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Invariant Subspaces, Quasi-invariant Subspaces, and Hankel Operators[1]

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In this paper, using the theory of Hilbert modules we study invariant subspaces of the Bergman spaces on bounded symmetric domains and quasi-invariant subspaces of the Segal–Bargmann spaces. We completely characterize small Hankel operators with finite rank on these spaces.

1. INTRODUCTION

In this paper, we study algebraic properties of small Hankel operators on Bergman spaces of bounded symmetric domains. Because of the connections between finite codimensional invariant subspaces and finite rank small Hankel operators we need to study invariant subspaces of the Bergman space on the bounded symmetric domains and the quasi-invariant subspaces of the Segal–Bargmann spaces.

Hankel operators on Bergman spaces have been extensively studied in recent years, see [Arz, Ax1, Ax2, BCZ, Guo4, JPR, SZh1, Zhu1, Zhu2] and references there. This theme is interesting because it exhibits the connection between function theory and operator theory, and strongly depends on the geometry of underlying domains.

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The Segal–Bargmann space, or the so-called Fock space, is the analogue of the Bergman space in the context of complex n-space C^n. It is a Hilbert space consisting of entire functions in C^n. This space is important because of the relationship between the operator theory on it and Weyl quantization [Cob, Fol]. However in this space, unlike in Bergman spaces, there exist no nontrivial invariant subspaces for all polynomials. Thus, an appropriate substitute for invariant subspaces, so-called quasi-invariant subspaces is needed. A (closed) subspace M of the Segal–Bargmann space $L_a^2(C^n)$ is called quasi-invariant if the relation $pf \in L_a^2(C^n)$ implies $pf \in M$ for any $f \in M$ and any polynomial p. Equivalently, a closed subspace M is quasi-invariant if and only if $pM \cap L_a^2(C^n) \subset M$ for every polynomial p.

The paper is arranged in the following manner. In Section 2 we introduce some background material and study some algebric properties of small Hankel operators. The proofs of results in this section depend on the explicit expressions of reproducing kernel functions of bounded symmetric domains. Section 3 treats finite codimensional invariant subspaces on bounded symmetric domains. By means of the automorphism group of a bounded symmetric domain, we can explicitly write out the structure of $M^\perp$ if M is finite codimensional. Applying the result of Section 3, we obtain a complete characterization of small Hankel operators of finite rank in Section 4.

In Section 5, we prove an algebraic reduction theorem for finite codimensional quasi-invariant subspaces. We apply this theorem to study when two quasi-invariant subspaces are similar (unitarily equivalent). Two cases are considered. First, on the complex plane C, it is shown that a quasi-invariant subspace M is similar to a finite codimensional quasi-invariant subspace N if and only if M is finite codimensional, and M and N have the same codimension. This is completely different from the case of the Bergman space. It is well known that all finite codimensional invariant subspaces of $L_a^2(D)$ (D: unit disk) are similar. Second, on n-dimensional complex space C^n ($n > 1$), an entirely new phenomenon occurs: a quasi-invariant subspace M is similar to a finite codimensional quasi-invariant subspace N only if $M = N$. In particular, if a quasi-invariant subspace M is similar to $L_a^2(C^n)$, then it must be $L_a^2(C^n)$. In contrast, for an analytic Hilbert space X on a bounded domain Ω, there exist many invariant subspaces which are similar to X.

In Section 6, we apply results in Section 5 to obtain a complete description for finite rank small Hankel operators on the Segal–Bargmann space $L_a^2(C^n)$. Namely, a small Hankel operator Γ_ϕ is of finite rank if and only if there exist points $\lambda_1, \lambda_2, ..., \lambda_l$ in C^n and polynomials $p_1, p_2, ..., p_l$ such that

$$\phi(z) = \sum_{i=1}^{l} p_i(z)\, e^{\lambda_i z}.$$

2. ALGEBRAIC PROPERTIES OF SMALL HANKEL OPERATORS

In this section we study algebraic properties of small Hankel operators. First let us recall the properties of bounded symmetric domains which will be used in subsequent sections.

Let Ω be a domain in C^n, $z_0 \in \Omega$. Ω is called to be symmetric with respect to z_0 if there exists a biholomorphic mapping ϕ of Ω onto Ω with $\phi \circ \phi =$ the identity and z_0 an isolated fixed point of ϕ. The domain is symmetric if it is symmetric in each of its points. The simplest examples of bounded symmetric domains are the unit ball B_n and the unit polydisk D^n.

Let Ω be a bounded symmetric domain. We always assume that Ω is circular and in its standard (Harish–Chandra) realization so that $0 \in \Omega$. Moreover Ω is also starlike; i.e., $z \in \Omega$ implies that $tz \in \Omega$ for all $t \in [0, 1]$. We can canonically define (see [Hel] or [BCZ]) for each $\lambda \in \Omega$, an automorphism ϕ_λ in $Aut(\Omega)$, the group of all automorphisms (biholomorphic mappings) of Ω such that

1. $\phi_\lambda \circ \phi_\lambda(z) = z$;
2. $\phi_\lambda(0) = \lambda$, $\phi_\lambda(\lambda) = 0$;
3. ϕ_λ has a unique fixed point in Ω.

Recall that the rank of the bounded symmetric domain Ω (assumed to be in its standard realization) is the largest positive integer m such that there exists an m-dimensional subspace V of C^n with the property that $\Omega \cap V$ is holomorphically equivalent to the m-dimensional polydisk D^m. It is well known that the rank of the Cartesian product of two bounded symmetric domains equals the sum of the ranks of the factors; and a bounded symmetric domain has rank one if and only if it is holomorphically equivalent to B_n, the unit ball of C^n. Therefore, on the complex plane C, the unit disk D is the unique bounded symmetric domain, and on C^2, by [Tim, Th. 2.5] there are only bounded symmetric domains the unit ball B_2 and the unit polydisk D^2 (under holomorphic equivalence).

Furthermore, Cartan [Car] proved that each bounded symmetric domain is (holomorphically equivalent to) a Cartesian product of irreducible bounded symmetric domains. Here "irreducible" means "not holomorphically equivalent to a cartesian product." Up to the two exceptional bounded symmetric domains in C^{16} and C^{27}, respectively, the irreducible bounded symmetric domains can be classified into four types that are called the classical domains [Car, Hua].

Let Ω be a bounded symmetric domain in C^n with dA the normalized Lebesgue measure on it. The Bergman space $L^2_a(\Omega, dA)$ (for short L^2_a) is the closed subspace of $L^2(\Omega, dA)$ consisting of analytic functions. For $\lambda \in \Omega$, the Bergman reproducing kernel is the function $K_\lambda \in L_a^2$ such that

$$f(\lambda) = \langle f, K_\lambda \rangle$$

for each $f \in L_a^2$. The normalized Bergman reproducing kernel k_λ is the function $K_\lambda / \|K_\lambda\|$. For the classical domains of types I–IV, Hua [Hua] explicity computed the reproducing kernels. Namely, for each classical domain, there is an analytic polynomial p such that $K_\lambda(z) = 1/p(z, \bar{\lambda})$. Since the reproducing kernel of the Cartesian product of two domains equals the product of the reproducing kernels of the factors, the reproducing kernel of each bounded symmetric domain which is (or is holomorphically equivalent to) a Cartesian product of classical domains is of the form above.

For $\phi \in L^2(\Omega)$, the Toeplitz operator T_ϕ with symbol ϕ is the operator densely defined in L_a^2 by

$$T_\phi f = P(\phi f), \qquad f \in H^\infty(\Omega),$$

where P is the orthogonal projection from $L^2(\Omega)$ onto $L_a^2(\Omega)$.

Let U: $L^2(\Omega) \to L^2(\Omega)$ be the unitary operator defined by

$$(Uf)(z) = f(\bar{z}) \stackrel{\text{def}}{=} \hat{f}(z), \qquad z \in \Omega.$$

For $\phi \in L^2(\Omega)$, the small Hankel operator Γ_ϕ with symbol ϕ is the operator densely defined in L_a^2 by

$$\Gamma_\phi f = P(\phi U f), \qquad f \in H^\infty(\Omega).$$

If Γ_ϕ as so defined is bounded related to the L^2 norm, it has a unique extension to a bounded operator on $L_a^2(\Omega)$ (since $H^\infty(\Omega)$ is dense in $L_a^2(\Omega)$) in $L_a^2(\Omega)$ if Γ_ϕ extends to a bounded operator on $L_a^2(\Omega)$.

In this paper, we consider bounded small Hankel operators possibly with unbounded symbols. It is easy to verify that a bounded small Hankel operator Γ is completely characterized by the algebraic equation

$$T_{z_i}^* \Gamma = \Gamma T_{z_i}, \qquad i = 1, 2, \ldots, n.$$

In [Zhu1], a reduced Hankel operator $\tilde{H}_\phi$ with symbol ϕ is defined by

$$\tilde{H}_\phi \colon L_a^2 \to \overline{L_a^2}, \tilde{H}_\phi h = \bar{P}(\phi h), \qquad h \in H^\infty(\Omega),$$

where $\bar{P}$ denotes the orthogonal projection from L^2 onto $\overline{L_a^2}$ (the complex conjugate of L_a^2). Similarly to the case of the unit ball B_n [Guo4], the small Hankel operator Γ_ϕ and the reduced Hankel operator $\tilde{H}_\phi$ are connected by the relation

$$\Gamma_\phi = U\tilde{H}_{\hat{\phi}}.$$

Let $BC(\Omega)$ be the space of bounded continuous functions on Ω, $C(\bar{\Omega})$ be the space of continuous functions on the closure $\bar{\Omega}$ of Ω, and $C_0(\Omega)$ be the space of the continuous functions on $\bar{\Omega}$ that vanish on the topological boundary $\partial\Omega$ of Ω. By the relation between small Hankel operators and reduced Hankel operators [Zhu1], one easily gets the following the elementary properties of small Hankel operators.

PROPOSITION 2.1. *Let ϕ be in $L^2(\Omega)$. Then*

1. *Γ_ϕ is bounded if and only if $\phi \in L^\infty(\Omega) + L_a^{2\perp}$, which happens if and only if $\phi \in BC(\Omega) + L_a^{2\perp}$;*
2. *Γ_ϕ is compact if and only if $\phi \in C(\bar{\Omega}) + L_a^{2\perp}$, which happens if and only if $\phi \in C_0(\Omega) + L_a^{2\perp}$;*
3. *$\Gamma_\phi^* = \Gamma_{\hat{\bar{\phi}}}$;*
4. *if f is a bounded analytic function on Ω, then $T_f^* \Gamma_\phi = \Gamma_\phi T_{\hat{\bar{f}}}$.*

Let $\beta(\Omega)$ be the Bloch space of Ω, and $B_0(\Omega)$ the little Bloch space of Ω(defined in [Tim]). Also note that a small Hankel operator Γ_ϕ depends only on the analytic part of ϕ, that is, $\Gamma_\phi = \Gamma_{P\phi}$. Therefore when discussing small Hankel operators, one often assumes that its symbol is analytic. Let ϕ be analytic. From [Zhul], we see that Γ_ϕ is bounded if $\phi \in \mathscr{B}(\Omega)$, and Γ_ϕ is compact if $\phi \in \mathscr{B}_0(\Omega)$. In particular, if $rank(\Omega) = 1$, then Γ_ϕ is bounded if and only if $\phi \in \mathscr{B}(\Omega\}$, and Γ_ϕ is compact if and only if $\phi \in \mathscr{B}_0(\Omega)$.

For $f, g \in L^2(\Omega)$, define the rank one operator $f \otimes g$ by

$$(f \otimes g)\, h = \langle h, g\rangle f$$

for $h \in L^2(\Omega)$. For $\lambda \in \Omega$, define the unitary operator U_λ on $L^2(\Omega)$ by

$$U_\lambda f = (f \circ \phi_\lambda)\, k_\lambda.$$

From [BCZ], U_λ has the following properties:

1. $U_\lambda^2 = I$;
2. $U_\lambda P = PU_\lambda$.

For a bounded operator S on L_a^2, we define the Berezin transform $\tilde{S}$ of S by

$$\tilde{S}(\lambda) = \langle Sk_\lambda, k_\lambda \rangle, \qquad \lambda \in \Omega.$$

In the case of the unit disk D, the following proposition has appeared in [SZh1, SZh2].

Proposition 2.2. *Let Ω be a bounded symmetric domain in its standard realization. Then there exist polynomials p_i, q_i, $i = 1, 2, \ldots, m$ such that*

$$k_\lambda \otimes k_\lambda = \sum_{i=1}^{m} T_{p_i \circ \phi_\lambda} T_{\overline{q_i \circ \phi_\lambda}}.$$

Proof. Since Ω is a bounded symmetric domain in its standard realization, from [Hua, Ko], we see that there exist polynomials p_i, q_i, $i = 1, 2, \ldots, m$ such that

$$K_\lambda(\lambda) = K(\lambda, \lambda) = 1 \Big/ \sum_{i=1}^{m} p_i(\lambda)\, \overline{q_i(\lambda)}.$$

An easy calculation gives

$$\begin{aligned} \langle (1 \otimes 1)\, k_\lambda, k_\lambda \rangle &= |\langle 1, k_\lambda \rangle|^2 = 1/K_\lambda(\lambda) \\ &= \sum_{i=1}^{m} p_i(\lambda)\, \overline{q_i(\lambda)} \\ &= \left\langle \left(\sum_{i=1}^{m} T_{p_i} T_{\overline{q_i}} \right) k_\lambda, k_\lambda \right\rangle. \end{aligned}$$

Note that a bounded linear operator is completely determined by its Berezin transform, that is, $S_1 = S_2$ if and only if $\tilde{S}_1 = \tilde{S}_2$. Set $S = \sum_{i=1}^{m} T_{p_i} T_{\overline{q_i}}$. Then

$$1 \otimes 1 = \sum_{i=1}^{m} T_{p_i} T_{\overline{q_i}}.$$

So applying the unitary operator U_λ to the above equation leads to

$$\begin{aligned} k_\lambda \otimes k_\lambda &= (U_\lambda 1) \otimes (U_\lambda 1) \\ &= U_\lambda (1 \otimes 1)\, U_\lambda^* \\ &= U_\lambda \left(\sum_{i=1}^{m} T_{p_i} T_{\overline{q_i}} \right) U_\lambda^* \\ &= \sum_{i=1}^{m} T_{p_i \circ \phi_\lambda} T_{\overline{q_i \circ \phi_\lambda}} \end{aligned}$$

as desired.

COROLLARY 2.3. *The w^*-closure of the linear manifold $\{\sum_{i=1}^{k} T_{f_i} T_{\overline{g_i}} : f_i, g_i \in H^\infty(\Omega)\}$ equals $B(L_a^2)$, where $B(L_a^2)$ is the set of all bounded linear operators on L_a^2.*

Proof. By Proposition 2.2, we need only to prove that the linear manifold $\{\sum_{i=1}^{k} c_i k_{\lambda_i} \otimes k_{\lambda_i} : \lambda_i \in \Omega\}$ is w^*-dense in the space $B(L_a^2)$ of bounded operators on L_a^2. Let S be in the trace class on L_a^2. Note that

$$\begin{aligned} \operatorname{tr}(S(k_\lambda \otimes k_\lambda)) &= \operatorname{tr}(sk_\lambda \otimes k_\lambda) \\ &= \langle Sk_\lambda, k_\lambda \rangle \\ &= \tilde{S}(\lambda). \end{aligned}$$

Since a bounded linear operator is completely determined by its Berezin transform, the above equation implies that the linear manifold $\{\sum_{k=1}^{k} c_i k_{\lambda_i} \otimes k_{\lambda_i} : \lambda_i \in \Omega\}$ is w^*-dense in $B(L_a^2)$. This gives the desired conclusion.

In the next theorem we characterize when a finite sum of the products of small Hankel operators is zero.

THEOREM 2.4. *Let $\phi_1, \psi_1, \ldots, \phi_n, \psi_n$ be analytic functions. Then the following are equivalent:*

1. $\sum_{k=1}^{n} \Gamma_{\phi_k} \Gamma_{\psi_k} = 0$;
2. $\sum_{k=1}^{n} \psi_k \otimes \widehat{\overline{\phi_k}} = 0$;
3. $\sum_{k=1}^{n} \phi_k(z)\, \psi_k(\bar{z}) = 0$, *for all* $z \in \Omega$;
4. $\sum_{k=1}^{n} \phi_k(z)\, \psi_k(w) = 0$, *for all* $z, w \in \Omega$.

Proof. For $h_1, h_2 \in H^\infty(\Omega)$, we have

$$\begin{aligned} \sum_{k=1}^{n} \langle \Gamma_{\phi_k} \Gamma_{\psi_k} \widehat{\overline{h_1}}, \widehat{\overline{h_2}} \rangle &= \sum_{k=1}^{n} \langle \Gamma_{\psi_k} \widehat{\overline{h_1}}, \Gamma_{\widehat{\overline{\phi_k}}} \widehat{\overline{h_2}} \rangle \\ &= \sum_{k=1}^{n} \langle T_{\overline{h_1}} \psi_k, T_{\overline{h_2}} \widehat{\overline{\phi_k}} \rangle \\ &= \sum_{k=1}^{n} \langle T_{h_2} T_{\overline{h_1}} \psi_k, \widehat{\overline{\phi_k}} \rangle \\ &= \operatorname{tr}\left(T_{h_2} T_{\overline{h_1}} \sum_{k=1}^{n} \psi_i \otimes \widehat{\overline{\phi_k}} \right). \end{aligned}$$

By Corollary 2.3, we see that

$$\sum_{k=1}^{n} \Gamma_{\phi_k} \Gamma_{\psi_k} = 0 \qquad \text{if and only if} \qquad \sum_{k=1}^{n} \psi_k \otimes \widehat{\overline{\phi_k}} = 0.$$

Set $A = \sum_{k=1}^{n} \psi_k \otimes \widehat{\overline{\phi_k}}$. Note that $A = 0$ if and only if

$$\langle Ak_z, k_w \rangle = 0, \qquad \text{for all} \quad z, w \in \Omega.$$

Thus the above equality holds if and only if

$$\tilde{A}(z) = \langle Ak_z, k_z \rangle = 0, \qquad \text{for all} \quad z \in \Omega.$$

Since

$$\langle Ak_{\bar{z}}, k_w \rangle = \frac{1}{\|K_{\bar{z}}\| \, \|K_w\|} \sum_{k=1}^{n} \phi_k(z) \, \psi_k(w)$$

and

$$\langle Ak_{\bar{z}}, k_{\bar{z}} \rangle = \frac{1}{\|K_{\bar{z}}\| \, \|K_{\bar{z}}\|} \sum_{k=1}^{n} \phi_k(z) \, \psi_k(\bar{z}),$$

we obtain the desired conclusion, completing the proof of the theorem.

Remark 2.5. In the Hardy space $H^2(D)$, Gu and Zheng [GZh] discussed when a finite sum of products of (big) Hankel operators is zero, for which they obtained an algebraic condition. In the context of the Hardy space, using the methods in this paper, one can obtain an analogue of Theorem 2.4.

For the product $H_f^* H_g$ of two Hankel operators on the Hardy space $H^2(D)$, Brown and Halmos [BH] proved that $H_f^* H_g$ is zero if and only if either H_f or H_g is zero. By Theorem 2.4, on the Bergman space we have

COROLLARY 2.6. *If $\Gamma_{\phi_1} \Gamma_{\phi_2} = 0$, then either Γ_{ϕ_1} or Γ_{ϕ_2} is zero.*

The following example shows that on the Bergman space $L_a^2(D)$ of the unit disk D, the product of three small Hanked operators can equal zero even if none of them is zero. The example is

$$\Gamma_z \Gamma_{z^3} \Gamma_z = 0.$$

On the Bergman space $L_a^2(\Omega)$ of the bounded symmetric domain Ω in $C^n (n > 1)$, it is not difficult to verify that

$$\Gamma_{z_1} \Gamma_{z_2} \Gamma_{z_1} = 0.$$

For more about products of Hankel operators on the Hardy space $H^2(D)$, see [XZ1, XZ2].

From Theorem 2.4 we have the following corollaries.

COROLLARY 2.7. *Let ϕ_1, ϕ_2 be analytic and not zero. Then $\Gamma_{\phi_1} \Gamma_{\phi_2} = \Gamma_{\phi_2} \Gamma_{\phi_1}$ if and only if there are constants c_1 and c_2, not both zero, such that $c_1 \phi_1 + c_2 \phi_2 = 0$.*

Proof. It is easy to see that the sufficiency is obvious. For the necessity, take w_0 such that both $\phi_1(w_0)$ and $\phi_2(w_0)$ are not zero. Then (3) of Theorem 2.4 gives the desired conclusion.

COROLLARY 2.8. *Let ϕ be a analytic function. Then Γ_ϕ is normal if and only if there are constants c_1 and c_2, not both zero, such that $c_1 \hat{\phi} + c_2 \bar{\phi} = 0$.*

On the Bergman space L_a^2, one easily verifies that

$$\Gamma_{k_\lambda} = k_\lambda \otimes K_{\bar{\lambda}},$$

and hence

$$\Gamma_{k_\lambda} \Gamma_{k_\lambda} = k_\lambda(\bar{\lambda})\, \Gamma_{k_\lambda}.$$

The next theorem shows that the product of two small Hankel operators equals a small Hankel operator only in the above case. On the Hardy space of the unit circle Yoshino obtained the analogous result in [Yos].

THEOREM 2.9. *Let ϕ_1, ϕ_2 be analytic functions, and neither the zero function. If $\Gamma_{\phi_1} \Gamma_{\phi_2} = \Gamma_\phi$, then there exist some $\lambda \in \Omega$ and constants c_1, c_2 such that $\phi_1 = c_1 k_\lambda$, and $\phi_2 = c_2 k_\lambda$.*

Proof. By the identity

$$1 \otimes 1 = \sum_{i=1}^{m} T_{p_i} T_{\overline{q_i}},$$

we have that

$$\phi_1 \otimes \widehat{\phi_2} = \Gamma_{\phi_1}(1 \otimes 1)\,\Gamma_{\phi_2} = \sum_{i=1}^{m} T_{\hat{p}_i}\Gamma_{\phi_1}\Gamma_{\phi_2}T_{\widehat{\phi_k}}.$$

Since

$$\Gamma_{\phi_1}\Gamma_{\phi_2} = \Gamma_{\phi},$$

we see that

$$\begin{aligned}\phi_1 \otimes \widehat{\phi_2} &= \sum_{i=1}^{m} T_{\hat{p}_i}\Gamma_{\phi}T_{\widehat{q_i}}\\ &= \sum_{i=1}^{m} \Gamma_{\phi}T_{p_i}T_{\widehat{q_i}}\\ &= \Gamma_{\phi}T_{\sum_{i=1}^{m} p_i\widehat{q_i}}\\ &= \Gamma_{\phi\sum_{i=1}^{m}\hat{p}_i\bar{q}_i}.\end{aligned}$$

Applaying the operators in the above equalities to the function 1, we get

$$\Gamma_{\phi\sum_{i=1}^{m}\hat{p}_i\bar{q}_i}1 = P\left(\phi\sum_{i=1}^{m}\hat{p}_i\bar{q}_i\right) = \phi_2(0)\,\phi_1.$$

Thus

$$\phi_1 \otimes \widehat{\phi_2} = \phi_2(0)\,\Gamma_{\phi_1}.$$

Since $\phi_i \neq 0$ for $i = 1, 2,$ we see that $\phi_2(0) \neq 0$. Note that the kernels of small Hankel operators are invariant subspaces of all the coordinate functions. By the equality

$$\phi_1 \otimes \widehat{\phi_2} = \phi_2(0)\,\Gamma_{\phi_1},$$

we see that *ker* Γ_{ϕ_1} is an invariant subspace of codimension 1. Therefore there exist some $\lambda \in \Omega$ and constant c such that

$$\widehat{\phi_2} = ck_{\lambda}, \quad \text{i.e.} \quad \phi_2 = \bar{c}k_{\bar{\lambda}}.$$

Similarly using the equality

$$\Gamma_{\widehat{\phi_2}}\Gamma_{\widehat{\phi_1}} = \Gamma_{\hat{\bar{\phi}}},$$

we have that

$$\widehat{\overline{\phi_2}} \otimes \phi_1 = \overline{\phi_1(0)} \Gamma_{\widehat{\overline{\phi_2}}}.$$

Conjugation of the above equality gives

$$\phi_1 \otimes \widehat{\overline{\phi_2}} = \phi_1(0)\, \Gamma_{\phi_2},$$

and hence

$$\phi_2(0)\, \Gamma_{\phi_1} = \phi_1(0)\, \Gamma_{\phi_2}.$$

So,

$$\phi_1 = \frac{\phi_1(0)}{\phi_2(0)}\, \phi_2.$$

This ensures that

$$\phi_1 = \frac{\bar{c}\phi_1(0)}{\phi_2(0)}\, k_{\bar{\lambda}},$$

completing the proof. Q.E.D.

3. FINITE CODIMENSIONAL INVARIANT SUBSPACES OF BERGMAN SPACES

For a closed subspace M of L_a^2, we say that M is invariant if it is invariant under multiplication by all the coordinate functions. To study Hankel operators (in the next section), our interest and results require us to study the structure of $M^\perp$ when M is of finite codimension.

Let Ω be a bounded symmetric domain in C^n (in its standard realization) and let *Aut*(Ω) be the automorphism group of Ω (all biholomorphic mappings of Ω onto Ω).

LEMMA 3.1. *For each point $\lambda \in \partial\Omega$, there exists no positive constant c, such that*

$$|p(\lambda)| \leqslant c\, \|p\|_2,$$

for each polynomial p.

Proof. Assume that there exists a positive constant c such that the estimate in Lemma 3.1 holds. Then the same estimate holds for any $f \in A(\Omega)$, where $A(\Omega)$ is the so-called Ω-algebra consists of all functions continuous on the closure Ω of Ω, and analytic on Ω.

Note that $\partial\Omega$ is the disjoint union of the $Aut(\Omega)$-orbits $\partial_j\Omega(=Aut(\Omega)\, u_j)$, $j=1, ..., r$, where u_j are tripotents of rank j [Loo].

Let $\lambda \in \partial\Omega$. We may assume that $\lambda \in \partial_j\Omega$ for some $1 \leqslant j \leqslant r$. By the K-invariance we can assume that

$$\lambda = u_j + w,$$

where w is in Ω and orthogonal to u_j (in the triple product sense). Let $0 < t < 1$, define

$$f_t = K_{tu_j}.$$

Clearly, the family functions f_t are in $A(\Omega)$. An easy calculation gives

$$f_t(\lambda) = (1-t)^{-jp},$$

where p is the genus of Ω, and

$$\|f_t\|_2 = (1-t^2)^{-jp/2}.$$

Hence

$$\frac{f_t(\lambda)}{\|f_t\|_2} = \left(\frac{1+t}{1-t}\right)^{jp/2}$$

is not bounded as $t \to 1$. So there is no constant c such that

$$|f(\lambda)| \leqslant c\, \|f\|_2$$

for $f \in A(\Omega)$.

We thank the referee for pointing out the above proof of Lemma 3.1.

Let $\mathscr{C}$ denote the ring $C[z_1, z_2, ..., z_n]$ of all polynomials on C^n. We say that $\lambda \in C^n$ is a *virtual point* of L^2_a provided the homomorphism

$$p \to p(\lambda)$$

defined on $\mathscr{C}$, extends to a bounded linear functional on L^2_a. From Lemma 3.1, we see that the set of all virtual points of L^2_a is exactly Ω. Thus, in the language in [Guo1], L^2_a is an analytic Hilbert module on Ω So we view every invariant subspace of L^2_a to be a submodule of L^2_a. We will use the Hilbert module theory to study invariant subspaces of L^2_a.

Let M be a finite codimensional invariant subspace of L_a^2. Then by [Guo1, Th. 4.1] or [DPSY, Cor. 2.8], the subspace M has only finitely many zero points $\lambda_1, \lambda_2, ..., \lambda_l$ in Ω, such that M can be uniquely represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where each M_i is a finite codimensional invariant subspace having a unique zero λ_i.

Let M be a submodule of L_a^2 and let $\lambda \in \Omega$, define

$$M_\lambda = \{q \in \mathscr{C} : [q(D) f](\lambda) = 0, \forall f \in M\};$$

here $q(D)$ denotes the differential operator $q(\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n})$ if q is the polynomial $q(z_1, ..., z_n)$. M_λ is called *the characteristic space* of M at λ. For more details about the theory of characteristic spaces, see [Guo1], [Guo2], and [Guo3].

The functions z^α, α ranging over all nonnegative multi-indices, are orthogonal in L_a^2 and the uniqueness of the Taylor expansion implies that $\{z^\alpha : \alpha \geqslant 0\}$ is a complete set [FaK, Upm]. Thus $\{z^\alpha / \|z^\alpha\|_2 : \alpha \geqslant 0\}$ is an orthogonal basis for L_a^2. This implies that for any polynomial p, the Toeplitz operator $T_{\bar{p}}$ maps $\mathscr{C}$ to $\mathscr{C}$. Now let $\mathscr{P}$ be a space consisting of polynomials. We say that $\mathscr{P}$ is an *invariant polynomial space*, if for any polynomial p, $\mathscr{P}$ is invariant under the action of $T_{\bar{p}}$. It is easy to see that in the case $n = 1$, an invariant polynomial space with the dimension m $(1 \leqslant m \leqslant \infty)$ is the linear space with the basis $\{1, z^1, ..., z^m\}$.

LEMMA 3.2. *Let M be a finite codimensional invariant subspace with a unique zero $\lambda = 0$. Then $M^\perp$ is a finite dimensional invariant polynomial space.*

Proof. Since M is an invariant subspace of L_a^2, M is also invariant under the action of the Toeplitz operator T_p for a polynomial p. Hence $M^\perp$ is invariant under the action of $T_{\bar{p}}$.

To prove the Lemma 3.2, we need the characteristic space theory [Guo1, Guo2]. Let M_0 be the characteristic space of M at $\lambda = 0$. Since $\{z^\alpha / \|z^\alpha\|_2 : \alpha \geqslant 0\}$ is an orthogonal basis for L_a^2, the Taylor expansion gives that

$$[\partial^\alpha f](0) = \frac{\alpha!}{\|z^\alpha\|_2^2} \langle f, z^\alpha \rangle,$$

for each $f \in L_a^2$, where $\alpha! = \prod_{i=1}^{n} \alpha_i!$. Let $p(z) = \sum a_\alpha z^\alpha$ be a polynomial. Then

$$[p(D)\,f](0) = \sum a_\alpha[\partial^\alpha f](0)$$
$$= \sum a_\alpha \frac{\alpha!}{\|z^\alpha\|_2^2}\langle f, z^\alpha\rangle$$
$$= \left\langle f, \sum \bar{a}_\alpha \frac{\alpha!}{\|z^\alpha\|_2^2} z^\alpha \right\rangle.$$

Define the conjugate linear map $\gamma\colon \mathscr{C} \to \mathscr{C}$ by

$$\gamma\left(\sum a_\alpha z^\alpha\right) = \sum \bar{a}_\alpha \frac{\alpha!}{\|z^\alpha\|_2^2} z^\alpha.$$

It is easy to verify that γ is one to one, and onto. Thus the image of M_0 under the conjugate linear operator γ is a subspace of $M^\perp$. By [Guo2],

$$\operatorname{codim} M = \dim M_0.$$

So we get that $M^\perp = \gamma(M_0)$. Hence $M^\perp$ is a finite dimensional invariant polynomial space, completing the proof. Q.E.D.

Let M be a finite codimensional invariant subspace of L_a^2. Then M has finitely many zero points $\lambda_1, \lambda_2, \ldots, \lambda_l$ in Ω such that M can be uniquely represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where M_i is a finite codimensional invariant subspace, and has a unique zero λ_i.

Let λ be a point in Ω. For each $f \in L_a^2$, define

$$U_\lambda f = f \circ \phi_\lambda k_\lambda,$$

where ϕ_λ is an element in *Aut*(Ω) such that $\phi_\lambda(0) = \lambda$ and $\phi_\lambda(\lambda) = 0$. Note that $\det[\phi'_\lambda(z)] = (-1)^n k_\lambda(z)$. Hence U_λ is a unitary operator from L_a^2 to L_a^2.

THEOREM 3.3. *Under the above assumption, there are invariant polynomial spaces $\mathscr{P}_i$, $i = 1, 2, \ldots, l$ such that*

$$M^\perp = \sum_{i=1}^{l} \mathscr{P}_i \circ \phi_{\lambda_i} k_{\lambda_i}.$$

Proof. Let N be an invariant subspace. We claim that $U_\lambda N = \{f \circ \phi_\lambda k_\lambda : f \in N\}$ is also invariant. In fact, by the identity $k_\lambda k_\lambda \circ \phi_\lambda = 1$ [BCZ], we get

$$U_\lambda N = \left\{\frac{f}{k_\lambda} \circ \phi_\lambda : f \in N\right\} = \{f \circ \phi_\lambda : f \in N\}.$$

From the equation

$$\phi_\lambda \circ \phi_\lambda(z) = z,$$

we see that each coordinate function $z_i = \phi_\lambda^{(i)} \circ \phi_\lambda(z)$, where $\phi_\lambda^{(i)}$ is the ith argument of ϕ_λ. This ensures that $U_\lambda N$ is invariant under the multiplication by all polynomials, and hence $U_\lambda N$ is invariant. Now assume that N is an invariant subspace of finite codimension with a unique zero point λ. Since

$$L_a^2 = N \oplus N^\perp = U_\lambda N \oplus U_\lambda N^\perp,$$

the invariant subspace $U_\lambda N$ is of the same codimension as N. Note that $U_\lambda N$ has only the zero point 0. Thus from Lemma 3.2, $U_\lambda N^\perp$ is a finite dimensional invariant polynomial space. We denote $U_\lambda N^\perp$ by $\mathscr{P}$ so,

$$N^\perp = U_\lambda \mathscr{P} = \mathscr{P} \circ \phi_\lambda k_\lambda.$$

Since $M = \bigcap_{i=1}^{l} M_i$, we have that

$$M^\perp = \sum_{i=1}^{k} M_i^\perp.$$

Thus there are finite dimensional invariant polynomial spaces $\mathscr{P}_i$, $i = 1, 2, \ldots, l$ such that

$$M^\perp = \sum_{i=1}^{l} \mathscr{P}_i \circ \phi_{\lambda_i} k_{\lambda_i}.$$

This completes the proof of the theorem. Q.E.D.

Combining the above theorem with the Grothendieck theorem [Gro], we get the following corollary. We thank D. Sarason for his pointing out the Grothendieck theorem and the Grothendieck paper [Gro].

COROLLARY 3.4. *Let M be an invariant subspace. Then M is of finite codimension if and only if $M^\perp \subset A(\Omega)$.*

Proof. The necessity is given by Theorem 3.3. Now by that assumption $M^\perp \subset A(\Omega)$, the Grothendieck theorem implies that every infinite dimensional subspace of L^2 contains unbounded functions [Gro]. Thus $M^\perp$ is of finite dimension. Q.E.D.

In the cases of the unit ball B_n and the unit polydisk D^n, their automorphisms and reproducing kernels are rational functions. The same reasoning shows the following result.

COROLLARY 3.5. *Let Ω be the unit ball B_n or the unit polydisk D^n, and let M be an invariant subspace of $L_a^2(\Omega)$. Then M is of finite codimension if and only if $M^\perp$ consists of rational functions.*

4. FINITE RANK SMALL HANKEL OPERATORS ON BERGMAN SPACES

Kronecker's well-known result on finite rank Hankel operator on the Hardy space $H^2(D)$ is as follows: if f is a analytic function, then $H_{\bar{f}}$ is of finite rank if and only if f is a rational function. On the multivariable Hardy spaces, there have been several generalizations of this result, see [Gu, Pow] and references there. Note that on the Bergman space L_a^2, each nonzero (big) Hankel operator H_ϕ (by definition, $H_\phi \colon L_a^2 \to L_a^{2\perp}$ is given by $H_\phi f = (I-P)(\phi f))$ has a trivial kernel. In fact, if there exists a nonzero f such that $H_\phi f = 0$, then ϕf is analytic, and hence ϕ is analytic. So, $H_\phi = 0$. This implies that there exist no nonzero (big) Hankel operators that are of finite rank. However, there are lots of small Hankel operators with finite rank. For example $\Gamma_{k_\lambda} = k_\lambda \otimes K_{\bar{\lambda}}$ with $\lambda \in \Omega$. In this section we will completely characterize small Hankel operators of finite rank on the Bergman space of bounded symmetric domains.

PROPOSITION 4.1. *On the bounded symmetric domain Ω,*

$$U_\lambda \Gamma_\psi = \Gamma_{(k_\lambda/\bar{k}_\lambda)\,\psi \circ \phi_\lambda} U_{\bar{\lambda}}.$$

Proof. Note that for each $f \in L_a^2$. We have

$$\begin{aligned} U_\lambda \Gamma_\psi f &= U_\lambda P(\psi \hat{f}) = P U_\lambda (\psi \hat{f}) \\ &= P(\psi \circ \phi_\lambda \hat{f} \circ \phi_\lambda k_\lambda) \\ &= P\left(\frac{k_\lambda}{\bar{k}_\lambda} \psi \circ \phi_\lambda \widehat{f \circ \phi_{\bar{\lambda}}}\, \bar{k}_\lambda\right) \\ &= P\left(\frac{k_\lambda}{\bar{k}_\lambda} \psi \circ \phi_\lambda U U_{\bar{\lambda}} f\right) \\ &= \Gamma_{(k_\lambda/\bar{k}_\lambda)\,\psi \circ \phi_\lambda} U_{\bar{\lambda}} f, \end{aligned}$$

as desired. This completes the proof of the proposition. Q.E.D.

THEOREM 4.2. *Let ψ be analytic. Then Γ_ψ is of finite rank if and only if there exist $\lambda_1, \lambda_2, ..., \lambda_l$ in Ω, and polynomials $p_1, p_2, ..., p_l$ such that*

$$\psi = \sum_{i=1}^{l} p_i \circ \phi_{\lambda_i} k_{\lambda_i}.$$

Proof. First we assume that Γ_ψ is of finite rank. Then $M = \ker \Gamma_\psi$ is a finite codimensional invariant subspace. Note that

$$\Gamma_\psi M = 0 \qquad \text{if and only if} \qquad \langle \psi \hat{M}, L_a^2 \rangle = 0,$$

where $\hat{M} = \{\hat{h}: h \in M\}$. The above equality holds if and only if

$$\langle \psi, \hat{\bar{M}} \rangle = 0;$$

here $\hat{\bar{M}} = \{\hat{\bar{h}}: h \in M\}$. Since $\hat{\bar{M}}$ is an invariant subspace of finite codimension, Theorem 3.3 ensures the desired conclusion.

Now using the identity $k_\lambda \circ \phi_\lambda k_\lambda = 1$, and the equations $\hat{\phi}_{\bar{\lambda}} = \bar{\phi}_\lambda$ and $\hat{k}_{\bar{\lambda}} = \bar{k}_\lambda$, for each nonnegative multi-index α, Proposition 4.1 implies that

$$\begin{aligned} U_\lambda \Gamma_{z^\alpha \circ \phi_\lambda k_\lambda} f &= \Gamma_{z^\alpha k_\lambda \circ \phi_\lambda (k_\lambda / \bar{k}_\lambda)} U_{\bar{\lambda}} f \\ &= P\left(\frac{z^\alpha}{\bar{k}_\lambda} \widehat{f \circ \phi_{\bar{\phi}}\, k_{\bar{\lambda}}}\right) \\ &= P(z^\alpha \widehat{f \circ \phi_{\bar{\phi}}}) \\ &= \Gamma_{z^\alpha} f \circ \phi_{\bar{\lambda}} \\ &= \Gamma_{z^\alpha} C_{\phi_{\hat{\lambda}}} f. \end{aligned}$$

Note that the composition operator $C_{\phi_{\hat{\lambda}}}$ is bounded, and Γ_{z^α} is of finite rank. So, $\Gamma_{z^\alpha \circ \phi_\lambda k_\lambda}$ is of finite rank and hence for each polynomial p, $\Gamma_{p \circ \phi_\lambda k_\lambda}$ is of finite rank. The desired conclusion follows. Q.E.D.

In the case of the unit ball B_n and the unit polydisk D^n, their automorphisms and reproducing kernels are rational functions. Thus from Theorem 4.2, we have the following result.

COROLLARY 4.3. *Let Ω be the unit ball B_n or the unit polydisk D^n, and let ψ be analytic. If Γ_ψ is of finite rank, then ψ is a rational function.*

Remark 4.4. The converse of Corollary 4.3 is not true in general. For example, take $\psi = \frac{1}{1-0.5z_1}$; then Γ_ψ is not of finite rank.

5. QUASI-INVARIANT SUBSPACES OF THE SEGAL–BARGMANN SPACE

The Segal–Bargmann space, or the so-called Fock space, is the analogue of the Bergan space in the context of complex n-space C^n. It is a Hilbert space consisting of entire functions in C^n. Let

$$d\mu(z) = e^{-|z|^2/2}\, dv(z)(2\pi)^{-n}$$

be the Gaussian measure on C^n (dv is ordinary Lebesgue measure). The Segal–Bargmann space $L_a^2(C^n, d\mu)$ (for short, $L_a^2(C^n)$), by definition, is the space of all μ-square-integrable entire functions on C^n. It is easy to see that $L_a^2(C^n)$ is a closed subspace of $L^2(C^n)$ with the reproducing kernel functions $K_\lambda(z) = e^{\bar{\lambda}z/2}$, and the normalized reproducing kernel functions $k_\lambda(z) = e^{\bar{\lambda}z/2 - |\lambda|^2/4}$ (here $\bar{\lambda}z = \sum_{i=1}^n \bar{\lambda}_i z_i$).

By Liouville's theorem, there are no nonconstant bounded entire functions on C^n, and on $L_a^2(C^n)$, multiplication operators with analytic symbols are unbounded unless their symbols are constants. More general, we have:

PROPOSITION 5.1. *Let M be a (closed) subspace of $L_a^2(C^n)$, and $M \neq \{0\}$. If f is an entire function on C^n such that $fM \subset M$, then f is a constant.*

Proof. The closed graph theorem implies that the multiplication by f on M, denoted by M_f, is a bounded operator. Use $\tilde{K}_\lambda$ to denote the reproducing kernel functions associated with M, and $\tilde{k}_\lambda$ the normalized reproducing kernel functions. Since

$$\langle f\tilde{k}_\lambda, \tilde{k}_\lambda \rangle = f(\lambda),$$

we see that

$$|f(\lambda)| = |\langle f\tilde{k}_\lambda, \tilde{k}_\lambda \rangle| \leqslant \|M_f\|,$$

and hence f is a bounded entire function on C^n. So, f is a constant.

Q.E.D.

From Proposition 5.1, there exist no nontrivial invariant subspaces for all coordinate functions. Thus, an appropriate substitute for invariant subspace, the so-called *quasi-invariant* subspace is needed. Namely, a (closed) subspace M of the Segal–Bargmann space $L_a^2(C^n)$ is called quasi-invariant if the relation $pf \in L_a^2(C^n)$ implies $pf \in M$ for any $f \in M$ and any polynomial p. Equivalently, a closed subspace M is quasi-invariant if and only if $pM \cap L_a^2(C^n) \subset M$ for each polynomial p.

It is difficult to characterize quasi-invariant subspaces completely. However, using the characteristic space theory developed in [Guo1, Guo2, Guo3], we can characterize finite codimensional quasi-invariant subspaces.

First let us note that along this line, in the case of the Hardy space $H^2(D^n)$, Ahern and Clark [AC] proved that there exists the bijective correspondence between the invariant subspaces of the Hardy space $H^2(D^n)$ of finite codimension on one hand, and the ideals in the polynomial ring $\mathscr{C}$ of finite codimension whose zero sets are contained in polydisk, D^n, on the other. The extension of the above results to general bounded domains in C^n was considered by Axler, Agrawal, Bourdon, Douglas, Paulsen, Putinar, and Salinas; see [AB, AS, ACD, DPSY, Pu] and references there.

Recall that $\mathscr{C}$ is the ring of polynomials in C^n. Let I be an ideal of $\mathscr{C}$. We use $Z(I)$ to denote the zero variety of I:

$$Z(I) = \{\lambda \in C^n : q(\lambda) = 0, \forall q \in I\}.$$

The characteristic space of I at $\lambda \in C^n$ is defined by

$$I_\lambda = \{q \in \mathscr{C} : [q(D)\, f](\lambda) = 0, \forall f \in I\}.$$

The envelope of I at λ is defined by

$$I_\lambda^e = \{q \in \mathscr{C} : [p(D)\, q](\lambda) = 0, \forall p \in I_\lambda\}.$$

It was proved in [Guo1] that the ideal of $\mathscr{C}$ is completely determined by its characteristic spaces on a characteristic set. More precisely, in [Guo1] Guo proved

$$I = \bigcap_{\lambda \in Z(I)} I_\lambda^e.$$

LEMMA 5.2. *Let I be an ideal in the polynomial ring $\mathscr{C}$, and let $[I]$ be the closure of I in $L_a^2(C^n)$. Then $[I] \cap \mathscr{C} = I$.*

Proof. Let B^n denote the unit ball in C^n. For a fixed λ in the zero variety $Z(I)$ of I, we may assume that $|\lambda| < r$ for some positive constant r. Now

$$f \to [\partial^\alpha f](\lambda)$$

are bounded linear functionals on the Bergman space $L_a^2(rB^n)$, for all index α. For each $p \in [I] \cap \mathscr{C}$, there are $\{p_n\} \subset I$ such that

$$\|p - p_n\|_2 \to 0$$

as $n \to \infty$. Note that for each $q \in I_\lambda$, there is a positive constant C_q,

$$\begin{aligned} |[q(D)\, p](\lambda)| &= |[q(D)(p - p_n)](\lambda)| \\ &\leqslant C_q \, \|p - p_n\|_{L_a^2(rB_n)} \leqslant C_q \, \|p - p_n\|_2 \to 0, \end{aligned}$$

we get

$$[q(D)\, p](\lambda) = 0.$$

Thus,

$$[q(D)\, p](\lambda) = 0, \forall q \in I_\lambda.$$

Hence p is in the envelope I_λ^e of I at λ. By [Guo1, Th. 2.1],

$$I = \bigcap_{\lambda \in Z(I)} I_\lambda^e.$$

Thus we get that $p \in I$, and hence $[I] \cap \mathscr{C} \subset I$. Obviously, $I \subset [I] \cap \mathscr{C}$. The desired conclusion follows. Q.E.D.

By Lemma 5.2, for each ideal I, we can establish a canonical linear map

$$\tau: \mathscr{C}/I \to L_a^2(C^n)/[I]$$

by $\tau(p+I) = p + [I]$.

Lemma 5.3. *Let I be an ideal of finite codimension. Then $[I]$ is a quasi-invariant subspace of $L_a^2(C^n)$ of finite codimension. Furthermore, the canonical map*

$$\tau = \mathscr{C}/I \to L_a^2(C^n)/[I]$$

is an isomorphism.

Proof. We express $\mathscr{C}$ as

$$\mathscr{C} = I \dotplus R,$$

where R is a linear space of polynomials with $\dim R = \dim \mathscr{C}/I$. Since the polynomial ring $\mathscr{C}$ is dense in $L_a^2(C^n)$, and $[I] + R$ is closed in $L_a^2(C^n)$, we have that

$$[I] + R = L_a^2(C^n).$$

By the equality

$$[I] \cap R = [I] \cap \mathscr{C} \cap R,$$

and Lemma 5.2, we get that

$$[I] \cap R = \{0\}.$$

So,

$$L_a^2(C^n)/[I] = ([I]+R)/[I] \simeq R/([I] \bigcap R)$$
$$= R/\{0\}$$
$$= R$$
$$\simeq \mathscr{C}/I.$$

It follows that

$$\tau: \mathscr{C}/I \to L_a^2(C^n)/[I]$$

is an isomorphism.

Suppose that $f \in [I]$, and p is a polynomial such that $pf \in L_a^2(C^n)$. We need to prove that $pf \in [I]$. Since $L_a^2(C^n) = [I] \dotplus R$, pf can be expressed as

$$pf = g + h,$$

where $g \in [I]$, and $h \in R$. Note that for each $\lambda \in Z(I)$, and any $q \in I_\lambda$,

$$[q(D)\, f](\lambda) = 0, \qquad \text{and} \qquad [q(D)\, g](\lambda) = 0.$$

Since I_λ is invariant under the action by the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, ..., \partial/\partial z_n\}$ [Guo1], it follows that

$$[q(D)\, pf](\lambda) = 0.$$

This implies that

$$[q(D)\, h](\lambda) = 0,$$

for each $\lambda \in Z(I)$ and any $q \in I_\lambda$. Thus h is in I_λ^e. Theorem 2.1 in [Guo1] implies that $h \in I$, and hence $h = 0$. So, $pf \in [I]$. We conclude that $[I]$ is quasi-invariant, completing the proof. Q.E.D.

LEMMA 5.4. *Let M be a quasi-invariant subspace of finite codimension in $L_a^2(C^n)$. Then $M \cap \mathscr{C}$ is an ideal of $\mathscr{C}$, and $M \cap \mathscr{C}$ is dense in M. Furthermore, the canonical map*

$$\tau': \mathscr{C}/M \cap \mathscr{C} \to L_a^2(C^n)/M$$

is an isomorphism, where $\tau'(p + M \cap \mathscr{C}) = p + M$.

Proof. Clearly, $M \cap \mathscr{C}$ is an ideal of $\mathscr{C}$ because M is quasi-invariant. It is easy to see that the map τ' is injective, and hence the ideal $M \cap \mathscr{C}$ is of finite codimension, and

$$\dim \mathscr{C}/M \cap \mathscr{C} \leqslant \dim L_a^2(C^n)/M.$$

By Lemma 5.3,

$$\dim \mathscr{C}/M \cap \mathscr{C} = \dim L_a^2(C^n)/[M \cap \mathscr{C}].$$

Since

$$[M \cap \mathscr{C}] \subset M,$$

we have

$$\dim L_a^2(C^n)/[M \cap \mathscr{C}] \geqslant \dim L_a^2(C^n)/M.$$

So,

$$\dim L_a^2(C^n)/[M \cap \mathscr{C}] = \dim L_a^2(C^n)/M.$$

This gives

$$[M \cap \mathscr{C}] = M.$$

Therefore, $M \cap \mathscr{C}$ is dense in M. From Lemma 5.3, we see the map τ' is an isomorphism. Q.E.D.

From Lemmas 5.2, 5.3, 5.4, we obtain an algebraic reduction theorem for finite codimension quasi-invariant subspaces.

THEOREM 5.5. *Let M be a quasi-invariant subspace of finite codimension. Then $\mathscr{C} \cap M$ is an ideal in the ring $\mathscr{C}$, and*

1. *$\mathscr{C} \cap M$ is dense in M;*
2. *the canonical map $\tau: \mathscr{C}/M \cap \mathscr{C} \to L_a^2(C^n)/M$ is an isomorphism, where $\tau(p + M \cap \mathscr{C}) = p + M$.*

Conversely, if I is an ideal in $\mathscr{C}$ of finite codimension, then $[I]$ is a quasi-invariant subspace of the same codimension and $[I] \cap \mathscr{C} = I$.

Remark 5.6. For bounded domains Ω in the complex plane, which satisfy certain technical hypotheses, Axler and Bourdon [AB] proved that each finite codimensional invariant subspace M has the form $M = pL_a^2$, where p is a polynomial with its zeros in Ω. Putinar [Pu] extended this result to some bounded pseudoconvex domains in C^n containing balls and polydisks. Namely, for such a domain Ω, Putinar proved that every finite codimension invariant subspace M has the form

$$M = \sum_{i=1}^{k} p_i L_a^2,$$

where p_i are polynomials having a finite number of common zero, all contained in Ω. However from Theorem 5.5, we see that a finite codimension quasi-invariant subspace need not have the above form. This may be an essential difference between analytic Hilbert spaces on bounded domains and those on unbounded domains.

Let M_1 and M_2 be quasi-invariant subspaces. We say that they are similar (unitarily equivalent) if there exists an invertible operator (a unitary operator) $A: M_1 \to M_2$ such that if $z_i f \in M_1$ with $f \in M_1$, then $A(z_i f) = z_i A(f)$. Thus, from the definition, the relation $z_i f \in M_1$ forces that $z_i A(f) \in M_2$. It is easy to check that similarity (unitary equivalence) is an equivalence relation in the set of all quasi-invariant subspaces.

There are two cases in studying similarity (unitary equivalence) of quasi-invariant subspaces.

Case 1. Under similarity (and under unitary equivalence), let us consider the equivalence classes of finite codimensional quasi-invariant subspaces of the Segal–Bargmann space $L_a^2(C)$ on the complex plane.

By Theorem 5.5, each finite codimensional quasi-invariant subspace M has the form

$$M = [I],$$

where I is a finite codimensional ideal with the same codimension as M. Note that on the complex plane C, every nonzero ideal I is principal; that is, there is a polynomial p such that $I = p\mathscr{C}$. It follows that each nonzero ideal I is of finite codimension. Therefore, on the Segal–Bargmann space of the complex plane, finite codimensional quasi-invariant subspaces are exactly the ideal $[p\mathscr{C}]$, where p range over all non-zero polynomials. Let $M = [p\mathscr{C}]$; it is easy to check that the codimension

$$\operatorname{codim} M = \dim L_a^2(C)/M = \deg p.$$

Theorem 5.7. *The quasi-invariant subspaces* $[p_1\mathscr{C}]$ *and* $[p_2\mathscr{C}]$ *are similar if and only if*

$$\deg p_1 = \deg p_2.$$

Hence, two finite codimensional quasi-invariant subspaces are similar if and only if they have the same codimension.

Remark 5.8. By Zhu's paper [Zhu3], all finite codimensional invariant subspaces of the Bergman space $L_a^2(D)$ are similar, and they are similar to $L_a^2(D)$. Therefore, Theorem 5.7 exhibits an important difference between Bergman spaces and the Segal–Bargmann space. Therefore, under similarity the equivalence classes of finite codimensional quasi-invariant subspaces of $L_a^2(C)$ exactly are

$$\{[\mathscr{C}], [z\mathscr{C}], ..., [z^n\mathscr{C}], ...\}.$$

Proof of Theorem 5.7. First assume that $\deg p_1 = \deg p_2$. For $f \in [p_1\mathscr{C}]$, it is easy to see that f/p_1 is an entire function, and hence $\frac{p_2}{p_1}f$ is an entire function on C. Since $\deg p_1 = \deg p_2$, there exists positive constants c_1, c_2 and r such that

$$c_1 \leqslant \frac{|p_2(z)|}{|p_1(z)|} \leqslant c_2$$

if $|z| \geqslant r$. This implies that $\frac{p_2}{p_1}f$ is in $L_a^2(C)$ for each $f \in [p_1\mathscr{C}]$. Hence we can define an operator A: $[p_1\mathscr{C}] \to L_a^2(C)$ by $Af = \frac{p_2}{p_1}f$ for $f \in [p_1\mathscr{C}]$. By a simple application of the closed graph theorem, the operator A is bonded. Since A maps $p_1\mathscr{C}$ onto $p_2\mathscr{C}$, this implies that A maps $[p_1\mathscr{C}]$ to $[p_2\mathscr{C}]$. So, $A: [p_1\mathscr{C}] \to [p_2\mathscr{C}]$ is a bounded operator. Clearly, $A(z_i f) = z_i A(f)$ if $z_i f \in [p_1\mathscr{C}]$ (here $f \in [p_1\mathscr{C}]$). Similarly we can show that the operator B: $[p_2\mathscr{C}] \to [p_1\mathscr{C}]$ defined by $Bf = \frac{p_1}{p_2} f$ for $f \in [p_2\mathscr{C}]$, is bounded. It is easy to see that

$$AB = \text{ the identity on } [p_2\mathscr{C}], \text{ and } BA = \text{ the identity on } [p_1\mathscr{C}].$$

We thus conclude that $[p_1\mathscr{C}$ and $[p_2\mathscr{C}]$ are similar if $\deg p_1 = \deg p_2$.

From the preceding proof, we need only to prove that $[z^m\mathscr{C}]$ and $[z^n\mathscr{C}]$ are not similar if $m \neq n$. We may assume that $m < n$, and hence $[z^m\mathscr{C}] \supset [z^n\mathscr{C}]$. Assume that there exists a similarity A: $[z^m\mathscr{C}] \to [z^n\mathscr{C}]$. Thus there is an entire function f on C such that

$$A(z^m) = z^n f.$$

So $A(h) = z^{n-m} f h$ for any $h \in [z^m\mathscr{C}]$. By Proposition 5.1, the entire function $z^{n-m} f$ is a constant. This is impossible, and hence $[z^m\mathscr{C}]$ and $[z^n\mathscr{C}]$ are not similar if $m \neq n$. This completes the proof of the theorem. Q.E.D.

Next we will show that finite codimensional quasi-invariant subspaces have strongly rigidity.

THEOREM 5.9. *Let M be quasi-invariant. If M is similar to a finite codimensional quasi-invariant subspace $[p\mathscr{C}]$, then there is a polynomial q with* $\deg q = \deg p$ *such that* $M = [q\mathscr{C}]$.

Remark 5.10. Combining Theorem 5.8 with Theorem 5.9, we see that if a quasi-invariant subspace is similar to a finite codimension quasi-invariant subspace $[p\mathscr{C}]$, then it is also a finite codimension quasi-invariant subspace with the codimension $\deg p$. In particular, the only quasi-invariant subspace that is similar to $L_a^2(C)$ is $L_a^2(C)$ itself.

To prove Theorem 5.9, the following lemma is needed. The proof of the lemma is left an exercise for readers.

LEMMA 5.11. *Let f be an entire function on C, and p a polynomial. If there exist a positive constant c and an r_0 such that*

$$\frac{|f(z)|}{|p(z)|} \leqslant c$$

for $|z| > r_0$, then f is a polynomial, and $\deg f \leqslant \deg p$.

Proof of Theorem 5.9. Let $\deg p = n$. Therefore by Theorem 5.7, M is similar to $[z^n\mathscr{C}]$. Suppose that A: $[z^n\mathscr{C}] \to M$ is a similarity. Set $r = A(z^n)$. Since for any polynomial p, we have that

$$A(z^n p) = rp = \frac{r}{z^n} z^n p,$$

for each $h \in [z^n\mathscr{C}]$

$$A(h) = \frac{r}{z^n} h.$$

Note that $L_a^2(C)$ has the standard orthogonal basis $\{e_0, e_1, ..., e_k, ...\}$, where $e_k(z) = z^k / \sqrt{2^k k!}$. It follows that the projection Q from $L_a^2(C)$ onto $[z^n\mathscr{C}]^\perp$ is

$$Q = \sum_{k=0}^{n-1} e_k \otimes e_k.$$

Thus the reproducing kernel $K_\lambda^{(n)}$ of the subspace $[z^n\mathscr{C}]$ is given by

$$\begin{aligned} K_\lambda^{(n)} &= (I - Q) K_\lambda \\ &= K_\lambda - \sum_{k=0}^{n-1} \langle K_\lambda, e_k \rangle e_k \\ &= K_\lambda - \sum_{k=0}^{n-1} \frac{\bar{\lambda}^k z^k}{2^k k!}. \end{aligned}$$

For $\lambda \neq 0$, we have that

$$\langle AK_\lambda^{(n)}, K_\lambda \rangle = \left\langle \frac{r}{z^n} \left(K_\lambda - \sum_{k=0}^{n-1} \frac{\bar{\lambda}^k z^k}{2^k k!} \right), K_\lambda \right\rangle$$
$$= \frac{r(\lambda)}{\lambda^n} \left(e^{|\lambda|^2/2} - \sum_{k=0}^{n-1} \frac{|\lambda|^{2k}}{2^k k!} \right).$$

Since A: $[z^n \mathscr{C}] \to L_a^2(C^n)$ is a bounded operator, we see that

$$|\langle AK_\lambda^{(n)}, K_\lambda \rangle| \leqslant \|A\| \, \|K_\lambda^{(n)}\| \, \|K_\lambda\|$$
$$\leqslant \|A\| \, \|K_\lambda\|^2$$
$$= e^{|\lambda|^2/2} \, \|A\|.$$

So, for $\lambda \neq 0$,

$$\left| \frac{r(\lambda)}{\lambda^n} \left(e^{|\lambda|^2/2} - \sum_{k=0}^{n-1} \frac{|\lambda|^{2k}}{2^k k!} \right) \right| \leqslant e^{|\lambda|^2/2} \, \|A\|,$$

and hence when $|\lambda|$ is sufficiently large,

$$\left(\frac{|r(\lambda)|}{|\lambda|^n} - \|A\| \right) e^{|\lambda|^2/2} \leqslant \frac{|r(\lambda)|}{|\lambda|^n} \sum_{k=0}^{n-1} \frac{|\lambda|^{2k}}{2^k k!}$$
$$\leqslant \gamma_0 \, |r(\lambda)| \, |\lambda|^n$$

for some constant γ_0.

Since $r \in L_a^2(C)$, we have that

$$|r(\lambda)| = |\langle r, K_\lambda \rangle| \leqslant \|r\| \, \|K_\lambda\| = \|r\| \, e^{|\lambda|^2/4}.$$

Therefore, when $|\lambda|$ is sufficiently large,

$$\frac{|r(\lambda)|}{|\lambda|^n} \leqslant \|A\| + \gamma_0 \, \|r\| \, |\lambda|^n \, e^{-|\lambda|^2/4}.$$

This implies that when $|\lambda|$ is sufficiently large, there exists a constant r_0 such that $\frac{|r(\lambda)|}{|\lambda|^n}$ is less than r_0. By Lemma 5.11, $A(z^n) = r$ is a polynomial, and $\deg r \leqslant n$. Note that $M = [r\mathscr{C}]$. Theorem 5.7 implies that

$$\deg r = \deg p.$$

This completes the proof of the theorem. Q.E.D.

The following theorem shows that two finite codimensional quasi-invariant subspaces are unitarily equivalent only if they are equal.

THEOREM 5.12. *On the complex plane, the quasi-invariant subspaces* $[p_1\mathscr{C}]$ *and unitarily equivalent only if* $[p_2\mathscr{C}]=[p_2\mathscr{C}]$, *and hence only if* $p_1=cp_2$ *for some constant c.*

Proof. Let $U: [p_1\mathscr{C}]\to[p_2\mathscr{C}]$ be a unitary equivalence. Set $g=Up_1$. Then we have

$$\|qp_1\|_2=\|qg\|_2$$

for any polynomial q, and it follows that

$$\frac{1}{2\pi}\int_C |q(z)|^2\,(|p_1(z)|^2-|g(z)|^2)\,e^{-|z|^2/2}\,dv(z)=0.$$

By the equality

$$p\bar{q}=\tfrac{1}{2}\,(|p+q|^2+i\,|p+iq|^2-(i+1)\,|p|^2-(i+1)\,|q|^2),$$

we get that

$$\int_C p(z)\,\overline{q(z)}\,e^{-|z|^2/2}(|p_1(z)|^2-|g(z)|^2)\,dv(z)=0,$$

for any polynomial p and q. Obviously, $p(z)\,\overline{q(z)}\,e^{-|z|^2/2}$ is in $C_0(C)$. Note that $(|p_1(z)|^2-|g(z)|^2)\,dv(z)$ is a regular Borel measure on C, and it annihilates $p(z)\,\overline{q(z)}\,e^{-|z|^2/2}$ for any polynomials p and q. Thus it is not difficult to verify that the above measure annihilates the subalgebra $\mathscr{A}$ of $C_0(C)$ generated by all $p(z)\,\overline{q(z)}\,e^{-|z|^2/2}$. The Stone–Weierstrass theorem [Con, p. 147, Cor. 8.3] implies that the subalgebra $\mathscr{A}$ equals $C_0(C)$. Now applying the Riesz representation theorem [Con, p. 383], we obtain that

$$(|p_1(z)|^2-|g(z)|^2)\,dv(z)=0,$$

and thus $p_1(z)=\gamma g(z)$ for some constant γ. Consequently,

$$[p_2\mathscr{C}]=[U(p_1\mathscr{C})]=[g\mathscr{C}]=[p_1\mathscr{C}],$$

and therefore $p_1=cp_2$ for some constant c. This completes the proof of the theorem. Q.E.D.

From Theorems 5.9 and 5.12, we immediately obtain the following corollary.

COROLLARY 5.13. *Let M be quasi-invariant. If M is unitarily equivalent to $[p\mathscr{C}]$ for some polynomial p, then $M = [p\mathscr{C}]$.*

Remark 5.14. Suppose that $f \in L^2_a(C)$ such that $zf \notin L^2_a(C)$. It is easy to check that the one dimensional subspace $\{cf: c \in C\}$ is quasi-invariant. We choose two such fuctions $f; g$ with $f/g \neq$ constant. Then quasi-invariant subspaces $\{cf: c \in C\}$ and $\{cg: c \in C\}$ are unitarily equivalent, but they are not equal. However, for the Bergman space of a bounded domain in the complex plane, Richter [Ric] proved that two invariant subspaces are unitarily equivalent only if they are equal. This phenomenon exhibits a new feature of the Segal–Bargmann space.

Case 2. Under similarity, let us consider that the classification of finite codimensional quasi-invariant subspaces of the Segal–Bargmann space on C^n. Here $n > 1$.

Note that each finite codimensional quasi-invariant subspace M has the form

$$M = [I],$$

where I is a finite codimensional ideal with the same codimension as M.

THEOREM 5.15. *Let N be a quasi-invariant subspace, and $M = [I]$ a finite codimensional quasi-invariant subspace. Then N is similar to M only if $N = M$.*

Proof. Let $A: M \to N$ be a similarity. Then for any $p, q \in I$, we have

$$A(pq) = pA(q) = qA(p),$$

and hence

$$\frac{A(p)}{p} = \frac{A(q)}{q}.$$

Thus we can define an analytic function on $C^n \setminus Z(I)$ by

$$\phi(z) = \frac{A(p)(z)}{p(z)}$$

for any $p \in I$ with $p(z) \neq 0$. Clearly ϕ is independent of p and is analytic on $C^n \setminus Z(I)$. Since I is finite codimensional, $Z(I)$ is a finite set. By Hartogs' extension theorem, $\phi(z)$ extends to an analytic function on C^n; that is, $\phi(z)$ is an entire function. It follows that

$$A(p) = \phi p$$

for any $p \in I$. Because I is dense in M, we conclude that

$$A(h) = \phi h$$

for any $h \in M$. We claim that $\phi h \in M$ if $h \in M$. From the proof of Lemma 5.3, there is a finite dimensional space R consisting of polynomials such that

$$L_a^2(C^n) = [I] \dot{+} R.$$

For $h \in M$, ϕh can be expressed as

$$\phi h = g_1 + g_2,$$

where $g_1 \in [I]$, and $g_2 \in R$. Note that for each $\lambda \in Z(I)$, and any $q \in I_\lambda$,

$$q(D)\, h|_\lambda = 0, \qquad \text{and} \qquad q(D)\, g_1|_\lambda = 0.$$

Since I_λ is invariant under the action by the basic partial differential operators $\{\partial/\partial z_1, \partial/\partial z_2, ..., \partial/\partial z_n\}$, it follows that

$$q(D)\, rh|_\lambda = 0,$$

for any polynomial r. We choose pynomials $\{r_n\}$ such that r_n uniformly converge to ϕ on some bounded neighborhood $\mathcal{O}$ of λ, as $n \to \infty$. Thus we have that

$$0 = \lim_{n \to \infty} [q(D)\, r_n h](\lambda) = [q(D)\, \phi h](\lambda).$$

This implies that

$$[q(D)\, g_2](\lambda) = 0,$$

for each $\lambda \in Z(I)$ and any $q \in I_\lambda$. Theorem 2.1 in [Guol] implies that $g_2 \in I$, and hence $g_2 = 0$. So, $\phi h \in [I]$. Consequently, $\phi h \in M$ if $h \in M$. This says that $M \supset N$. Now by Proposition 5.1, ϕ is a constant. Thus we conclude that $M = N$. This completes the proof of the theorem. Q.E.D.

Remark 5.16. On the bidisk D^2, it is easy to check that $L_a^2(D^2)$ is similar to $z_1 L_a^2(D^2)$. Hence Theorem 5.14 again points out difference between the Bergman spaces and the Segal–Bargmann spaces.

Before ending this section, let us look at the structure of $M^\perp$ if M is finite codimensional. Let I be the finite codimensional ideal such that $M = [I]$. Since I is of finite codimension, $Z(I)$ is a finite set, say, $Z(I) = \{\lambda_1, \lambda_2, ..., \lambda_l\}$. I can then be uniquely decomposed as

$$I = \bigcap_{k=1}^{l} I_k,$$

where I_k are ideals with the unique zero λ_k. Clearly,

$$[I] \subset \bigcap_{k=1}^{k} [I_k].$$

Note that $\bigcap_{k=1}^{k} [I_k]$ is quasi-invariant, and is of finite codimension. From Lemma 5.2 and Lemma 5.4,

$$\mathscr{C} \cap \left(\bigcap_{k=1}^{k} [I_k] \right) = \bigcap_{k=1}^{l} I_k = I$$

is dense in $\bigcap_{k=1}^{k} [I_k]$. So,

$$[I] = \bigcap_{k=1}^{k} [I_k].$$

Thus each finite codimensional quasi-invariant subspace M can be uniquely decomposed as

$$M = \bigcap_{k=1}^{l} M_k,$$

where M_k are quasi-invariant, and are determined by a unique zero point.

It is not difficult to see that $\{z^\alpha/\|z^\alpha\|_2 : \alpha$ ranging over all non-negative indices$\}$ is an orthogonal basis of $L_a^2(C^n)$. From this, we see that, although for each polynomial p the Toeplitz operator $T_{\bar{p}}$ is unbounded on $L_a^2(C^n)$, $T_{\bar{p}}$ maps $\mathscr{C}$ to $\mathscr{C}$. Now let $\mathscr{P}$ be a linear space consisting of polynomials. We say that $\mathscr{P}$ is an *invariant polynomial space*, if for any polynomial p, $\mathscr{P}$ is invariant under the action by $T_{\bar{p}}$.

Using the same proof for Lemma 3.2, we have the following result.

LEMMA 5.17. *Let M be a finite codimensional quasi-invariant subspace with a unique zero $\lambda = 0$. Then $M^\perp$ is a finite dimensional invariant polynomial space.*

Consider the parallel shifts on C^n

$$\gamma_\lambda(z) = \lambda - z.$$

These maps determine unitary operators on $L^2(C^n)$ given by

$$V_\lambda f = f \circ \gamma_\lambda k_\lambda.$$

It is easy to verify that V_λ commute with the Segal–Bargmann projection P, and $V_\lambda^2 = I$. [Cob, BC].

Let M be a finite codimensional quasi-invariant subspace. Then M has finitely many zero points $\lambda_1, \lambda_2, ..., \lambda_l$ such that M can uniquely be represented as

$$M = \bigcap_{i=1}^{l} M_i,$$

where M_i are quasi-invariant, and are determined by a unique zero λ_i.

THEOREM 5.18. *Under the above assumption, there are finite dimensional invariant polynomial spaces* $\mathscr{P}_i$, $i = 1, 2, ..., l$ *such that*

$$M^\perp = \sum_{i=1}^{l} \mathscr{P}_i \circ \gamma_{\lambda_i} k_{\lambda_i}.$$

Proof. Let N be a quasi-invariant subspace of finite codimension with a unique zero point λ. We claim that $V_\lambda N = \{f \circ \gamma_\lambda k_\lambda : f \in N\}$ is quasi-invariant. In fact, suppose there is a polynomial q and some $f \in N$ such that $q(z)\, f(\lambda - z)\, k_\lambda(z) \in L_a^2(C^n)$. Write $q(z) = p(\lambda - z) + c$, where p is a polynomial, and c a constant. Since

$$q(z)\, f(\lambda - z)\, k_\lambda(z) = V_\lambda((p+c)\, f)(z),$$

we have

$$(p+c)\, f = V_\lambda V_\lambda((p+c)\, f) \in L_a^2(C^n).$$

Hence $pf \in L_a^2(C^n)$. Note that N is quasi-invariant. We have $pf \in N$. It follows that $(p+c)\, f \in N$, and hence

$$q(z)\, f(\lambda - z)\, k_\lambda(z) = V_\lambda((p+c)\, f)(z) \in V_\lambda N.$$

This insures that $V_\lambda N$ is quasi-invariant. From the equality,

$$L_a^2(C^n) = N \oplus N^\perp = V_\lambda N \oplus V_\lambda N^\perp,$$

we see that $V_\lambda N$ is a finite codimensional quasi-invariant subspace, and $V_\lambda N$ is of the same codimension as N. Note that $V_\lambda N$ has only the zero point 0. Thus from Lemma 5.17, $V_\lambda N^\perp$ is a finite dimensional invariant polynomial space. We denote $V_\lambda N^\perp$ by $\mathscr{P}$. So,

$$N^\perp = V_\lambda \mathscr{P} = \mathscr{P} \circ \gamma_\lambda k_\lambda.$$

Since $M = \bigcap_{i=1}^{l} M_i$, this gives that

$$M^\perp = \sum_{i=1}^{l} M_i^\perp.$$

Thus there are finite dimensional invariant polynomial spaces $\mathscr{P}_i$, $i = 1, 2, \ldots, l$ such that

$$M^{\perp} = \sum_{i=1}^{l} P_i \circ \gamma_{\lambda_i} k_{\lambda_i}.$$

This completes the proof of the theorem. Q.E.D.

6. FINITE RANK SMALL HANKEL OPERATORS ON THE SEGAL–BARGMANN SPACE

On the Hilbert space $L^2(C^n)$, define the unitary operator U by

$$(Uf)(z) = f(\bar{z}) \overset{\text{def}}{=} \hat{f}(z).$$

Similarly to what is done in Bergman spaces, for $\phi \in L^2(C^n)$, define the small Hankel operator in $L_a^2(C^n)$, Γ_ϕ, by

$$\Gamma_\phi f = P(\phi U f) = P(\phi \hat{f})$$

if $\phi \hat{f} \in L^2(C^n)$. Therefore the domain of Γ_ϕ is

$$\mathscr{D}_\phi = \{f \in L_a^2(C^n) : \phi \hat{f} \in L^2(C^n)\}.$$

If $\mathscr{D}_\phi$ is dense in $L_a^2(C^n)$, then Γ_ϕ is densely defined. If Γ_ϕ is densely defined, and Γ_ϕ extends to a bounded operator on $L_a^2(C^n)$, then Γ_ϕ is a bounded small Hankel operator. In this section, we consider bounded small Hankel operators with finite rank on $L_a^2(C^n)$.

Let Γ_ϕ be a bounded small Hankel operator. Then it is easy to see that $\Gamma_\phi = \Gamma_{P\phi}$; that is, the small Hankel operator Γ_ϕ only depends on the analytic part of ϕ. Thus when discussing small Hankel operators, we assume that their symbols are analytic. The reader will easily also verify that kernels of small Hankel operators are quasi-invariant.

Using the same proof as for Proposition 4.1, we can obtain the following result.

Proposition 6.1. *The following equality holds:*

$$V_\lambda \Gamma_\phi = \Gamma_{\frac{k_\lambda}{k_{\bar{\lambda}}} \phi \circ \gamma_\lambda} V_{\bar{\lambda}} = \Gamma_{e^{iIm\bar{\lambda}z} \phi \circ \gamma_\lambda} V_{\bar{\lambda}}.$$

THEOREM 6.2. *Let ϕ be an entire function on C^n. Then the small Hankel operator Γ_ϕ is of finite rank if and only if there are $\lambda_1, \lambda_2, ..., \lambda_l$ in C^n, and polynomials $p_1, p_2, ..., p_l$ such that*

$$\phi(z) = \sum_{i=1}^{l} p_i(z)\, e^{\lambda_i z}.$$

Proof. First we assume that Γ_ϕ is of finite rank. Then $M = \ker \Gamma_\phi$ is a finite codimensional quasi-invariant subspace. Note that $\Gamma_\phi M = 0$ if and only if there exists a dense subset L of M such that

$$\langle \phi \hat{L}, L_a^2(C^n) \rangle = 0,$$

where $\hat{L} = \{\hat{h} : h \in L\}$. The above equality holds if and only if

$$\langle \phi, \hat{\bar{L}} \rangle = 0,$$

where $\hat{\bar{L}} = \{\hat{\bar{h}}: h \in L\}$. Since the closure of $\hat{\bar{L}}$ equals $\hat{\bar{M}}$, it follows that

$$\langle \phi, \hat{\bar{M}} \rangle = 0.$$

Note that $\hat{\bar{M}}$ is quasi-invariant and of the same codimension as M. Theorem 5.18 gives the desired conclusion.

In the opposite direction, by Proposition 6.1, it is easy to check that

$$V_{2\bar{\lambda}} \Gamma_{pe^{\lambda z}} q = e^{|\lambda|^2} \Gamma_{p \circ \gamma_{2\bar{\lambda}}} q \circ \gamma_{2\lambda}$$

for each polynomial p and q. Since $p \circ \gamma_{2\bar{\lambda}}$ is a polynomial, $\Gamma_{p \circ \gamma_{2\bar{\lambda}}}$ is of finite rank. Thus it is easy to see that

$$\mathscr{C} \cap \ker \Gamma_{pe^{\lambda z}} = \{q \circ \gamma_{2\lambda} : q \in \mathscr{C} \cap \ker \Gamma_{p \circ \gamma_{2\bar{\lambda}}}\}.$$

By Lemma 5.4, the ideal $\mathscr{C} \cap \ker \Gamma_{p \circ \gamma_{2\bar{\lambda}}}$ is of finite codimension, and hence the ideal $\mathscr{C} \cap \ker \Gamma_{pe^{\lambda z}}$ is of finite codimension. So, by Lemma 5.3, $\ker \Gamma_{pe^{\lambda z}}$ is a quasi-invariant finite codimensional subspace. We thus conclude that $\Gamma_{pe^{\lambda z}}$ is of finite rank. Q.E.D.

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