Finite Rank Commutator of Toeplitz Operators or Hankel Operators

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Abstract. In this paper we completely characterize when the commutator of two Toeplitz operators or two Hankel operators on the Hardy space has finite rank.

1 Introduction

Let D be the open unit disk in the complex plane and ∂D the unit circle. Let L^2 denote the Lebesgue square integrable functions on the unit circle. The Hardy space H^2 is the Hilbert space consisting of the analytic functions on the unit disk D whose boundary functions are also in L^2. Let H^∞ denote the set of bounded analytic functions on the unit disk.

Let P be the orthogonal projection of L^2 onto H^2. For $f \in L^\infty$, the space of essentially bounded measurable functions on the unit circle, ∂D, the Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined by

$$T_f h = P(fh),$$

and

$$H_f h = P(Ufh),$$

for h in H^2. Here U is the unitary operator on L^2 defined by

$$Uh(w) = \bar{w}\widetilde{h}(w)$$

where $\tilde{f}(w)$ denotes the function $f(\bar{w})$. Clearly,

$$H_f^* = H_{f^*}$$

where $f^*(w) = \bar{f}(\bar{w})$. In fact, U is a unitary operator which maps H^2 onto $(H^2)^\perp$ and has the following useful property : $UP = (1-P)U$, and $U^* = U$.

An operator on the Hardy space H^2 is said to have finite rank if the range of the operator has finite dimension. In this paper, we will study the following problem:

Problem 1.1 *When does the commutator $[T_f, T_g] = T_f T_g - T_g T_f$ of two Toeplitz operators T_f and T_g have finite rank?*

Brown and Halmos [2] have shown that the commutator $T_f T_g - T_g T_f$ is zero if and only if one of the following conditions holds:

1. $f \in H^\infty$ and $g \in H^\infty$;
2. $\tilde{f} \in H^\infty$ and $\tilde{g} \in H^\infty$;
3. There exist constants a, b, not both zero, such that $af + bg$ is constant.

Gorkin and the second author [5] have shown that the commutator of two Toeplitz operators is compact on H^2 if and only if for each support set S, one of the following holds:

1. $f|_S$ and $g|_S$ are in $H^\infty|_S$;
2. $\tilde{f}|_S$ and $\tilde{g}|_S$ are in $H^\infty|_S$;
3. There exist constants a, b, not both zero, such that $af + bg|_S$ is constant.

The Toeplitz operators and Hankel operators are connected by the following relations: for f and g in L^∞,

$$T_{fg} - T_f T_g = H_{\tilde{f}} H_g \tag{1.2}$$

and

$$H_{fg} = H_f T_g + T_{\tilde{f}} H_g. \tag{1.3}$$

(1.3) gives that if f is in H^∞ then

$$T_{\tilde{f}} H_g = H_{fg} = H_g T_f. \tag{1.4}$$

One of the well-known results about Hankel matrices was Kronecker's theorem that describes the Hankel matrices of finite rank. Kronecker's theorem states that for $f \in L^\infty$, H_f is of finite rank if and only if f is the sum of an analytic function h and a rational function $r(z)$. Thus for a rational function $r(z) \in L^\infty$, $H_{r(z)}$ and $H_{\tilde{r}(z)}$ both are finite rank operators. In fact, the following theorem is another form [7] of Kronecker's theorem, which we will use often in this paper.

Theorem 1.5 (**Kronecker's theorem**) *Suppose that f is in L^∞. H_f has finite rank if and only if there exists a nonzero analytic polynomial $p(z)$ such that $pf \in H^\infty$.*

(1.2) gives immediately that the semicommutator $(T_f, T_g](:= T_{fg} - T_f T_g) = H_{\tilde{f}} H_g$ and the commutator

$$[T_f, T_g] = H_{\tilde{g}} H_f - H_{\tilde{f}} H_g.$$

So Problem 1.1 is equivalent to the problem of when $H_{\tilde{g}} H_f - H_{\tilde{f}} H_g$ has finite rank. If the commutator $[T_f, T_g]$ is replaced by the semicommutator

$$(T_f, T_g] = H_{\tilde{f}} H_g$$

in Problem 1.1, the complete solution of the problem was obtained by S. Axler, A. Chang and D. Sarason in [1] and is stated in the following theorem.

Theorem 1.6 *(**The Axler-Chang-Sarason Theorem**) Suppose that f and g are in L^∞. The semicommutator $(T_f, T_g]$ has finite rank if and only if one of the operators $H_{\tilde{f}}$ or H_g does.*

Some purely algebraic proofs of the theorem were obtained in [8], [9]. But the analysis of the commutator turned out to be more difficult than that of the semicommutator. The reason is that the semicommutator equals the product of two Hankel operators while the commutator is the sum of two products of Hankel operators. So one needs to find out precisely how the two summands cancel each other to make the product have finite rank. The commutator is the sum of two products of Hankel operators. Clearly, a much more involved cancellation may happen. The Kronecker theorem, the Axler-Chang-Sarason theorem, the Brown-Halmos theorem [2] and the result in [5] on compact commutators of two Toeplitz operators suggest the following theorem, which completely solves Problem 1.1.

Theorem 1.7 *For f, g in L^∞, the commutator $T_fT_g - T_gT_f$ has finite rank if and only if one of the following conditions holds:*

1. *there is a nonzero analytic polynomial p such that $pf \in H^\infty$ and $pg \in H^\infty$;*
2. *there is a nonzero analytic polynomial q such that $q\tilde{f} \in H^\infty$ and $q\tilde{g} \in H^\infty$;*
3. *there are analytic polynomials A_1, A_2, B_1 and B_2 with $|A_1|+|A_2| \neq 0$ and $|B_1| + |B_2| \neq 0$, such that*

$$A_1(z)\tilde{B}_1(z) = A_2(z)\tilde{B}_2(z),$$
$$A_1\tilde{g} + A_2\tilde{f} \in H^\infty \quad \text{and} \quad B_1 f + B_2 g \in H^\infty.$$

Another natural question is about the commutator of two Hankel operators.

Problem 1.8 *When does the commutator $H_fH_g - H_gH_f$ of two Hankel operators H_f and H_g have finite rank?*

The solution of Problem 1.8 is given in the following theorem.

Theorem 1.9 *For f and g in L^∞, the commutator $H_fH_g - H_gH_f$ has finite rank if and only if one the following conditions holds:*

1. *there is a nonzero analytic polynomial p such that $pf \in H^\infty$;*
2. *there is a nonzero analytic polynomial q such that $qg \in H^\infty$;*
3. *there are nonzero analytic polynomials $A(z)$ and $B(z)$ such that*

$$A(z)\widetilde{B}(z) = B(z)\widetilde{A}(z)$$

and

$$Af + Bg \in H^\infty.$$

Recently, the following problem was studied in [3]:

Problem 1.10 *When is the product $H_f H_g$ of two Hankel operators equal to a compact perturbation of a Hankel operator?*

Problem was solved 1.10 if both f and g are complex conjugates of inner functions [3]. The authors showed that there are products of two Hankel operators which are compact perturbations of noncompact Hankel operators. So Problem 1.10 turns out to be quite subtle. In this paper, we will show that if we replace "a compact perturbation" by "a perturbation of finite rank operators" in Problem 1.10, the problem has a trivial solution, i.e., $H_f H_g - H_h$ has finite rank if and only if both $H_f H_g$ and H_h have finite rank.

In this paper we use

$$A = B \quad mod \;\; (F)$$

to denote that the operator $A - B$ has finite rank.

2 Some lemmas

We begin with a (possibly known) lemma.

Lemma 2.1 *Let A be a bounded linear operator on H^2. Suppose that $p(z)$ and $q(z)$ are nonzero analytic polynomials. If $T_p^* A T_q$ has finite rank, then A has finite rank.*

PROOF: Factorize $q(z)$ as the product $q(z) = B(z)F(z)$ of a finite Blaschke product $B(z)$ and an outer function $F(z)$. Let $M = T_p^* A T_q H^2$. Since $T_p^* A T_q$ has finite rank, M is a finite dimension subspace of H^2. Since $F(z)$ is an outer function ,

$$closure\{T_F H^2\} = H^2.$$

Thus

$$\begin{aligned} T_p^* A T_B H^2 &= closure\{T_p^* A T_B T_F H^2\} \\ &= M. \end{aligned}$$

This gives that $T_p^* A T_B$ has finite rank and then

$$T_p^* A = T_p^* A T_{\overline{B}} T_B = T_p^* A T_B T_{\overline{B}} \qquad mod \quad (F).$$

So $T_p^* A$ has finite rank. By the same argument, we have that A^* has finite rank. Hence A has finite rank also. This completes the proof.

Gu and the second author [6] obtained a necessary and sufficient for the sum $\sum_{i=1}^n H_{f_i} H_{g_i}$ to be zero. It remains open to characterize when the sum has finite rank. The next result gives a necessary condition for the sum to have finite rank.

For x and y in H^2, the operator $x \otimes y$ of rank one is defined by

$$(x \otimes y)(h) = \langle h, y \rangle x,$$

for $h \in H^2$.

Lemma 2.2 *For f_i, g_i in $L^\infty, i = 1, 2, ..., n$, if $\sum_{i=1}^n H_{f_i} H_{g_i}$ has rank k, then there are analytic polynomials $A_i(z), B_i(z)$ with*

$$\max\{degA_i(z) : 1 \le i \le n\} = k, \quad and \quad max\{degB_i(z) : 1 \le i \le n\} = k,$$

such that

$$\sum_{i=1}^n A_i f_i \in H^\infty \text{ or } \sum_{i=1}^n B_i g_i \in H^\infty.$$

PROOF: Let K be the rank of $\sum_{i=1}^n H_{f_i} H_{g_i}$. We prove the result by induction on the rank K.

Assume that the rank $K = 0$. Then

$$\sum_{i=1}^n H_{f_i} H_{g_i} = 0.$$

If one of $H_{f_1}, \cdots, H_{f_n}$ and $H_{g_1}, \cdots, H_{g_n}$ is zero, obviously there are constants a_i or b_i with $\sum_{i=1}^n |a_i| > 0$, and $\sum_{i=1}^n |b_i| > 0$ such that

$$\sum_{i=1}^n a_i f_i \in H^\infty \quad \text{or} \quad \sum_{i=1}^n b_i g_i \in H^\infty.$$

If none of $H_{f_1}, \cdots, H_{f_n}$ and $H_{g_1}, \cdots, H_{g_n}$ is zero, let $K_\lambda(z)$ be the reproducing kernel $1/(1 - \bar{\lambda} z)$ at $\lambda \in D$. Noting

$$\begin{aligned} \sum_{i=1}^n H_{f_i} 1 \otimes H_{g_i^*} 1 &= \sum_{i=1}^n H_{f_i}(1 - T_z T_{\bar{z}}) H_{g_i} \\ &= \sum_{i=1}^n H_{f_i} H_{g_i} - T_{\bar{z}} (\sum_{i=1}^n H_{f_i} H_{g_i}) T_z \\ &= 0, \end{aligned}$$

we have

$$\begin{aligned} \sum_{i=1}^n \overline{H_{g_i^*} 1(\lambda)} H_{f_i} 1 &= \sum_{i=1}^n \langle K_\lambda, H_{g_i^*} 1 \rangle H_{f_i} 1 \\ &= [\sum_{i=1}^n H_{f_i} 1 \otimes H_{g_i^*} 1] K_\lambda \\ &= 0. \end{aligned}$$

Because none of $H_{g_1}, \cdots, H_{g_n}$ is zero, there is a $\lambda_0 \in D$ such that

$$a_i = \overline{H_{g_i^*} 1(\lambda_0)} \neq 0$$

for all i. Thus we have

$$\sum_{i=1}^{n} |a_i| > 0,$$

and

$$\sum_{i=1}^{n} a_i H_{f_i} 1 = 0,$$

and so

$$\sum_{i=1}^{n} a_i f_i \in H^\infty.$$

Assume that the result is true if the rank $K < k$. We need to show that the result is true for $K = k$.

Write

$$\sum_{i=1}^{n} H_{f_i} H_{g_i} = \sum_{j=1}^{k} x_j \otimes y_j$$

where x_j, y_j are in H^2 and

$$dimspan\{x_1, \ \cdots, x_k\} = dimspan\{y_1, \ \cdots, \ y_k\} = k.$$

If $T_{\bar{z}} y_1, \cdots, T_{\bar{z}} y_k$ are linearly dependent, without loss of generality, we assume

$$T_{\bar{z}} y_k = c_1 T_{\bar{z}} y_1 + \cdots + c_{k-1} T_{\bar{z}} y_{k-1},$$

for some constants $c_1, \cdots, c_{k-1}$. Then

$$\begin{aligned} T_{\bar{z}} \sum_{i=1}^{n} H_{f_i} H_{g_i} T_z &= \sum_{i=1}^{n} H_{zf_i} H_{zg_i} \\ &= \sum_{j=1}^{k} T_{\bar{z}} x_j \otimes T_{\bar{z}} y_j \\ &= \sum_{j=1}^{k-1} T_{\bar{z}} x_j \otimes T_{\bar{z}} y_j + T_{\bar{z}} x_K \otimes T_{\bar{z}} y_k \\ &= \sum_{j=1}^{k-1} T_{\bar{z}} x_j \otimes T_{\bar{z}} y_j + \sum_{j=1}^{k-1} \bar{c}_j T_{\bar{z}} x_k \otimes T_{\bar{z}} y_j \\ &= \sum_{j=1}^{k-1} T_{\bar{z}} (x_j + \bar{c}_j x_k) \otimes T_{\bar{z}} y_j. \end{aligned}$$

Thus the rank of $\sum_{i=1}^{n} H_{zf_i} H_{zg_i}$ is at most $k-1$. So by the induction hypothesis, there are analytic polynomials $a_i(z)$, $b_i(z)$ with

$$\max\{\max_i\{dega_i(z)\}, \max_i\{degb_i(z)\}\} \leq k-1$$

and

$$\sum_{i=1}^{n} |a_i(z)| \neq 0, \quad \text{and} \quad \sum_{i=1}^{n} |b_i(z)| \neq 0,$$

such that

$$\sum_{i=1}^{n} a_i(z) z f_i(z) \in H^\infty$$

or

$$\sum_{i=1}^{n} b_i(z) z g_i(z) \in H^\infty.$$

Let $l = max\{dega_i(z)\}$ or $l = max\{degb_i(z)\}$. Set $A_i(z) = z^{k-l} a_i(z)$, or $B_i(z) = z^{k-l} b_i(z)$. Then $\max\{degA_i(z)\} = k$, and $max\{degB_i(z)\} = k$. We have

$$\sum_{i=1}^{n} A_i(z) f_i(z) \in H^\infty$$

or

$$\sum_{i=1}^{n} B_i(z) g_i(z) \in H^\infty.$$

Thus the result is true in this case.

If $T_{\bar{z}} x_1, \cdots, T_{\bar{z}} x_k$ are linearly dependent, by the same argument as above, we obtain that the result is true.

To finish the proof, we may assume that $T_{\bar{z}} y_1, \cdots, T_{\bar{z}} y_k$ are linearly independent and $T_{\bar{z}} x_1, \cdots, T_{\bar{z}} x_k$ also are linearly independent. By

$$\sum_{i=1}^{n} H_{f_i} H_{g_i} = \sum_{j=1}^{k} x_j \otimes y_j,$$

we have

$$\sum_{i=1}^{n} H_{f_i} 1 \otimes H_{g_i^*} 1 = \sum_{j=1}^{k} x_j \otimes y_j - \sum_{j=1}^{k} T_{\bar{z}} x_j \otimes T_{\bar{z}} y_j. \tag{2.3}$$

Noting

$$[\sum_{i=1}^{n} H_{f_i} 1 \otimes H_{g_i^*} 1](T_{\bar{z}} y_l) = \sum_{i=1}^{n} \langle T_{\bar{z}} y_l, H_{g_i^*} 1 \rangle H_{f_i} 1,$$

and

$$[\sum_{j=1}^{k} x_j \otimes y_j - \sum_{j=1}^{k} T_{\bar{z}} x_j \otimes T_{\bar{z}} y_j](T_{\bar{z}} y_l) = \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, y_j \rangle x_j - \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, T_{\bar{z}} y_j \rangle T_{\bar{z}} x_j,$$

we have

$$\sum_{i=1}^{n} \langle T_{\bar{z}} y_l, H_{g_i^*} 1 \rangle H_{f_i} 1 = \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, y_j \rangle x_j - \sum_{j=1}^{k} \langle T_{\bar{z}} y_l, T_{\bar{z}} y_j \rangle T_{\bar{z}} x_j,$$

for $l=1,2,...,k$. Let

$$a_{lj}=\langle T_{\bar z}y_l, T_{\bar z}y_j\rangle,$$

$$b_{lj}=\langle T_{\bar z}y_l, y_j\rangle - a_{lj}\bar z,$$

and

$$c_{lj}=\langle T_{\bar z}y_l, H_{g_j^*}1\rangle.$$

Since

$$T_{\bar z}x_j=\bar z x_j-\bar z x_j(0),$$

we have

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k1} & c_{k2} & \cdot & c_{kn} \end{pmatrix}\begin{pmatrix} H_{f_1}1 \\ H_{f_2}1 \\ \vdots \\ H_{f_n}1 \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_k(0) \end{pmatrix}\bar z.$$

Let $A=(a_{ij})$, $B=(b_{lj})$, $C=(c_{lj})$ and

$$H_f1=(H_{f1}1,\cdots,H_{fn}1)^T,$$

and

$$X=(x_1,\cdots,x_k)^T.$$

The above system is equivalent to

$$CH_f1=BX+\bar z AX(0). \tag{2.4}$$

The determinant of matrix $B=(b_{lj})_{k\times k}$ is

$$B(z)=det(b_{lj})=(-1)^k a\bar z^k+a_1\bar z^{k-1}+\cdots+a_k,$$

where $a=det(a_{lj})\neq 0$, and a_i are constants since $T_{\bar z}y_1\cdots,T_{\bar z}y_k$ are linearly independent . Hence $degB(z)=k$. The adjugate of the matrix B is

$$adjB=\begin{pmatrix} B_{11} & B_{21} & \cdots & B_{k1} \\ B_{12} & B_{22} & \cdots & B_{k2} \\ \cdots & \cdots & \cdots & \cdots \\ B_{1k} & B_{2k} & \cdots & B_{kk} \end{pmatrix}$$

where B_{lj} denotes the cofactor of b_{lj} and is a co-analytic polynomial in z with degree at most $k-1$. Equation (2.4) gives

$$(adjB)CH_f1=B(z)X+(adjB)AX(0)\bar z. \tag{2.5}$$

Let

$$(C_{lj}(z)) = (adjB)C,$$

where $\{C_{lj}(z)\}$ are co-analytic polynomials in z with degree at most $k-1$. Applying the projection P to both sides of the above equation gives

$$P[(C_{lj}(z))H_f 1] = PB(z)X.$$

Thus

$$\begin{pmatrix} H_{\sum_{i=1}^n \tilde{C}_{1i}(z)f_i}1 \\ \vdots \\ H_{\sum_{i=1}^n \tilde{C}_{ki}(z)f_i}1 \end{pmatrix} = \begin{pmatrix} T_{B(z)}x_1 \\ \vdots \\ T_{B(z)}x_k \end{pmatrix}.$$

Similarly, we also have

$$\begin{pmatrix} H_{\sum_{i=1}^n \tilde{u}_{1i}(z)g_i^*}1 \\ \vdots \\ H_{\sum_{i=1}^n \tilde{u}_{ki}(z)g_i^*}1 \end{pmatrix} = \begin{pmatrix} T_{E(z)}y_1 \\ \vdots \\ T_{E(z)}y_k \end{pmatrix}$$

where $u_{lj}(z)$ are co-analytic polynomial in z with degree at most $k-1$ and $E(z)$ is a co-analytic polynomial in z with degree k.

If $H_{g_1^*}1$, $H_{g_2^*}1$, $\cdots$,$H_{g_n^*}1$, y_1, y_2, $\cdots$, y_k are linearly dependent, without loss of generality, assume

$$H_{g_n^*}1 = c_1 H_{g_1^*}1 + \cdots + c_{n-1}H_{g_{n-1}^*}1 + b_1 y_1 + \cdots + b_k y_k,$$

then

$$\begin{aligned} T_E H_{g_n^*}1 &= H_{g_n^*\tilde{E}}1 \\ &= \sum_{i=1}^{n-1} H_{g_i^* c_i \tilde{E}}1 + \sum_{j=1}^{k} b_j T_E y_j \\ &= \sum_{i=1}^{n-1} H_{g_i^* c_i \tilde{E}}1 + \sum_{j=1}^{k} b_j \sum_{l=1}^{n} H_{\tilde{u}_{jl}g_l^*}1 \\ &= \sum_{i=1}^{n-1} H_{g_i^* c_i \tilde{E}}1 + \sum_{l=1}^{n} H_{\sum_{j=1}^k b_j \tilde{u}_{jl} g_l^*}1. \end{aligned}$$

This gives

$$H^*_{g_n[\tilde{E}-\sum_{j=1}^k b_j\tilde{u}_{jn}]^* + \sum_{l=1}^{n-1} g_l[c_l\tilde{E}-\sum_{j=1}^k b_j\tilde{u}_{jl}]^*}1 = 0.$$

Thus

$$\sum_{i=1}^{n} B_i(z)g_i(z) \in H^\infty$$

where

$$B_l = [c_l\tilde{E} - \sum_{j=1}^{k} b_j\tilde{u}_{jl}]^* \in H^\infty$$

and deg $B_l \leq k$, $1 \leq l \leq n-1$, and

$$B_n = [\tilde{E} - \sum_{j=1}^{k} b_j \tilde{u}_{jn}]^*$$

is an analytic polynomial with degree k. So this is the result as desired.

By the same argument, if $H_{f_1}1, \cdots, H_{f_n}1, x_1, \cdots, x_k$ are linearly dependent, we also have that there are analytic polynomials $A_i(z)$ with

$$max\{degA_i(z)\} = k,$$

such that

$$\sum_{i=1}^{n} A_i(z) f_i(z) \in H^\infty.$$

Next we assume $H_{f_1}1, \cdots, H_{f_n}1, x_1, \cdots, x_k$ are linearly independent and $H_{g_1^*}1, \cdots, H_{g_n^*}1, y_1, \cdots, y_k$ are linearly independent also. We will derive a contradiction. First we claim that

$$dimspan\{x_1, \cdots x_k, T_{\bar{z}}x_1, \cdots, T_{\bar{z}}x_k\} \geq k+n.$$

In fact, since $H_{g_1^*}1, \cdots, H_{g_n^*}1$ are linearly independent, there is a vector $\xi \in H^2$, such that

$$\langle \xi, H_{g_i^*}1 \rangle = 1,$$

and

$$\langle \xi, H_{g_j^*}1 \rangle = 0,$$

for $j \neq i$. Hence, by (2.3),

$$H_{f_i}1 = \sum_{j=1}^{k} \langle \xi, y_j \rangle x_j - \sum_{j=1}^{k} \langle \xi, T_{\bar{z}} y_j \rangle T_{\bar{z}} x_j$$

Thus

$$H_{f_i}1 \in span\{x_1 \cdots x_k, T_{\bar{z}}x_1 \cdots T_{\bar{z}}x_k\}$$

for all $i = 1, \cdots n$. This gives

$$span\{H_{f_1}1, \cdots, H_{f_n}1, x_1 \cdots x_k\} \subseteq span\{x_1, \cdots, x_k, T_{\bar{z}}x_1, \cdots, T_{\bar{z}}x_k\}.$$

So

$$dimspan\{x_1 \cdots, x_k, T_{\bar{E}}x_1, \cdots, T_{\bar{E}}x_k\} \geq k+n.$$

This shows that our claim is true.

Since

$$\begin{aligned} dimspan\{T_{\bar{z}}x_1, \cdots, T_{\bar{z}}x_k\} &\leq k < k+n \\ &\leq dimspan\{x_1, \cdots, x_k, T_{\bar{z}}x_1, \cdots, T_{\bar{z}}x_k\}, \end{aligned}$$

there is a nonzero vector ξ in $span\{x_1, \ \cdots, \ x_k, \ T_{\bar{z}}x_1, \ \cdots, \ T_{\bar{z}}x_k\}$ such that

$$\xi \perp \{T_{\bar{z}}x_1, \ \cdot\cdot\cdot, \ T_{\bar{z}}x_k\}.$$

By (2.3), we have

$$\sum_{i=1}^{n} \langle \xi, H_{f_i}1 \rangle H_{g_i^*}1 = \sum_{i=1}^{k} \langle \xi, x_i \rangle y_i.$$

We claim that not all of $\{\langle \xi, x_i \rangle\}_1^k$ are zero. Otherwise ξ is orthogonal to $\{x_1, \ ..., \ x_k, \ T_{\bar{z}}x_1, \ \cdot\cdot\cdot, \ T_{\bar{z}}x_k\}$. This would imply that $\xi = 0$. Note that not all of $\{\langle \xi, H_{f_i}1 \rangle\}_1^k$ are zero since $y_1, \ y_2, \ \cdots, \ y_k$ are linearly independent. This gives that $H_{g_1^*}1, \ \cdots, \ H_{g_n^*}1, \ y_1, \ \cdots, \ y_k$ are linearly dependent. We have obtained a contradiction to complete the proof.

As an application of the above two lemmas, we present a proof of the Axler-Chang-Sarason theorem, which is the motivation for the proofs of our main results in next section.

Proof of Theorem 1.6. We only have to prove the "only if" part. Suppose $H_{\bar{f}}H_g$ has finite rank. By Lemma 2.2, then there are nonzero analytic polynomials $A(z), \ B(z)$ such that

$$A(z)f(z) \in H^\infty$$

or

$$B(z)g(z) \in H^\infty.$$

If $A(z)f(z)$ is in H^∞, then

$$H_{Af} = H_f T_A = 0.$$

By Lemma 2.1, H_f has finite rank.

If $B(z)g(z)$ is in H^∞, then

$$H_{Bf} = H_g T_B = 0.$$

By Lemma 2.1 again, H_g has finite rank. This completes the proof.

3 Proofs of main results

First we prove the following theorem which encompasses the difficulty in the proofs of our main results.

Theorem 3.1 *Suppose that f_1, f_2, g_1, g_2 are in L^∞. If none of H_{f_1}, H_{f_2}, H_{g_1} and H_{g_2} has finite rank, then*

$$H_{f_1}H_{g_1} = H_{f_2}H_{g_2} \qquad mod \quad (F)$$

if and only if there are nonzero analytic polynomials $A_1(z)$, $A_2(z)$, $B_1(z)$ and $B_2(z)$ such that

$$A_1(z)f_1(z) + A_2(z)f_2(z) \in H^\infty,$$

$$B_1(z)g_1(z) + B_2(z)g_2(z) \in H^\infty$$

and

$$A_1(z)\tilde{B}_1(z) = A_2(z)\tilde{B}_2(z).$$

Proof: First we prove the "only if " part. Suppose

$$H_{f_1}H_{g_1} = H_{f_2}H_{g_2} \qquad mod \quad (F).$$

By Lemma 2.2, there are analytic polynomials $A_1(z)$, $A_2(z)$, $B_1(z)$ and $B_2(z)$ such that

$$A_1f_1 + A_2f_2 \in H^\infty,$$

or

$$B_1g_1 + B_2g_2 \in H^\infty.$$

Here $A_1(z)$ and $A_2(z)$ are not both zero and $B_1(z)$ and $B_2(z)$ are not both zero. Assume that

$$A_1f_1 + A_2f_2 \in H^\infty,$$

and $A_1(z)$, and $A_2(z)$ are not both zero. Thus

$$H_{A_1f_1} = -H_{A_2f_2}.$$

Since neither H_{f_1} nor H_{f_2} has finite rank, none of $A_1(z)$ and $A_2(z)$ is zero. Let $max\{degA_i(z)\} = n$. Write $A_1(z)$ and $A_2(z)$ as

$$A_1(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0$$

and

$$A_2(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_0$$

where $|a_n| + |b_n| > 0$.

$$\begin{aligned}
& T_{\widetilde{A_1(z)}}[H_{f_1}H_{g_1} - H_{f_2}H_{g_2}] \\
= \;& H_{A_1f_1}H_{g_1} - H_{A_1f_2}H_{g_2} \\
= \;& -H_{A_2f_2}H_{g_1} - H_{A_1f_2}H_{g_2} \\
= \;& -\{H_{f_2}T_{z^n\sum_{j=0}^n b_j\bar{z}^{n-j}}H_{g_1} + H_{f_2}T_{z^n\sum_{j=0}^n a_j\bar{z}^{n-j}}H_{g_2}\} \\
= \;& -\{H_{f_2}T_{z^n}T_{\sum_{j=0}^n b_j\bar{z}^{n-j}}H_{g_1} + H_{f_2}T_{z^n}T_{\sum_{j=0}^n a_j\bar{z}^{n-j}}H_{g_2}\} \qquad mod \quad (F) \\
= \;& -\{H_{z^nf_2}H_{g_1\sum_{j=0}^n b_jz^{n-j}} + H_{z^nf_2}H_{g_2\sum_{j=0}^n a_jz^{n-j}}\} \\
= \;& -H_{z^nf_2}H_{[g_1\sum_{j=0}^n b_jz^{n-j} + g_2\sum_{j=0}^n a_jz^{n-j}]}.
\end{aligned}$$

This gives that $H_{z^n f_2}H_{[g_1\sum_{j=0}^n b_j z^{n-j}+g_2\sum_{j=0}^n a_j z^{n-j}]}$ has finite rank. Since H_{f_2} is not of finite rank, by Lemma 2.1, $H_{z^n f_2}=H_{f_2}T_{z^n}$ is not of finite rank. Thus by Lemma 2.2, there is a nonzero analytic polynomial $q(z)$ such that

$$q(z)[g_1\sum_{j=0}^n b_j z^{n-j}+g_2\sum_{j=0}^n a_j z^{n-j}]\in H^\infty.$$

Let

$$B_1(z)=q(z)\sum_{j=0}^n b_j z^{n-j},\ \ B_2(z)=q(z)\sum_{j=0}^n a_j z^{n-j}.$$

So $B_1(z)$ and $B_2(z)$ are both nonzero analytic polynomials such that

$$B_1g_1+B_2g_2\in H^\infty.$$

Also we have

$$\begin{aligned}\widetilde{B_1}(z) &= \widetilde{q}(z)\sum_{j=0}^n b_j\bar{z}^{n-j}\\ &= \widetilde{q}(z)\bar{z}^n\sum_{j=0}^n b_j z^j\\ &= \tilde{q}(z)\bar{z}^n A_2(z),\\ \widetilde{B_2}(z) &= \widetilde{q}(z)\sum_{j=0}^n a_j\bar{z}^{n-j}\\ &= \widetilde{q}(z)\bar{z}^n\sum_{j=0}^n a_j z^j\\ &= \tilde{q}(z)\bar{z}^n A_1(z).\end{aligned}$$

Thus

$$A_1(z)\widetilde{B_1}(z)=A_1(z)A_2(z)\widetilde{q}(z)\bar{z}^n=A_2(z)\widetilde{B_2}(z).$$

Next we prove the "if" part. Assume there are nonzero analytic polynomials $A_1(z),\ A_2(z),\ B_1(z),\ B_2(z)$ such that

$$A_1f_1+A_2f_2\in H^\infty,$$

$$B_1g_1+B_2g_2\in H^\infty$$

and

$$A_1(z)\widetilde{B_1}(z)=A_2(z)\widetilde{B_2}(z).$$

Thus

$$\begin{aligned}T_{\widetilde{A_1}}[H_{f_1}H_{g_1}-H_{f_2}H_{g_2}]T_{B_1} &= H_{A_1f_1}H_{g_1B_1}-H_{A_1f_2}H_{g_2B_1}\\ &= H_{A_2f_2}H_{g_2B_2}-H_{A_1f_2}H_{g_2B_1}\\ &= H_{f_2}[T_{A_2}T_{\widetilde{B_2}}-T_{A_1}T_{\widetilde{B_1}}]H_{g_2}\\ &= H_{f_2}[T_{\widetilde{B_2}}T_{A_2}-T_{\widetilde{B_1}}T_{A_1}]H_{g_2}\qquad mod\quad (F)\\ &= H_{f_2}T_{[A_2\widetilde{B_2}-\widetilde{B_1}A_1]}H_{g_2}=0.\end{aligned}$$

This gives that $T_{\widetilde{A_1}}[H_{f_1}H_{g_1} - H_{f_2}H_{g_2}]T_{B_1}$ has finite rank. Thus by Lemma 2.1, $H_{f_1}H_{g_1} - H_{f_2}H_{g_2}$ has finite rank to complete the proof.

We are ready to prove our main theorems.

Proof of Theorem 1.7. Suppose that one of Conditions (1)-(3) holds. We are going to show that the commutator $T_fT_g - T_gT_f$ has finite rank. Noting

$$T_fT_g - T_gT_f = H_{\tilde{g}}H_f - H_{\tilde{f}}H_g,$$

for two analytic polynomials p and q, we have

$$\begin{aligned} T_q[T_fT_g - T_gT_f]T_p &= T_qH_{\tilde{g}}H_fT_p - T_qH_{\tilde{f}}H_gT_p \\ &= H_{q\tilde{g}}H_{fp} - H_{q\tilde{f}}H_{gp}. \end{aligned} \tag{3.2}$$

If one of Conditions (1)-(2) holds, by Kronecker's Theorem, Equation (3.2) gives $T_q[T_fT_g - T_gT_f]T_p$ has finite rank. By Lemma 2.1, we have that the commutator $T_fT_g - T_gT_f$ has finite rank.

If Condition (3) holds, then

$$\begin{aligned} T_{A_1}T_{\tilde{B}_1} - T_{A_2}T_{\tilde{B}_2} &= T_{A_1}T_{\tilde{B}_1} - T_{A_1\tilde{B}_1} + T_{A_1\tilde{B}_1 - A_2\tilde{B}_2} + T_{A_2\tilde{B}_2} - T_{A_2}T_{\tilde{B}_2} \\ &= -H_{\tilde{A}_1}H_{\tilde{B}_1} + H_{\tilde{A}_2}H_{\tilde{B}_2} \end{aligned}$$

has finite rank. Equation (3.2) gives

$$\begin{aligned} T_{\tilde{A}_2}[T_fT_g - T_gT_f]T_{B_1} &= H_{A_2\tilde{g}}H_{fB_1} - H_{A_2\tilde{f}}H_{gB_1} \\ &= H_{A_2\tilde{g}}H_{-gB_2} - H_{-\tilde{g}A_1}H_{gB_1} \\ &= H_{\tilde{g}}[T_{A_1}T_{\tilde{B}_1} - T_{A_2}T_{\tilde{B}_2}]H_g \\ &= 0 \qquad mod \quad (F). \end{aligned}$$

The second equality follows because both $A_1\tilde{g} + A_2\tilde{f}$ and $B_1f + B_2g$ are in H^∞ and the last equality follows because $T_{A_1}T_{\tilde{B}_1} - T_{A_2}T_{\tilde{B}_2}$ has finite rank. By Lemma 2.1, $T_fT_g - T_gT_f$ has finite rank.

Conversely, suppose that $T_fT_g - T_gT_f$ has finite rank.

If H_f has finite rank, by the Kronecker theorem (Theorem 1.5), there is a nonzero analytic polynomial p_1 such that $p_1f \in H^\infty$. Equation (3.2) gives that

$$H_{\tilde{f}}H_{gp_1} = 0 \qquad mod \quad (F).$$

By the Axler-Chang-Sarason Theorem (Theorem 1.6), there is a nonzero polynomial q_1 such that either $q_1\tilde{f} \in H^\infty$ or $q_1p_1g \in H^\infty$. Let $p = p_1q_1$, $B_1 = p_1$, $B_2 = 0$, $A_1 = 0$ $A_2 = q_1$. Either Condition (1) or (3) holds.

If H_g has finite rank, by the Kronecker theorem, there is a nonzero analytic polynomial q_1 such that $q_1g \in H^\infty$. Equation (3.2) gives that

$$H_{\tilde{g}}H_{fq_1} = 0 \qquad mod \quad (F).$$

Using the same method as above, we have that either Condition (2) or (3) holds.

Similarly, if either $H_{\tilde{f}}$ or $H_{\tilde{g}}$ has finite rank, we will obtain that one of Conditions (1), (2), and (3) holds.

If none of H_f, H_g, $H_{\tilde{f}}$ and $H_{\tilde{g}}$ has finite rank, by Theorem 3.1, there are nonzero polynomials A_1, A_2, B_1 and B_2 such that

$$A_1\tilde{g} + A_2\tilde{f} \in H^\infty$$

and

$$B_1 f + B_2 g \in H^\infty.$$

To finish the proof, we need only to show that

$$A_1(z)\tilde{B}_1(z) = A_2(z)\tilde{B}_2(z).$$

Let n denote the maximal degree of B_1 and B_2. Equation (3.2) gives

$$\begin{aligned}
T_{\tilde{A}_2}[T_f T_g - T_g T_f]T_{B_1} &= H_{A_2\tilde{g}} H_{fB_1} - H_{\tilde{f}A_2} H_{gB_1} \\
&= H_{A_2\tilde{g}} H_{-gB_2} - H_{-\tilde{g}A_1} H_{gB_1} \\
&= -H_{\tilde{g}} T_{A_2} H_g T_{B_2} + H_{\tilde{g}} T_{A_1} H_g T_{B_1} \\
&= H_{\tilde{g}} T_{z^n} T_{\widetilde{z^n\tilde{A}_1}} H_g T_{B_1} - H_{\tilde{g}} T_{z^n} T_{\widetilde{z^n\tilde{A}_2}} H_g T_{B_2} \quad mod \quad (F) \\
&= H_{\tilde{g}} T_{z^n} H_g [T_{z^n\tilde{A}_1} T_{B_1} - T_{z^n\tilde{A}_2} T_{B_2}] \\
&= T_{\bar{z}^n} H_{\tilde{g}} H_g [T_{z^n\tilde{A}_1} T_{B_1} - T_{z^n\tilde{A}_2} T_{B_2}] \\
&= T_{\bar{z}^n} H_{\tilde{g}} H_g T_{[z^n\tilde{A}_1 B_1 - z^n\tilde{A}_2 B_2]}
\end{aligned}$$

has finite rank. The fourth equality follows because the Toeplitz operators whose symbols are analytic polynomials commute with each other modulo finite rank operators and the last equality follows because the semicommutator of two Toeplitz operators whose symbols are polynomials in z and $\bar{z}$ has finite rank. By the Axler-Chang-Sarason Theorem and Lemma 2.1, we have

$$z^n\tilde{A}_1 B_1 - z^n\tilde{A}_2 B_2 = 0.$$

This completes the proof.

Proof of Theorem 1.9. First we prove necessity. Suppose $H_f H_g - H_g H_f$ has finite rank.

If H_f has finite rank, then by the Kronecker theorem, there is a polynomial $p(z)$ such that $pf \in H^\infty$. This gives Condition (1).

If H_g has finite rank, similarly, there is a polynomial $q(z)$ such that $qg \in H^\infty$. This gives Condition (2).

If neither of H_f nor H_g has finite rank, by Theorem 3.1, there are nonzero analytic polynomials $A(z)$ and $B(z)$ with degrees at most n such that

$$Af + Bg \in H^\infty.$$

Hence

$$\begin{aligned}
T_{\widetilde{A}}[H_f H_g - H_g H_f]T_B &= H_{Af} H_{gB} - H_{Ag} H_{fB} \\
&= H_{-Bg} H_{-fA} - H_{Ag} H_{fB} \\
&= H_g [T_B H_{fA} - T_A H_{fB}] \\
&= H_{z^n g}[H_{fz^n A\widetilde{B}} - H_{fBz^n\widetilde{A}}] \qquad mod \quad (F).
\end{aligned}$$

Let

$$A(z) = \sum_{i=0}^{n} a_i z^i, B(z) = \sum_{i=0}^{n} b_i z^i.$$

Then

$$\begin{aligned}\widetilde{A}(z) &= \sum_{i=0}^{n} a_i \bar{z}^i = \bar{z}^n \sum_{i=0}^{n} a_i z^{n-i}, \\ \widetilde{B}(z) &= \sum_{i=0}^{n} b_i \bar{z}^i = \bar{z}^n \sum_{i=0}^{n} a_i z^{n-i}.\end{aligned}$$

Thus

$$T_{\widetilde{A}}[H_f H_g - H_g H_f] T_B = H_{z^n g} H_f T_{[A \sum_{i=0}^{n} b_i z^{n-i} - B \sum_{i=0}^{n} a_i z^{n-i}]} \qquad mod \quad (F).$$

But neither $H_{z^n g}$ nor H_f has finite rank. So by the Axler-Chang-Sarason Theorem and Lemma 2.1,

$$A(z) \sum_{i=0}^{n} b_i z^{n-i} - B(z) \sum_{i=0}^{n} a_i z^{n-i} = 0.$$

This gives

$$A(z) \sum_{i=0}^{n} b_i z^{n-i} = B(z) \sum_{i=0}^{n} a_i z^{n-i}.$$

Thus

$$\bar{z}^n A(z) \sum_{i=0}^{n} b_i z^{n-i} = \bar{z}^n B(z) \sum_{i=0}^{n} a_i z^{n-i},$$

and so

$$A(z) \sum_{i=0}^{n} b_i \bar{z}^i = B(z) \sum_{i=0}^{n} a_i \bar{z}^i.$$

This means

$$A(z)\widetilde{B}(z) = B(z)\widetilde{A}(z).$$

Hence we obtain Condition (3).

Now we prove sufficiency. If either Condition (1) or Condition (2) holds, then by Kronecker's theorem, $H_f H_g - H_f H_g$ has finite rank.

If Condition (3) holds, then we have

$$Af + Bg \in H^\infty$$

and

$$A(z)\widetilde{B}(z) = B(z)\widetilde{A}(z).$$

Thus

$$\begin{aligned} T_{\widetilde{A}}[H_fH_g - H_gH_f]T_B &= H_{Af}H_{gB} - H_{Ag}H_{fB} \\ &= H_{Bg}H_{fA} - H_{Ag}H_{fB} \\ &= H_g[T_BT_{\widetilde{A}} - T_AT_{\widetilde{B}}]H_f \\ &= H_g[T_{\widetilde{A}}T_B - T_{\widetilde{B}}T_A]H_f \qquad mod \quad (F) \\ &= 0. \end{aligned}$$

So by Lemma 2.1, $H_fH_g - H_gH_f$ has finite rank. This completes the proof.

The following theorem gives a complete solution to the version of Problem 1.10 when "a compact perturbation" is replaced by "a perturbation of finite rank".

Theorem 3.3 *Suppose that f, g, and h are in L^∞. The following are equivalent:*

1. *H_fH_g is a perturbation of H_h of a finite rank;*
2. *H_fH_g and H_h both are finite rank operators;*
3. *there are nonzero analytic polynomials $A(z)$, $B(z)$ and $C(z)$ such that*

$$Ch \in H^\infty \quad \text{and} \quad Af \in H^\infty$$

or

$$Ch \in H^\infty \quad \text{and} \quad Bg \in H^\infty.$$

Proof: Clearly, we need only to prove that (1) implies (2). Suppose

$$H_fH_g = H_h \qquad mod \quad (F).$$

Since

$$\begin{aligned} H_f1 \otimes H_{g^*}1 &= H_f[1 \otimes 1]H_g \\ &= H_f(1 - T_zT_{\bar{z}})H_g \\ &= H_fH_g - T_{\bar{z}}H_fH_gT_z \\ &= H_h - T_{\bar{z}}H_hT_z \qquad mod \quad (F) \\ &= H_{h(1-z^2)} = H_hT_{1-z^2}, \end{aligned}$$

$H_hT_{1-z^2}$ has finite rank. Thus by Lemma 2.1, H_h has finite rank also, and so H_fH_g has finite rank. This completes the proof.

Theorem 3.4 *Suppose that f_i, g_i and h are in L^∞ for $i = 1, 2$. The following are equivalent:*

1. $H_{f_1}H_{g_1} - H_{f_2}H_{g_2} = H_h \quad mod \quad (F)$;
2. *H_h and $H_{f_1}H_{g_1} - H_{f_2}H_{g_2}$ have finite rank;*
3. *H_h, $H_{f_1}H_{g_1}$ and $H_{f_2}H_{g_2}$ have finite rank or there are nonzero analytic polynomials $A_i(z)$, $B_i(z)$, $C(z)$ such that*

$$A_1(z)\tilde{B_1}(z) = A_2(z)\tilde{B_2}(z)$$

and

$$Ch \in H^\infty, \quad A_1f_1 + A_2f_2 \in H^\infty \text{ and } B_1g_1 + B_2g_2 \in H^\infty.$$

PROOF: First we show that (2) implies (3). Suppose that $H_{f_1}H_{g_1} - H_{f_2}H_{g_2}$ has finite rank. If $H_{f_1}H_{g_1}$ has finite rank, then $H_{f_2}H_{g_2}$ does. If neither of $H_{f_1}H_{g_1}$ nor $H_{f_2}H_{g_2}$ has finite rank, by Theorem 3.1 there are nonzero analytic polynomials $A_i(z)$, $B_i(z)$ such that

$$A_1(z)\tilde{B_1}(z) = A_2(z)\tilde{B_2}(z),$$

$$A_1f_1 + A_2f_2 \in H^\infty \text{ and } B_1g_1 + B_2g_2 \in H^\infty$$

This is the desired result.

Next we prove that (1) implies (2). Suppose

$$H_{f_1}H_{g_1} - H_{f_2}H_{g_2} = H_h.$$

An easy calculation gives

$$\begin{aligned} H_{f_1}1 \otimes H_{g_1^*}1 - H_{f_2}1 \otimes H_{g_2^*}1 &= H_{f_1}(1 - T_zT_{\bar{z}})H_{g_1} - H_{f_2}(1 - T_zT_{\bar{z}})H_{g_2} \\ &= H_{f_1}H_{g_1} - H_{f_2}H_{g_2} - T_{\bar{z}}(H_{f_1}H_{g_1} - H_{f_2}H_{g_2})T_z \\ &= H_h - T_{\bar{z}}H_hT_z \quad mod \quad (F) \\ &= H_hT_{1-z^2}. \end{aligned}$$

Thus $H_hT_{1-z^2}$ has finite rank. By Lemma 2.1, H_h has finite rank and so $H_{f_1}H_{g_1} - H_{f_2}H_{g_2}$ has finite rank.

As in the proof of Theorem 3.1, it is easy to see that (3) implies (1). We leave details for the reader. This completes the proof.

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