The topic of my talk is: Some examples of the multivariable situation of Jim Agler's approach to dilation theory.

The Sz. Nagy dilation theorem states that every contraction on a Hilbert space has an isometric dilation. In fact, by considering adjoints one has an even better theorem:

$\|T\| \leq 1 \Rightarrow \exists\, V \in B(\mathcal{K}),\ \mathcal{H} \subseteq \mathcal{K},\ V\mathcal{H} \subseteq \mathcal{H}$
$T = V|\mathcal{H}$ and V^* is isometric

The Wold decomposition implies $V = S^* \oplus U$ where S is a unilateral shift of some multiplicity and U is unitary.

There is a special case of this, where $T^n \to 0$ SOT, then U is absent, and the proof technique of de Branges - Rovnyak is well-known to have many useful extensions to more general situations and the multivariable context.

That goes back to many including Drury (1978) Agler (1982), and there is now a nice 2002 article in JOT by Ambrozie, Englis, Müller. That is <u>not</u> what this talk is about. In this talk I will specifically be interested in analogues of the general situation, where the unitary U may appear.

I will start with Agler's 1988 definitions and his theorem. The theory is quite abstract, in order to avoid dealing with limiting ordinals and transfinite induction I will assume in this talk that all Hilbert spaces are separable.

Def: (Agler) A family is a collection $\mathcal{F}$ of Hilbert space operators. $(T, \mathcal{H})$ such that

(a) $\mathcal{F}$ is bounded, i.e. $\exists c \ni \|T\| \le c \;\forall T \in \mathcal{F}$

(b) $\mathcal{F}$ is closed under restrictions to invariant subspaces, i.e. $T \in \mathcal{F}$, $\mathcal{M} \in \operatorname{Lat} T \Rightarrow T|_{\mathcal{M}} \in \mathcal{F}$

(c) $\mathcal{F}$ is closed under taking direct sums, i.e.

$$\forall n \;\; T_n \in \mathcal{F} \Rightarrow \bigoplus_n T_n \in \mathcal{F}$$

(d) closed under unital $*$ representations of $\mathcal{B}(\mathcal{H})$, i.e.
$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$, $\pi(I) = I$, $A \in \mathcal{F} \Rightarrow \pi(A) \in \mathcal{F}$

Examples:

$\mathcal{F}$ = contractions

$\mathcal{F}$ = isometries

$\mathcal{F}$ = selfadjoint contractions

$\mathcal{F}$ = subnormal contractions

$\mathcal{F}$ = hyponormal contractions

There is a <u>partial ordering</u> on $\mathcal{F}$. We say $(T, \mathcal{H}) \le (S, \mathcal{K})$ or $T \le S$, if $\mathcal{H} \subseteq \mathcal{K}$ is a subspace, $\mathcal{H} \in \operatorname{Lat} S$, and $T = S|\mathcal{H}$

<u>Def</u>: (Agler) T is extremal for $\mathcal{F}$ ($T \in \operatorname{ext} \mathcal{F}$)

$$\iff S \ge T,\ S \in \mathcal{F} \implies S = T \oplus T_1$$

$$\iff S = \begin{pmatrix} T & A \\ 0 & T_1 \end{pmatrix} \in \mathcal{F} \implies A = 0$$

<u>Theorem</u> (Agler) If $\mathcal{F}$ is a family, if $T \in \mathcal{F}$, then $\exists\, S \in \operatorname{ext}(\mathcal{F}) \ni T = S|\mathcal{H}$

The proof of this theorem is omitted in Agler's 1988 paper. In 2005 McCullough-Dritschel published a proof giving Agler all the credit. Also, Arveson has an ~~a~~ unpublished manuscript on his webpage (Notes on the unique extension property, 2003) He refers to the Dritschel-McCullough paper.

Proof idea of Agler's theorem

<u>Lemma 1</u>: If $\mathcal{F}$ is a family, if $T_n \in \mathcal{F}$ $\forall n$ and if $T_n \le T_{n+1}$, then $T = \sup T_n \in \mathcal{F}$

(here $T_n \in \mathcal{B}(\mathcal{H}_n)$, $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$, $\mathcal{H} = \overline{\bigcup_n \mathcal{H}_n}$ and $Tx = T_n x$ whenever $x \in \mathcal{H}_n$)

The proof uses Agler's hereditary calculus and positivity. Mathematical machinery is involved.

<u>Lemma 2</u>: If $\mathcal{F}$ is bounded and if whenever $T_n \in \mathcal{F}$, $T_n \le T_{n+1}$, then $T = \sup T_n \in \mathcal{F}$, then the conclusion of Agler's theorem holds, i.e. $\forall T \in \mathcal{F}\ \exists S \in \operatorname{ext}\mathcal{F} \ni T = S|\mathcal{H}$.

The proof of Lemma 2 is ingenious, but elementary. I will give the main idea. Note that if $S \ge T$, then $T^* = P_{\mathcal{H}} S^* | \mathcal{H}$, so $\|T^* x\| \le \|S^* x\|$ $\forall x \in \mathcal{H}$. Hence

$$T \in \operatorname{ext}(\mathcal{F}) \iff S \ge T,\ S \in \mathcal{F} \Rightarrow S^* \mathcal{H} \subseteq \mathcal{H}$$

$$\iff S \ge T,\ S \in \mathcal{F} \Rightarrow \|S^* x\| = \|T^* x\| \quad \forall x \in \mathcal{H}$$

Now let $T \in \mathcal{F}$ and fix $x \in \mathcal{H}$. Consider

$$(*) \qquad d(x) = \sup_{\substack{S \geq T \\ S \in \mathcal{F}}} \| S^* x \|$$

$d(x) < \infty$, since $\mathcal{F}$ is bdd and one can use the sup-property of $\mathcal{F}$ to show that the sup in $*$ is attained, i.e. $\exists\, S_0 \in \mathcal{B}(\mathcal{H}_0) \ni S_0 \geq T$, $\| S_0^* x \| = d(x)$.

Then, whenever $S \geq S_0$ we have $S \geq T$, so $\| S^* x \| \leq d(x) = \| S_0^* x \| \leq \| S^* x \|$, i.e. $\| S_0^* x \| = \| S^* x \|$

which means that S_0 does the right thing on x.

Next take a dense set $\{ x_1, x_2, x_3, \dots \} \subseteq \mathcal{H}$ and inductively construct $T \leq S_1 \leq S_2 \leq \dots$

such that $\quad S \geq S_n \Rightarrow \| S^* x_k \| = \| S_n^* x_k \|$

for $k = 1, \dots n$. Set $T_0 = \sup S_n \in \mathcal{B}(\mathcal{H}_0)$.

So $T_0 \geq T$ and $\| T_0^* x \| = \sup_{\substack{S \geq T \\ S \in \mathcal{F}}} \| S^* x \| \quad \forall x \in \mathcal{H} \subseteq \mathcal{H}_0$

Continue, get $T_1 \geq T_0$, $T_1 \in \mathcal{B}(\mathcal{H}_1)$, $\| T_1^* x \| = \sup_{S \geq T_0} \| S^* x \| \quad \forall x \in \mathcal{H}_0 \subseteq \mathcal{H}_1$

etc, $T \leq T_0 \leq T_1 \leq \dots$, $\tilde{T} = \sup T_n$ will do the trick.

Examples:

① Sz-Nagy : $\mathcal{F}$ = contractions

$\operatorname{ext}(\mathcal{F})$ = co-isometries = $\{T : T^* \text{ iso}\}$

pf: $S, T \in \mathcal{F}$, $S \geq T$ so

$$S = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}, \quad \|S\| \leq 1 \iff SS^* \leq I$$

$$SS^* = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix} \begin{pmatrix} T^* & 0 \\ A^* & B^* \end{pmatrix} = \begin{pmatrix} TT^* + AA^* & AB^* \\ BA^* & BB^* \end{pmatrix} \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If $TT^* = I$ (i.e. if T^* is an isometry), then $AA^* = 0$, so $A = 0$, so $T \in \operatorname{ext}(\mathcal{F})$

If $TT^* \neq I$, then $A = (I - TT^*)^{\frac{1}{2}}$, $B = 0$ will work, so $T \notin \operatorname{ext}(\mathcal{F})$.

② $\mathcal{F}$ = isometries $\Rightarrow \operatorname{ext}(\mathcal{F})$ = unitaries

similar proof

③ $\mathcal{F}$ = self-adjoint contractions
$\Rightarrow \operatorname{ext}(\mathcal{F}) = \mathcal{F}$

④ $\mathcal{F}$ = subnormal contractions
$\Rightarrow \operatorname{ext}(\mathcal{F})$ = normal contractions

The Theorem works for tuples

$$\mathcal{F} \ni T = (T_1, \ldots, T_d), \quad T_i \in \mathcal{B}(\mathcal{H})$$

$$T \leq S \iff T_i \leq S_i \quad \forall i = 1, \ldots, d$$

$$T \text{ extremal} \iff S \geq T, S \in \mathcal{F} \Rightarrow S_i = T_i \oplus T_i' \quad \forall i$$

In the following I will discuss examples, ~~...~~ ~~...~~ in the polydisc, in the unit ball, commutative, and non-commutative.

First the non-commutative polydisc

(1) (see Frazho (82, JFA) or Durszt, Sz Nagy (83, JFA)

$$\mathcal{F} = \{ (T_1 \ldots T_d) : \|T_i\| \leq 1 \} \text{ not necessarily commuting}$$

$$\Rightarrow \operatorname{ext}(\mathcal{F}) = \{ (V_1, \ldots V_d) : \text{each } V_i^* \text{ is isometric} \}$$

If all V_i^* are isometric, none of the V_i can be extended (~~a~~ as a contraction) except with direct sums, so such ~~is~~ a tuple is extremal.
If one of the V_i^* is not isometric, then

one can extend that one nontrivially, and do a direct sum extension of the others, so such tuple is not extremal.

(2) $\mathcal{F} = \{ (V_1, \dots V_d) : \text{each } V_i \text{ isometric} \}$

$\text{ext}(\mathcal{F}) = \{ (U_1, \dots U_d) : \text{unitary} \}$

Same argument

The commutative polydisc:

(3) $\mathcal{F} = \{ (T_1, \dots T_d) : \|T_i\| \le 1,\ T_i T_j = T_j T_i \ \forall i,j \}$

$\Rightarrow \text{ext}(\mathcal{F}) \supseteq \{ (V_1^*, \dots V_d^*) : V_i \text{ iso}, V_i V_j = V_j V_i \ \forall i,j \}$

$\neq$ $d \geq 3$ (Varopoulos)

$=$ $d = 1, 2$ ($d=2$ follows from Ando's theorem. I have not tried to give a new proof of Ando's theorem by use of this approach)

So for $d \geq 3$ it is open to determine $\text{ext}(\mathcal{F})$.

(4) $\mathcal{F} = \{ (S_1, \dots, S_d) : S_i^* S_i = 1,\ S_i S_j = S_j S_i \ \forall i,j \}$

$\Rightarrow$ $\mathrm{ext}(\mathcal{F})$ = commuting unitaries

proof: $\Leftarrow$ follows from the one variable result.

$\Rightarrow$ Each S_i is an isometry, so each S_i has a unitary extension. We must show that the tuple $(S_1, \dots, S_d)$ has a commuting unitary extension.

Fix the minimal unitary extension $U_1 \in B(\mathcal{K})$ of S_1. By minimality

$$\mathcal{K} = \overline{\left\{ \sum_n U_1^{*n} x_n : x_n \in \mathcal{H} \right\}}, \quad S_i \in B(\mathcal{H})$$

Note that

$$\Big\| \sum_n U_1^{*n} x_n \Big\|^2 = \Big\| \sum_n U_1^{*n} S_i x_n \Big\|^2 \quad \forall i$$

for all $x_n \in \mathcal{H}$. This is easy to check using that U_1 is normal and S_i is isometric.

Thus, one checks that

$$U_i \Big(\sum_n U_1^{*n} x_n \Big) := \sum_n U_1^{*n} S_i x_n \qquad i = 2, \dots d$$

is well-defined and extends to an isometry on $\mathcal{K}$ for each i. One also checks $U_i U_j = U_j U_i$

Hence $U = (U_1, U_2, \ldots U_d)$ extends $S = (S_1, \ldots S_d)$

If S is extremal, then $U_1^* \mathcal{H} \subseteq \mathcal{H}$, so

$\mathcal{K} = \mathcal{H}$ and hence $U_1 = S_1$ is unitary.

Similarly, $U_j = S_j$ must be unitary $\forall j$.

The non-commutative ball

5 (see Popescu, Trans. Amer. Math. Soc. 1989)

$$\mathcal{F} = \left\{ (T_1, \ldots T_d) : \sum_{i=1}^{d} T_i^* T_i \leq I \right\}$$

So $(T_1 \ldots T_d) \in \mathcal{F}$, iff $(T_1^*, \ldots T_d^*) \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} \leq I$

These are also called spherical contractions.

Then $\quad ext(\mathcal{F}) = \left\{ (T_1 \ldots T_d) : T_j T_i^* = \delta_{ij} I \right\}$

i.e. $(T_1 \ldots T_d) \in ext(\mathcal{F}) \iff \begin{pmatrix} T_1 \\ \vdots \\ T_d \end{pmatrix} (T_1^* \ldots T_d^*) = \begin{pmatrix} I & & 0 \\ & \ddots & \\ 0 & & I \end{pmatrix}$

so the T_i^* are isometries with orthogonal ranges

(6) $\mathcal{F} = \{ (T_1 \ldots T_d) : \sum_{i=1}^{d} T_i^* T_i = I \}$

spherical isometries

$\Longrightarrow \operatorname{ext}(\mathcal{F}) = \{ (T_1 \ldots T_d) : T_i T_j^* = \delta_{ij} I$ and $\sum_{i=1}^{d} T_i^* T_i = I \}$

i.e. the T_i^* satisfy the Cuntz-relations

(5) and (6) can be proved by elementary means. Also note that Popescu considers infinite sequences $(T_1, \ldots, T_n, \ldots)$ and also proves further structure theorems (e.g. an analogue of the Wold decomposition)

The commutative ball

(7) (Athavale)

$\mathcal{F} = \{ (T_1, \ldots, T_d) : \sum T_i^* T_i = I, \ T_i T_j = T_j T_i \}$

commuting spherical isometries

$\Longrightarrow \operatorname{ext}(\mathcal{F}) = \{ (U_1, \ldots, U_d) : \sum_{i=1}^{d} U_i^* U_i = I, \ U_i$ normal, $U_i U_j = U_j U_i \}$

(spherical unitaries)

One can give a simple proof of this that differs from Athavale's proof. In fact, one can proceed very similar to the approach in (4), this is based on an idea of Attele and Lubin, JFA, 1996.

(8) (Drury, Proc. AMS, 1978 – the case of no U, Müller – Vasilescu, Proc. AMS, 1993 Arveson, Acta Math, 1998)

$$\mathcal{F} = \{ (T_1, \dots T_d) : \sum T_i^* T_i \le I,\ T_i T_j = T_j T_i \}$$

$$\Rightarrow \operatorname{ext}(\mathcal{F}) = \{ T = S^* \oplus U \}$$

U – commuting spherical unitary

S – d-shift of some multiplicity, i.e. $S = M_z$ on $H_d^2 \otimes \mathcal{D}$, where H_d^2 is the Drury–Arveson–Hardy space defined by the reproducing kernel $k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle}$

$\mathcal{D}$ is some Hilbert space and

$M_z = (M_{z_1}, \dots M_{z_d}) \otimes I$

Richter and Sundberg found it useful to give an intrinsic characterization of operators of the type $S^{*} \oplus U$.

TFAE

(a) $T \in \operatorname{ext}(\mathcal{F})$

(b) $T = S^{*} \oplus U$ (as above)

(c) ① $\sum T_i^{*} T_i =$ projection

② $\sum T_i T_i^{*} \geq I$

③ if $x_1 \ldots x_d \in \mathcal{H}$ with $T_i x_j = T_j x_i$ then $\exists\, x \in \mathcal{H}$ with $x_i = T_i x$

The last condition states that the Koszul complex for T is exact at a certain stage.