$\|C_\psi^* C_\varphi\|_{e,\alpha}^2 \leq \limsup \left\{ \frac{N_{\varphi,\alpha+2}(\varphi(z)) N_{\psi,\alpha+2}(\psi(z))}{(\log|\varphi(z)| \log|\psi(z)|)^{\alpha+2}} \right\}$$

as $|\varphi(z)| \to 1^-$or $|\psi(z)| \to 1^-$.

This completes the proof.

We now turn to the proof of Theorem 1.3. The equivalence of (b) and (c) and the equivalence of (c) and (d) are established by Lemma 3.1, and Theorem 1.2 immediately proves $(b) \Rightarrow (a)$. Thus to finish the proof of Theorem 1.3 we only need to prove $(a) \Rightarrow (c)$. We start with the following technical lemmas.

Lemma 5.1 *Suppose φ is a holomorphic self-map of D. If $\varphi(1) = 1$ and $\varphi'(1) = 1$ then $\lim_{r\to 1^-} \langle C_\varphi k_r, k_r \rangle = 1$.*

PROOF: Let $0 < r < 1$ and using the fact that

$$\langle C_\varphi k_r, k_r \rangle_\alpha = \langle K_r \circ \varphi, K_r \rangle_\alpha (1 - r^2)^{\alpha+2}$$

we see

$$\begin{aligned}
\frac{1}{\langle C_\varphi k_r, k_r \rangle_\alpha} &= \left(\frac{1 - r\varphi(r)}{1 - r^2} \right)^{\alpha+2} \\
&= \frac{1}{(1+r)^{\alpha+2}} \left(\frac{1 - \varphi(r) + \varphi(r) - r\varphi(r)}{1 - r} \right)^{\alpha+2} \\
&= \frac{1}{(1+r)^{\alpha+2}} \left(\frac{1 - \varphi(r)}{1 - r} + \varphi(r) \right)^{\alpha+2}.
\end{aligned}$$

Since $\varphi(1) = 1$ and $\varphi'(1) = 1$ we see that $\lim_{r\to 1^-} \langle C_\varphi k_r, k_r \rangle_\alpha = 1$ and this completes the proof.

Lemma 5.2 *Suppose ψ is a holomorphic self-map of D. If ψ has a finite angular derivative at 1 then* $\lim_{r\to 1^-} \|C_\psi^* k_r\|_\alpha = \frac{1}{|\psi'(1)|^{1+\alpha/2}}$.

PROOF: Since $C_\psi^* K_r = K_{\psi(r)}$ and $\psi'(1)$ exist we see by the Julia-Carathéodory theorem that

$$\lim_{r\to 1^-} \|C_\psi^* k_r\|_\alpha = \lim_{r\to 1^-} \left(\frac{1-r^2}{1-|\psi(r)|^2}\right)^{1+\alpha/2} = \left(\frac{1}{|\psi'(1)|}\right)^{1+\alpha/2}.$$

This completes the proof.

Lemma 5.3 *Suppose φ is an univalent self-map of D. If $\varphi(1) = 1$, $\varphi'(1) = 1$, and $|\varphi(\zeta)|$ is less than 1 on $\partial D \setminus \{1\}$ then $\|C_\varphi\|_{e,\alpha} \leq 1$.*

PROOF: In [10] and [11] it shown, for $\alpha \geq -1$, that

$$\|C_\varphi\|_{e,\alpha}^2 \leq \limsup \frac{N_{\alpha+2}(w)}{\left(\log \frac{1}{|w|}\right)^{\alpha+2}} \quad \text{as} \quad |w| \to 1^-.$$

For φ univalent this simplifies to

$$\|C_\varphi\|_{e,\alpha}^2 \leq \sup_{\zeta\in\partial D} \frac{1}{|\varphi'(\zeta)|^{\alpha+2}}.$$

Since φ only has a finite angular derivative at 1 and we see

$$\|C_\varphi\|_{e,\alpha}^2 \leq \sup_{\zeta\in\partial D} \frac{1}{|\varphi'(\zeta)|^{\alpha+2}} = \frac{1}{|\varphi'(1)|^{\alpha+2}} = 1.$$

This completes the proof.

We will prove the contrapositive of $(a) \Rightarrow (b)$, in Theorem 1.3, which by Lemma 3.1 is:

Theorem 5.4 *Suppose φ and ψ are univalent self-maps of the disc and there exists a point ζ on the unit circle such that φ and ψ have finite angular derivatives at ζ. Then $C_\psi^* C_\varphi$ is not compact on $L_a^2(dA_\alpha)$ for $\alpha > -1$.*

PROOF: Let $\zeta \in \partial D$. Without loss of generality we may assume that $\zeta = 1$ so that φ and ψ have an angular derivative at the point 1. We may also assume that $\varphi(1) = 1$, $\varphi'(1) = 1$, and $\|C_\varphi\|_{e,\alpha} \leq 1$.

These reductions are accomplished by considering the operator

$$C_\psi^* C_\rho^* C_\rho C_\varphi C_\beta C_\tau C_\gamma = C_{\psi\circ\rho}^* C_{\gamma\circ\tau\circ\beta\circ\varphi\circ\rho},$$

where

$$\begin{aligned}
\rho(z) &= \zeta z \quad \text{a rotation of } D \text{ mapping point 1 to } \zeta, \\
\beta(z) &= \overline{\varphi(\zeta)}\, z \quad \text{a rotation of } D \text{ mapping } \varphi(\zeta) \text{ to the point 1} \\
\tau(z) &= \frac{(1+s)z+(1-s)}{(1-s)z+(1+s)} \quad \text{where } s = 1/(\beta\circ\varphi\circ\rho)'(1), \text{ and,} \\
\gamma(z) &= \frac{1+z}{3-z}
\end{aligned}$$

The mapping τ is a hyperbolic automorphism of D such that $(\tau\circ\beta\circ\varphi\circ\rho)'(1) = 1$. Finally γ is a parabolic non-automorphism of D with fixed point 1. Since γ is a linear fractional non-automorphism it does not map onto D. Thus it only has an angular derivative at the point 1, which implies $\gamma\circ\tau\circ\beta\circ\varphi\circ\rho$ only has an angular derivative at the point 1. Since $\gamma\circ\tau\circ\beta\circ\varphi\circ\rho$ is univalent we see by Lemma 5.3 that

$$\|C_{\gamma\circ\tau\circ\beta\circ\varphi\circ\rho}\|_{e,\alpha} \le 1.$$

Hence the inducing maps $\psi\circ\rho$ and $\gamma\circ\tau\circ\beta\circ\varphi\circ\rho$ have the desired properties and if $C_{\psi\circ\rho}^* C_{\gamma\circ\tau\circ\beta\circ\varphi\circ\rho}$ is not compact then clearly $C_\psi^* C_\varphi$ is not compact. We will assume φ and ψ have the desired properties.

Since the normalized reproducing kernels $k_w(z)$ converge weakly to zero as $|w| \to 1^-$, it will suffice to show that

$$\limsup_{r\to 1^-} \|C_\psi^* C_\varphi k_r\|_\alpha > 0,$$

in order to conclude that $C_\psi^* C_\varphi$ is not compact.

We now add and subtract the term $\langle C_\varphi k_r, k_r\rangle_\alpha C_\psi^* k_r$ to $C_\psi^* C_\varphi k_r$ and then apply the reverse triangle inequality to obtain,

$$\begin{aligned}
\|C_\psi^* C_\varphi k_r\|_\alpha &= \|\langle C_\varphi k_r, k_r\rangle_\alpha C_\psi^* k_r + C_\psi^*(C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r)\|_\alpha \\
&\ge |\langle C_\varphi k_r, k_r\rangle_\alpha| \|C_\psi^* k_r\|_\alpha - \|C_\psi^*\|_\alpha \|C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r\|_\alpha. \qquad (13)
\end{aligned}$$

By Lemma 5.1 and 5.2 we see that

$$\lim_{r\to 1^-} |\langle C_\varphi k_r, k_r\rangle_\alpha| \|C_\psi^* k_r\|_\alpha = 1/|\psi'(1)|^{1+\alpha/2},$$

thus we only need to show that $\lim_{r\to 1^-} \|C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r\|_\alpha = 0$ to finish the proof.

Expanding the norm by using the inner product we see,

$$\begin{aligned}
&\|C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r\|_\alpha^2 \\
&\quad = \|C_\varphi k_r\|_\alpha^2 + |\langle C_\varphi k_r, k_r\rangle_\alpha|^2 - 2Re\overline{\langle C_\varphi k_r, k_r\rangle_\alpha}\langle C_\varphi k_r, k_r\rangle_\alpha \\
&\quad = \|C_\varphi k_r\|_\alpha^2 - |\langle C_\varphi k_r, k_r\rangle_\alpha|^2.
\end{aligned}$$

By the reduction at the beginning of the proof $\|C_\varphi\|_{e,\alpha}^2 \le 1$ and since $\limsup_{r\to 1^-} \|C_\varphi k_r\|_\alpha^2 \le \|C_\varphi\|_{e,\alpha}^2 \le 1$ and $\lim_{r\to 1^-} \langle C_\varphi k_r, k_r\rangle_\alpha = 1$ we see

$$\limsup_{r\to 1^-} \|C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r\|_\alpha \le \|C_\varphi\|_{e,\alpha}^2 - 1 = 0.$$

Thus

$$\lim_{r\to 1^-} \|C_\varphi k_r - \langle C_\varphi k_r, k_r\rangle_\alpha k_r\|_\alpha = 0.$$

Therefore

$$\limsup_{r\to 1^-} \|C_\psi^* C_\varphi k_r\|_\alpha > \frac{1}{|\psi'(1)|^{1+\alpha/2}}.$$

Hence $C_\psi^* C_\varphi$ is not compact.

References

[1] S. Axler and D. Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J., **47** (1998), 387-400.

[2] J. H. Clifford, *The product of a composition operator with the adjoint of a composition operator*, Thesis, Michigan State University, 1998.

[3] J. H. Clifford and D. Zheng, *Composition operators on the Hardy space*, Indiana Univ. Math. J., **48**(1999), 387-400.

[4] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, (1995).

[5] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics 199, Springer, New York, 2000

[6] T. L. Kriete and B. D. MacCluer, *Composition operators on large weighted Bergman spaces*, Indiana Univ. Math. J. **41**(1992)

[7] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23**(1925).

[8] D. Luecking and K. Zhu, *Composition operators belonging to the Schatten ideals*, Amer. J. Math. **114**(1992), 1127-1145.

[9] B. D. MacCluer and J.H. Shapiro, *Angular derivatives and compact composition operators on Hardy and Bergman spaces*, Cand. J. Math., **39**(1986), 878-906.

[10] P. Poggi-Corradini, *The essential norm of composition operators revisited*, Contemporary Math., AMS, **213**(1997), 167-173.

[11] J. H. Shapiro, *The essential norm of a composition operator*, Annals of Math. **125**(1987), 375-404.

[12] J. H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, 1993.

[13] K. Stroethoff and D. Zheng, *Products of Hankel and Toeplitz Operators on the Bergman space, J. Functional Analysis* **169** (1999), no. 1, 289–313.

[14] K. Zhu, *Operator theory in function spaces,* Marcel Dekker, 1990.

Department of Mathematics and Statistics, University of Michigan at Dearborn, Dearborn, Michigan 48128, USA

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA