CALDERÓN-ZYGMUND CAPACITIES AND NON-LINEAR CAPACITIES

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ABSTRACT. There are many interesting questions concerning capacities with positive kernels. This is especially so if we are dealing with non-linear capacities introduced by Maz'ya. Non-linear capacities were extensively investigated by Maz'ya, Khavin, Hedberg, they play important part in Analysis and PDE. For example, they found important application in the solution the problem of spectral synthesis for Sobolev spaces given by Hedberg and Wolff. Recently the new type of capacities appear. For them the defining kernels are not positive. They are typically odd kernels of Calderón-Zygmund type. We see some of them below as well as their relations with non-linear capacities.

1. INTRODUCTION

Consider the following kernel in $\mathbb{R}^d$: $K_s(x) := \frac{x}{|x|^{1+s}}$. Here $x := (x_1, ..., x_d)$, $s \in (0, d]$. Consider also its positive counterpart $k_s(x) := \frac{1}{|x|^s}$.

Fix a compact set $K \subset \mathbb{R}^d$. Recall that usual (linear) capacity $C_s(K)$ is defined as follows:

$$C_s(K) = \sup\{\|\mu\| : \mu \in M_+(K) : U_s(\mu)(x) := \int k_s(x-y)\, d\mu(y) \leq 1\}\,.$$

Here $M_+(K)$ stands for all positive measures supported in K. We will also use symbols $M_r(K)$, $M_c(K)$, $D(K)$ for real measures, complex measures, and distributions with compact supported in K correspondingly.

The Calderón-Zygmund capacity $\gamma_{s,+}$ is defined totally similarly:

$$\gamma_{s,+}(K) := \sup \mu(K),$$

and the supremum is taken over all positive measures μ supported by K such that $\mu * \dfrac{x}{|x|^{s+1}}$ is a function in $L^\infty(\mathbb{R}^d)$ with $\left\|\mu * \dfrac{x}{|x|^{s+1}}\right\|_\infty \leq 1$.

Let us also define "similar" quantities $\gamma_{s,r}$, $\gamma_{s,c}$, γ_s. The definition is the same but we replace the assumption $\mu \in M_+(K)$ by $\mu \in M_r(K)$, $\mu \in M_c(K)$, and $\mu \in D(K)$ correspondingly. Also we natuarally replace the quantity $\mu(K)$ by $|\mu(K)| = |\langle \mu, 1\rangle|$.

Obviously

$$\gamma_{s,+} \leq \gamma_{s,r}(K) \leq \gamma_{s,c}(K) \leq \gamma_s(K)\,. \tag{1.1}$$

A natural question arise:

Question 1. Why not to introduce, say, $C_{s,r}$ along with C_s?

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The answer is simple, $C_s = C_{s,r}$. In fact, this can be derived easily from p. 136 of Landkof's book on potential theory. Let us give this simple reasoning. Introduce the class of positive measures $\mathcal{E}_+$ as $\mu \in M_+(K)$ such that the energy is finite

$$|\mu|^2 := \int U_s(\mu)\, d\mu < \infty\,.$$

Then let $\mathcal{E}$ denote the class of real measures μ such that $\mu_+, \mu_- \in \mathcal{E}_+$. It is obvious that

$$\int U_s(\mu_+)\, d\mu_- \le |\mu_+||\mu_-|\,.$$

Therefore, $|\cdot|$ is well defined on $\mathcal{E}$, and provide it with the Hilbert space structure. The space $\mathcal{E}$ is a linear pre-Hilbert space.

Now suppose $\mu \in M_r$ is in $\mathcal{E}$. Suppose also that

$$U_s(\mu) \le 1 \tag{1.2}$$

on K. Let λ be the equilibrium measure (see [14]). Then $U_s(\lambda) \le 1$ everywhere on K but also $U_s(\lambda) \ge 1$ quasieverywhere on K meaning that the set K' where this inequality fails is of zero C_s capacity. In particular, $\mu(K') = 0$ if $\mu \in \mathcal{E}$. Finally, $|\lambda|^2 = C_s(K)$. Using all that (and (1.2)) we get

$$|\mu - \lambda|^2 = |\mu|^2 - 2\int U_s(\lambda)\, d\mu + |\lambda|^2 \le \mu(K) - 2\mu(K) + C_s(K)\,.$$

Therefore,

$$\mu(K) \le C_s(K) - |\mu - \lambda|^2 \le C_s(K)\,.$$

2. Energy, pairs interactions, triplets interaction, and semiadditivity

Capacity C_s is trivially semiadditive. In fact, let $K = K_1 \cup K_2$, and let μ be a positive measure that saturates the assumptions for $C_s(K)$. In particular, $\mu * \frac{1}{|x|^s} \le 1$ on K. Then, for $\mu_1 := \mu|K_1$ we have $\mu_1 * \frac{1}{|x|^s} \le 1$ on K_1. And the same for $\mu_2 = \mu - \mu_1$ on K_2. This is trivial consequence of the positivity of the kernel. Then just by definition we get

$$C_s(K) = \mu(K) = \mu(K_1) + \mu(K_2) \le C_s(K_1) + C_s(K_2)\,.$$

Vitushkin's problem stated that analytic capacity $\gamma = \gamma_1$ on $\mathbb{R}^2$ is semiadditive as well. This is unknown and unlikely. What has been proved in Tolsa's celebrated paper [29] is that there exists an absolute constant A such that

$$\gamma(K_1 \cup K_2) \le A(\gamma(K_1) + \gamma(K_2))\,. \tag{2.1}$$

The self-contained exposition of this fact can be found in [31]. Also this book contains the proof in $\mathbb{R}^d$, $d > 2$ of (here $A = A(d)$)

$$\gamma_{d-1}(K_1 \cup K_2) \le A(\gamma_{d-1}(K_1) + \gamma_{d-1}(K_2))\,. \tag{2.2}$$

This is a generalization of (2.1) from $s = 1, d = 2$ case to $s = d-1, d > 2$ case. What are other indecies $s \in (0, d]$ for which this semiadditivity with constant holds? In other words, what are s such that

$$\gamma_s(K_1 \cup K_2) \le A(\gamma_s(K_1) + \gamma_s(K_2))\,? \tag{2.3}$$

We already said that $s = d-1$ works. There is a trivial case $s = d$, for which (2.3) works because $\gamma_d(E) \asymp |E|$, whre $|E|$ stands for Lebesgue measure of E. There are two non-trivial cases when this is true and known:

- We already said that $s = d - 1$ works.
- It works for $0 < s < 1$, see [27].

Strangely enough for all other s it seems to be an open problem.

How the proof of (2.1) can be organized? Suppose (see the next section about that) we already know that

$$\gamma \asymp \gamma_+ \,.$$

Then the proof of (2.1) can be deduced immediately from the fact

$$\gamma_+(K_1 \cup K_2) \le A(\gamma_+(K_1) + \gamma_+(K_2))\,. \tag{2.4}$$

If we try to repeat the proof that we have for C_s above, we will come immediately to the difficulty: if $\left|\mu * \frac{1}{z}\right| \le 1$ on $K_1 \cup K_2$ then this *does not mean* anymore that $\left|\mu_1 * \frac{1}{z}\right| \le 1$ on K_1, where $\mu_1 := \mu|K_1$.

In fact, the proof of (2.4) is a challenging task in its own right, and it is not known whether (2.4) holds with $A = 1$.

But there is a capacity for which we can claim the analog of (2.4), and moreover with $A = 1$!

We need to introduce the notion of energy into our considerations. In the case of C_s (positive kernel), the energy was $|\mu|^2 = \int \frac{d\mu(x)\,d\mu(y)}{|x-y|^s}$ which stands for the energy of interaction between all pairs of points.

In the case of γ_s there is no completely understood notion of energy, but the best known replacement for it is

$$E(\mu) := \|\mathfrak{R}^s_\mu : L^2(\mu) \to L^2(\mu)\|\,.$$

Here $\mathfrak{R}^s_\mu$ stand for the operator acting by the formula

$$(\mathfrak{R}^s_\mu f, g)_{L^2(\mu)} := \frac{1}{2}\int\int K_s(x-y)[f(x)g(y) - f(y)g(x)]\,d\mu(x)\,d\mu(y)$$

for all smooth functions f, g. We are interested in μ for which this energy (norm) is finite, that is $\mathfrak{R}^s_\mu$ can be extended from the lineal of smooth functions to $L^2(\mu)$ as a bounded operator into $L^2(\mu)$.

Here is the notion of capacity related to this energy:

$$\gamma_{s,op}(E) := \sup\{\|\mu\| : \mu \in \Sigma_s,\ \operatorname{supp}\mu \subset E,\ E(\mu) \le 1\}\,. \tag{2.5}$$

Here $\mu \in \Sigma_s$ means that $\mu(B(x,r)) \le r^s\ \forall x \in \mathbb{R}^d$ and all $r > 0$.

Now we can split the proof of (2.4) to two parts (see [22],[23]):

$$\gamma_{s,op}(E) \asymp \gamma_{s,+}\,. \tag{2.6}$$

$$\gamma_{s,op}(K_1 \cup K_2) \le \gamma_{s,op}(K_1) + \gamma_{s,op}(K_2)\,. \tag{2.7}$$

The first relationship is very non-trivial, and it is essentially non-homogeneous $T1$ theorem, see [22],[23], [28].

However, the second relationship is obvious! In fact, let μ staurate the left hand side. In particular, $\|\mathfrak{R}^s_\mu : L^2(\mu) \to L^2(\mu)\| \le 1$. But then obviously $\|\mathfrak{R}^s_{\mu_1} : L^2(\mu_1) \to L^2(\mu_1)\| \le 1$

and $\|\mathfrak{R}^s_{\mu_2} : L^2(\mu_2) \to L^2(\mu_2)\| \le 1$, where $\mu_1 := \mu|K_1, \mu_2 := \mu - \mu_1$! We just compress the operator. So its norm can only drop. Now we have a trivial chain of inequalities:

$$\gamma_{s,op}(K_1 \cup K_2) = \mu(K_1 \cup K_2) = \mu_1(K_1) + \mu_2(K_2) \le \gamma_{s,op}(K_1) + \gamma_{s,op}(K_2)\,.$$

For the special case $s = 1, d = 2$ one has another, more conspicous capacity and energy. It was introduced by M. Melnikov. Let $R(x, y, z)$ stands for the radius of the disc passing through points $x, y, z \in \mathbb{C}$. Here is the "energy of interaction of all triples", called Menger's curvature

$$c^2(\mu) := \int\int\int \frac{d\mu(x)d\mu(y)d\mu(z)}{R^2(x, y, z)}\,.$$

Corresponding capacity is

$$\gamma_{curv}(E) := \sup\{\|\mu\| : \mu \in \Sigma_1(E)\,,\, c^2(\mu) \le \mu(E)\}\,.$$

It is also equivalent to the capacity γ_+, see [20], [21], [28], [22], and the book [31].

Unfortunately, we do not know such geometrically meaningful energy for $d > 2$ or for $d = 2, s > 1$. The curvature tool is "cruelly missing" as Guy David puts it. At least this is so today.

3. Two thirds/three halves

Above we saw an almost obvious relationship $C_s = C_{s,r}$. We cannot use the argument as simple as above to prove

$$\gamma_{s,+}(E) = \gamma_{s,r}(E)\,.$$

Moreover, this is most probably false. The only thing that *may* and actually *is* true is the following equivalence

$$\gamma_{s,r}(E) \le A\,\gamma_{s,+}(E)\,.$$

The weak form of it was proved independently by Guy David and Pertti Mattila [6], and by Nazarov-Treil-Volberg, see [25], [24] (see also the exposition in [31]). Namely, the weak form means $\gamma_{s,+}(E) = 0 \Rightarrow \gamma_{s,r}(E) = 0$. Actually, the equivalence is also true, but it is the essence of Tolsa's theorem. More precisely Tolsa [29] famous result states

$$\gamma_{s,c}(E) \le A\,\gamma_{s,+}(E)\,.$$

The weak form $\gamma_{s,+}(E) = 0 \Rightarrow \gamma_{s,c}(E) = 0$ was proved by Guy David [7] and by Nazarov-Treil-Volberg, see [25], [24] (also is in [31]).

Interestingly $\gamma_{s,+}(E)$ is related to the Riesz capacity $C_{\alpha,p}$ in non-linear potential theory. One of a number of equivalent definitions is the following equality (see [1], p. 34, Theorem 2.5.1):

$$C_{\alpha,p}(E) = \sup_{\mu\in M_+(E)} \left(\frac{\mu(E)}{\|I_\alpha * \mu\|_{p'}}\right)^p, \quad I_\alpha(x) = \frac{A_{d,\alpha}}{|x|^{d-\alpha}}, \quad \frac{1}{p'} + \frac{1}{p} = 1,$$

where $1 < p < \infty$, $0 < \alpha p \le d$, $\|\cdot\|_{p'}$ is the $L^{p'}$-norm with respect to the Lebesque measure in $\mathbb{R}^d$, and $A_{d,\alpha}$ is the certain constant depending on d and α. It was proved in [15] that

$$\gamma_{s,+}(E) \asymp \gamma_s(E) \asymp C_{\frac{2}{3}(d-s),\frac{3}{2}}(E), \quad 0 < s < 1\,. \tag{3.1}$$

In [10] we proved that the inequality $\gamma_{s,+}(E) \ge C \cdot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E)$ holds for $0 < s < d$.

Main results of this section relate to connections between Hausdorff content M_h and the capacity $\gamma_{s,+}$, as well as between $\gamma_{s,+}$ and $C_{\alpha,p}$.

When talking about M_h we will use the following assumption on the gauge function h:

$$\int_0 \left(\frac{h(t)}{t^s}\right)^2 \frac{dt}{t} < \infty\,, \tag{3.2}$$

We need also the following important characterization of $\gamma_{s,+}$ obtained essentially in [22], [23] but explicitly formulated in [31], Chapter 5:

$$\gamma_{s,+}(E) \asymp \gamma_{op}(E) := \sup\{\|\mu\| : \mu \in \Sigma_s,\ \operatorname{supp}\mu \subset E,\ |\mathfrak{R}^s_\mu| \le 1\}, \quad 0 < s < d. \tag{3.3}$$

Theorem 3.1. *Under assumption (3.2), for each compact set $E \subset \mathbb{R}^d$,*

$$\gamma_{s,+}(E) \ge C M_h(E) \left[\int_0^{t_2} \left(\frac{h(t)}{t^s}\right)^2 \frac{dt}{t}\right]^{-1/2}, \quad 0 < s < d, \tag{3.4}$$

where C depends only on d, s, and t_2 is defined by the equality $h(t_2) = M_h(E)$.

Proof. By Frostman's theorem (see [3], p. 7) there is a positive measure μ such that
$\operatorname{supp}\mu \subset E$,
$\mu(B(x,r)) \le h(r)$ for each ball $B(x,r) \subset \mathbb{R}^d$,
$\mu(E) \ge C M_h(E)$ with C depending only on d.
Without loss of generality we can assume that $\|\mu\| \le M_h(E)$ (otherwise we divide μ by the constant $\|\mu\|/M_h(E) > 1$). We define a_h and η as in Section 7 of [10] with $t_1 = 0$ and $t_2 = M_h(E)$, namely,

$$a_h := C_3 \left[\int_0^{t_2} \left(\frac{h(t)}{t^s}\right)^2 \frac{dt}{t}\right]^{1/2}, \quad \eta := a_h^{-1}\mu.$$

Then $\eta \in \Sigma_s$. Theorem 4.6 of [10] and (3.3) yield

$$\gamma_{s,+}(E) \ge C\eta(E) \ge C' a_h^{-1} M_h(E)$$

with C, C' depending only on d and s. ☐

For $h(t) = t^\beta$ easy calculations give the following result.

Corollary 3.2. *For each compact set $E \subset \mathbb{R}^d$,*

$$\gamma_{s,+}(E) \ge C(\beta - s)^{1/2}[M_h(E)]^{s/\beta}, \quad \text{where} \quad 0 < s < d, \quad h(t) = t^\beta, \quad \beta > s,$$

and C depends only on d and s.

The next statement can be viewed as a counterpart of the classical Frostman's theorem on connections between capacities generated by potentials with positive kernels and Hausdorff measure $\Lambda_h(E)$ (see, for example, [3], Section IV, Theorem 1).

Corollary 3.3. *For each compact set $E \subset \mathbb{R}^d$, the condition $\gamma_{s,+}(E) > 0$ implies $\Lambda_h(E) > 0$ for $h(t) = t^s$. On the other hand, if $\Lambda_h(E) > 0$ for a measuring function h satisfying* (3.2) *then $\gamma_{s,+}(E) > 0$.*

Remark. Notice that Carleson's book has a a very similar condition

$$\int_0 \frac{h(t)}{t^s} \frac{dt}{t} < \infty\,, \tag{3.5}$$

which guaranteed the positivity of $C_s(E)$ if $\Lambda_h(E)$ is already positive. The cancellation inherent in Calderón-Zygmund capacities reveals itself in this difference between these two conditions (3.2) and (3.5). The Calderón-Zygmund capacity behaves in much more "stochastic" fashion.

Proof. The first part of Corollary 3.3 is a direct consequence of the following result by Prat [27], p. 946: for $0 < s < d$

$$C_\varepsilon [M_{t^{s+\varepsilon}}(E)]^{s/(s+\varepsilon)} \leq \gamma_s(E) \leq C M_{t^s}(E)$$

(we need the second inequality). Indeed, by definition $\gamma_{s,+}(E) \leq \gamma_s(E)$, and $M_h(E)$, $\Lambda_h(E)$ vanish simultaneously. (We remark that for $0 < s < 1$, Prat [27] has obtained the following essentially stronger result: if $\gamma_s(E) > 0$ then $\Lambda_h(E) = \infty$.)

The second part is an immediate consequence of (3.4). $\square$

Obviously, there is a gap between the assumptions about h in the first and the second parts of Corollary 3.3. We claim that this gap cannot be reduced, that is, both parts are sharp. Concerning the first part it means that if $\liminf_{t\to 0} h(t)t^{-s} = 0$, then there is a compact set E for which $\gamma_{s,+}(E) > 0$ but $\Lambda_h(E) = 0$. This assertion follows from the more general and strong result [3], p. 34, Theorem 4: for any positive kernel $K(r)$ and any measuring function $h(r)$ such that

$$\liminf_{r\to 0} h(r)\overline{K}(r) = 0,$$

there is a Cantor type set E with $C_K(E) > 0$ and $\Lambda_h(E) = 0$. Here

$$\overline{K}(r) = \frac{1}{r^d}\int_0^r K(t)t^{d-1}\,dt,$$

$$C_K(E) := \sup\left\{ \|\mu\| : \mu \in M_+(E),\ \int_{\mathbb{R}^d} K(|x-y|)\,d\mu(y) \leq 1 \text{ on } E \right\}.$$

For $K(r) = r^{-s}$ we have $\overline{K}(r) = \frac{1}{d-s}r^{-s}$. Trivial estimate $\gamma_{s,+}(E) \geq C \cdot C_K(E)$ gets us the needed assertion.

The second part of Corollary 3.3 is also precise: if the integral in (3.2) is divergent, then there exists a compact set E for which $\Lambda_h(E) > 0$ but $\gamma_{s,+}(E) = 0$. One can derive this statement from the estimate for the capacity γ_s of Cantor sets given at the end of [16]. A simpler (but probabilistic) computation is in Section 8 of [10].

The results of this section mentioned above generalize the corresponding results in [9], Section 12.

In conclusion we prove Proposition 3.4 from [10] and so complement the relations (3.1) between the capacities $\gamma_{s,+}$ and $C_{\frac{2}{3}(d-s),\frac{3}{2}}$.

Proposition 3.4. *For $0 < s < d$, one has*

$$\gamma_{s,+}(E) \geq c \cdot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E). \tag{3.6}$$

Proof. We may assume that $C_{\frac{2}{3}(d-s),\frac{3}{2}}(E) > 0$. As in [15], our proof is based on the following Wolff's equality [1], p. 110, Theorem 4.5.4: for any $\mu \in M_+(\mathbb{R}^d)$ and $1 < p < \infty$, $0 < \alpha p \leq d$,

$$\int_{\mathbb{R}^d} W^\mu_{\alpha,p}(x)\,d\mu(x) \asymp \|I_\alpha * \mu\|^{p'}_{p'}, \quad W^\mu_{\alpha,p}(x) := \int_0^\infty \left[\frac{\mu(B(x,r))}{r^{d-\alpha p}}\right]^{p'-1} \frac{dr}{r}. \tag{3.7}$$

Take $\alpha = \frac{2}{3}(d-s)$, $p = \frac{3}{2}$. Then $p' = 3$, $d - \alpha p = s$, and

$$W^{\mu}_{\alpha,p}(x) = \int_0^\infty \frac{[\mu(B(x,r))]^2}{r^{2s+1}}\,dr =: W^\mu(x).$$

Choose $\mu \in M_+(E)$ for which

$$C_{\alpha,p}(E) < 2\|\mu\|^p \|I_\alpha * \mu\|_{p'}^{-p}. \tag{3.8}$$

Set

$$G := \left\{ x \in \mathbb{R}^d : W^\mu(x) > \frac{2}{\|\mu\|}\int_{\mathbb{R}^d} W^\mu(x)\,d\mu(x) \right\}.$$

It is easy to see that G is open and

$$\mu(G) \le \frac{1}{2}\|\mu\|.$$

Let

$$\mu^* = \mu|(\mathbb{R}^d \setminus G), \quad \mathbf{S} = \sup_{x \in \operatorname{supp}\mu^*} W^{\mu^*}(x).$$

We claim that

$$W^{\mu^*}(x) \le 2^{2s+1}\mathbf{S} \quad \text{for all } x \in \mathbb{R}^d. \tag{3.9}$$

It is enough to consider x with $\delta := \operatorname{dist}(x, \operatorname{supp}\mu^*) > 0$. Let x' be such that $x' \in \operatorname{supp}\mu^*$ and $|x - x'| = \delta$. Then

$$\begin{aligned} W^{\mu^*}(x) &= \int_\delta^\infty \frac{[\mu^*(B(x,r))]^2}{r^{2s+1}}\,dr \le \int_\delta^\infty \frac{[\mu^*(B(x',r+\delta))]^2}{r^{2s+1}}\,dr \\ &= \int_{2\delta}^\infty \frac{[\mu^*(B(x',t))]^2}{(t-\delta)^{2s+1}}\,dt < 2^{2s+1}\int_{2\delta}^\infty \frac{[\mu^*(B(x',t))]^2}{t^{2s+1}}\,dt \le 2^{2s+1}\mathbf{S}, \end{aligned}$$

and we get (3.9).

Let $\eta = (2^{2s+2}s\mathbf{S})^{-1/2}\mu^*$. Since for each ball $B(x,r)$

$$2^{2s+1}\mathbf{S} \ge \int_0^\infty \frac{[\mu^*(B(x,t))]^2}{t^{2s+1}}\,dt \ge \int_r^\infty \frac{[\mu^*(B(x,t))]^2}{t^{2s+1}}\,dt \ge \frac{[\mu^*(B(x,r))]^2}{2sr^{2s}},$$

we see that $\eta \in \Sigma_s$. Then we have

$$|\mathfrak{R}^s_\eta|^2 \le C(2^{2s+2}s\mathbf{S})^{-1}\mathbf{S} = C'.$$

Relations (3.3) and $\|\mu^*\| \ge \frac{1}{2}\|\mu\|$ yield

$$\gamma_{s,+}(E) \ge C\eta(E) \ge C'\|\mu\|\mathbf{S}^{-1/2}.$$

Since

$$\mathbf{S} \le \sup_{x \in \operatorname{supp}\mu^*} W^\mu(x) \le \frac{2}{\|\mu\|}\int_{\mathbb{R}^d} W^\mu(x)\,d\mu(x),$$

we have

$$\gamma_{s,+}(E) \ge C\|\mu\|^{3/2}\left[\int_{\mathbb{R}^d} W^\mu(x)\,d\mu(x)\right]^{-1/2} \overset{(3.7)}{\asymp} \|\mu\|^{3/2}\|I_\alpha * \mu\|_{p'}^{-p'/2} \overset{(3.8)}{>} \frac{1}{2}C_{\frac{2}{3}(d-s),\frac{3}{2}}(E),$$

and we get (3.6). $\square$

For integer $s \in (0,d)$ the opposite inequality $\gamma_{s,+}(E) \leq C \cdot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E)$ is false. In fact, for a smooth s-dimensional manifold E in $\mathbb{R}^d$ we have $\gamma_{s,+}(E) > 0$ by the obvious reason that natural Lebesgue measure on it gives bounded Riesz transform operator (this is from the classical Calderón-Zygmund theory). On the other hand, it has been noticed (for example in [15]) that any measure μ with finite Wolff's energy should have $\mu(B(x,r)) = o(r^s)$ for μ a. e. x. On a smooth s-dimensional manifold it can be only zero measure, so $C_{\frac{2}{3}(d-s),\frac{3}{2}}(E) = 0$.

Question 2. The question about validity of the inequality $\gamma_{s,+}(E) \leq C \cdot C_{\frac{2}{3}(d-s),\frac{3}{2}}(E)$ for all non-integer $s \in (0,d)$ remains open. We believe that this is the case.

4. Strange Sobolev embedding.

Previous result has a very strange interpretation. Recall that

$$I_\alpha(x) = \frac{A_{d,\alpha}}{|x|^{d-\alpha}}\,,$$

and that $I_\alpha * I_\beta = I_{\alpha+\beta}$.

Also recall the Sobolev embedding

$$I_\alpha * L^p_0 \subset L^q_{loc}\,, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}\,.$$

Notice now that $C_{2/3(d-s),3/2}(E) > 0$ means that there exists $\mu \in M_+(E)$ such that

$$I_{2/3(d-s)} * \mu \in L^3(\mathbb{R}^d)\,.$$

Then Sobolev embeding gives

$$I_{d-s} * \mu = I_{1/3(d-s)} * I_{2/3(d-s)}\mu \in L^q_{loc}\,,$$

where

$$\frac{1}{q} = \frac{1}{3} - \frac{d-s}{3d} = \frac{s}{3d}\,.$$

We get that

$$\int \frac{d\mu(x)}{|x-y|^s} \in L^{\frac{3d}{s}}_{loc}\,.$$

Then classical Calderón-Zygmund theory implies

$$\int \frac{x_i - y_i}{|x-y|^{1+s}}\, d\mu(x) \in L^{\frac{3d}{s}}_{loc}\,,\ i = 1, ..., d\,.$$

There is nothing strange in this. What is strange is that Proposition 3.4 then claims the following corollary

Theorem 4.1. *For any $\delta > 0$ there exists a function $h, 0 \leq h \leq 1$, $\int_E h d\mu \geq (1-\delta)\mu(E)$ such that measure $\mu' := h\, d\mu$ has the following property:*

$$\int \frac{x_i - y_i}{|x-y|^{1+s}} d\mu(x) \in L^\infty\,,\ i = 1, ..., d\,.$$

Kernels in the last three display relationships are of the same singularity s. But positive kernel gives us potentil of μ only in L^6 if $s = 1, d = 2$, the singular kernel gives us "almost the same" potential of "almost the same" measure in L^∞! But we required to "correct" measure μ a little bit. Still the jump from L^6 to L^∞ seeems to be very huge, and the price is just discarding a small (δ) piece of the measure.

The proof of the last theorem is scattered through [22], [23], [24], [25], [28], [29], and can be found in a self-contained form in the book [31].

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