CLASSIFICATION OF REDUCING SUBSPACES OF A CLASS OF MULTIPLICATION OPERATORS ON THE BERGMAN SPACE VIA THE HARDY SPACE OF THE BIDISK

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ABSTRACT. In this paper we obtain a complete description of nontrivial minimal reducing subspaces of the multiplication operator by a Blaschke product with four zeros on the Bergman space of the unit disk via the Hardy space of the bidisk.

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space L_a^2 is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on $\mathbb{D}$. For a bounded analytic function ϕ on the unit disk, the multiplication operator M_ϕ with symbol ϕ is defined on the Bergman space L_a^2 given by

$$M_\phi h = \phi h$$

for $h \in L_a^2$. On the basis $\{e_n\}_{n=0}^\infty$, where e_n is equal to $\sqrt{n+1}z^n$, the multiplication operator M_z by z is a weighted shift operator, said to be the Bergman shift:

$$M_z e_n = \sqrt{\frac{n+1}{n+2}} e_{n+1}.$$

A reducing subspace M for an operator T on a Hilbert space H is a subspace M of H such that $TM \subset M$ and $T^*M \subset M$. A reducing subspace M of T is called minimal if M does not have any nontrivial subspaces which are reducing subspaces. The goal of this paper is to classify reducing subspaces of M_ϕ for the Blaschke product ϕ with four zeros by identifying its minimal reducing subspaces. Our main idea is to lift the Bergman shift up as a compression of a commuting pair of isometries on a nice subspace of the Hardy space of the bidisk. This idea was used in studying the Hilbert modules by R. Douglas and V. Paulsen [5], operator theory in the Hardy space over the bidisk by R. Dougals and R. Yang [6], [18], [19] and [20]; the higher-order Hankel forms by S. Ferguson and R. Rochberg [7] and [8] and and the lattice of the invariant subspaces of the Bergman shift by S. Richter [12].

On the Hardy space of the unit disk, for an inner function ϕ, the multiplication operator by ϕ is a pure isometry. So its reducing subspaces are in one-to-one correspondence with the closed subspaces of $H^2 \ominus \phi H^2$ [4], [10]. Therefore, it has infinitely many reducing subspaces provided that ϕ is any inner function other than a Möbius function. Many people have studied the problem of determining reducing subspaces of a multiplication operator on the Hardy space of the unit circle [1], [2] and [11].

The multiplication operators on the Bergman space possess a very rich structure theory. Even the lattice of the invariant subspaces of the Bergman shift M_z is huge [3]. But the lattice of reducing subspaces of the multiplication operator by a finite Blaschke on the

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Bergman space seems to be simple. On the Bergman space, Zhu [21] showed that for a Blaschke product ϕ with two zeros, the multiplication operator M_ϕ has exact two nontrivial reducing subspaces $\mathcal{M}_0$ and $\mathcal{M}_0^\perp$. In fact, the restriction of the multiplication operator on $\mathcal{M}_0$ is unitarily equivalent to the Bergman shift. Using the Hardy space of bidisk in [9], we show that the multiplication operator with a finite Blaschke product ϕ has a unique reducing subspace $\mathcal{M}_0(\phi)$, on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift and if a multiplication operator has a such reducing subspace, then its symbol must be a finite Blaschke product. The space $\mathcal{M}_0(\phi)$ is called the distinguished reducing subspace of M_ϕ and is equal to

$$\bigvee\{\phi'\phi^n : n = 0, 1, \cdots, m, \cdots\}$$

if ϕ vanishes at 0 in [15], i.e,

$$\phi(z) = cz\prod_{k=1}^{n}\frac{z-\alpha_k}{1-\overline{\alpha_k}z},$$

for some points $\{\alpha_k\}$ in the unit disk and a unimodular constant c. The space has played an important role in classifying reducing subspaces of M_ϕ. In [9], we have shown that for a Blaschke product ϕ of the third order, except for a scalar multiple of the third power of a Möbius transform, M_ϕ has exactly two nontrivial minimal reducing subspaces $\mathcal{M}_0(\phi)$ and $\mathcal{M}_0(\phi)^\perp$. This paper continues our study on reducing subspaces of the multiplication operators M_ϕ on the Bergman space in [9] by using the Hardy space of the bidisk. We will obtain a complete description of nontrivial minimal reducing subspaces of M_ϕ for the fourth order Blaschke product ϕ.

This paper is organized as follows. In Section 1 we introduce some notation to lift the Bergman shift as the compression of some isometry on a subspace of the Hardy space of the bidisk and state some theorems in [9] which will be used later. In Section 2 we state the main result and present its proof. Since the proof is long, two difficult cases in the proof are considered in the last two sections.

1. BERGMAN SPACE VIA HARDY SPACE

Let $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T}\times\mathbb{T}$. Let $d\sigma$ be the rotation invariant Lebesgue measure on $\mathbb{T}^2$. The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2,\ d\sigma)$, where functions in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in the bidisc $\mathbb{D}^2$ with the square summable Fourier coefficients. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2(\mathbb{T}^2)$ where P is the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$.

For each integer $n \geq 0$, let

$$p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i}.$$

Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^\infty$. Thus

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus cl\{(z-w)H^2(\mathbb{T}^2)\}.$$

Let

$$\mathcal{B} = P_{\mathcal{H}}T_z|_{\mathcal{H}} = P_{\mathcal{H}}T_w|_{\mathcal{H}}$$

where $P_{\mathcal{H}}$ is the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $\mathcal{H}$. So $\mathcal{B}$ is unitarily equivalent to the *Bergman shift* M_z on the Bergman space L_a^2 via the following unitary operator $U : L_a^2(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z,w)}{n+1}.$$

This implies that the Bergman shift is lifted up as the compression of an isometry on a nice subspace of $H^2(\mathbb{T}^2)$. Indeed, for each finite Blaschke product $\phi(z)$, the multiplication operator M_ϕ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

Let L_0 be $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H}$. In [9], for each $e \in L_0$, we construct functions $\{d_e^k\}$ and d_e^0 such that for each $l \geq 1$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d_e^{l-k} \in \mathcal{H}$$

and

$$p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))d_e^0 \in \mathcal{H}.$$

On one hand, we have a precise formula of d_e^0:

$$d_e^0(z,w) = we(0,w)e_0(z,w) - w\phi_0(w)e(z,w), \tag{1.1}$$

where e_0 is the function $\frac{\phi(z)-\phi(w)}{z-w}$. On the other hand, d_e^k is orthogonal to

$$kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)} \cap \mathcal{H},$$

and for a reducing subspace $\mathcal{M}$ and $e \in \mathcal{M}$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d_e^{l-k} \in \mathcal{M}.$$

Moreover, the relation between d_e^1 and d_e^0 is given by Theorem 1 in [9] as follows:

Theorem 1.1. *If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each $e \in \mathcal{M} \cap L_0$, then there is an element $\tilde{e} \in \mathcal{M} \cap L_0$ and a number λ such that*

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0. \tag{1.2}$$

Since for Blaschke products with smaller order, it is not difficult to calculate $\tilde{e}$ and λ precisely, we are able to classify minimal reducing subspaces of a multiplication operator by a Blaschke product of the fourth order. Main ideas in the proof of Theorems 3.1 and 4.1 are that by complicated computations we use (1.2) to derive conditions on zeros of the Blaschke product of the fourth order.

In this paper we often use Theorem 1.1 and Theorems 1 and 25 in [9] stated as follows.

Theorem 1.2. *There is a unique reducing subspace $\mathcal{M}_0$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0}$ is unitarily equivalent to the Bergman shift. In fact,*

$$\mathcal{M}_0 = \bigvee_{l \geq 0} \{p_l(\phi(z), \phi(w))e_0\},$$

and $\{\frac{p_l(\phi(z),\phi(w))e_0}{\sqrt{l+1}\|e_0\|}\}_0^\infty$ form an orthonormal basis of $\mathcal{M}_0$.

We call $\mathcal{M}_0$ to be the distinguished reducing subspace for $\phi(\mathcal{B})$. $\mathcal{M}_0$ is unitarily equivalent to a reducing subspace of M_ϕ contained in the Bergman space, denoted by $\mathcal{M}_0(\phi)$. The space plays an important role in classifying the minimal reducing subspaces of M_ϕ in Theorem 2.1.

In [9] we showed that for a nontrivial minimal reducing subspace Ω for $\phi(\mathcal{B})$, either Ω equals $\mathcal{M}_0$ or Ω is a subspace of $\mathcal{M}_0^\perp$. The condition in the following theorem is natural.

Theorem 1.3. *Suppose that Ω, $\mathcal{M}$ and $\mathcal{N}$ are three distinct nontrivial minimal reducing subspaces for $\phi(\mathcal{B})$ and*

$$\Omega \subset \mathcal{M} \oplus \mathcal{N}.$$

If they are contained in $\mathcal{M}_0^\perp$, then there is a unitary operator $U : \mathcal{M} \to \mathcal{N}$ such that U commutes with $\phi(\mathcal{B})$ and $\phi(\mathcal{B})^*$.

2. Main result

Let ϕ be a Blaschke product with four zeros. In this section we will obtain a complete description of minimal reducing subspaces of the multiplication operator M_ϕ. First observe that the multiplication operator M_{z^4} is a weighted shift with multiplicity 4:

$$M_{z^4} e_n = \sqrt{\frac{n+1}{n+5}} e_{n+4}$$

where e_n equals $\sqrt{n+1} z^n$. By Theorem B [14], M_{z^4} has exact four nontrivial minimal reducing subspaces:

$$\mathcal{M}_j = \bigvee \{z^n : n \equiv j \bmod 4\}$$

for $j = 1, 2, 3, 4$.

Before stating the main result of this paper we need some notation. It is not difficult to see that the set of finite Blaschke products forms a semigroup under composition of two functions. For a finite Blaschke product ϕ we say that ϕ is decomposable if there are two Blaschke products ψ_1 and ψ_2 with orders greater than 1 such that

$$\phi(z) = \psi_1 \circ \psi_2(z).$$

For each λ in $\mathbb{D}$, let ϕ_λ denote the Möbius transform:

$$\phi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}.$$

Define the operator U_λ on the Bergman space as follows:

$$U_\lambda f = f \circ \phi_\lambda k_\lambda$$

for f in L^2_a where k_λ is the normalized reproducing kernel $\frac{(1-|\lambda|^2)}{(1-\bar{\lambda} z)^2}$. Clearly, U_λ is a self-adjoint unitary operator on the Bergman space. Using the unitary operator U_λ we have

$$\mathcal{M}_0(\phi) = U_\lambda \mathcal{M}_0(\phi \circ \phi_\lambda)$$

where λ is a zero of the finite Blaschke product ϕ. This easily follows from that $\phi \circ \phi_\lambda$ vanishes at 0 and

$$U_\lambda^* M_\phi U_\lambda = M_{\phi \circ \phi_\lambda}.$$

We say that two Blaschke products ϕ_1 and ϕ_2 are equivalent if there is a complex number λ in $\mathbb{D}$ such that

$$\phi_1 = \phi_\lambda \circ \phi_2.$$

For two equivalent Blaschke products ϕ_1 and ϕ_2, M_{ϕ_1} and M_{ϕ_2} are mutually analytic function calculus of each other and hence share reducing subspaces. The following main result of this paper gives a complete description of minimal reducing subspaces.

Theorem 2.1. *Let ϕ be a Blaschke product with four zeros. One of the following holds.*

(1) If ϕ is equivalent to z^4, i.e., ϕ is a scalar multiple of the fourth power ϕ_c^4 of the Möbius transform ϕ_c for some complex number c in the unit disk, M_ϕ has exact four nontrivial minimal reducing subspaces

$$\{U_c\mathcal{M}_1, U_c\mathcal{M}_2, U_c\mathcal{M}_3, U_c\mathcal{M}_4\}.$$

(2) If ϕ is is decomposable but not equivalent to z^4, i.e, $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with orders 2 but not both of ψ_1 and ψ_2 are a scalar multiple of z^2, then M_ϕ has exact three nontrivial minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2)^\perp\}.$$

(3) If ϕ is not decomposable, then M_ϕ has exact two nontrivial minimal reducing subspaces

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}.$$

To prove the above theorem we need the following two lemmas which tell us when a Blaschke product with order 4 is decomposable.

Lemma 2.2. *If a Blaschke product ϕ with order four is decomposable, then the numerator of the rational function $\phi(z) - \phi(w)$ has at least three irreducible factors.*

Proof. Suppose that ϕ is the Blaschke product with order four. Let $f(z, w)$ be the numerator of the rational function $\phi(z) - \phi(w)$. If ϕ is decomposable, then $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with order two. Let $g(z, w)$ be the numerator of the rational function $\psi_1(z) - \psi_1(w)$. Clearly, $z - w$ is a factor of $g(z, w)$. Thus we can write

$$g(z, w) = (z - w)p(z, w)$$

for some polynomial $p(z, w)$ of z and w to get

$$g(\psi_2(z), \psi_2(w)) = (\psi_2(z) - \psi_2(w))p(\psi_2(z), \psi_2(w)).$$

On the other hand, we also have

$$\psi_2(z) - \psi_2(w) = \frac{(z - w)p_2(z, w)}{q_2(z, w)}$$

for two polynomials $p_2(z, w)$ and $q_2(z, w)$ which $p_2(z, w)$ and $q_2(z, w)$ do not have common factor. In fact, $q_2(z, w)$ and the numerator of the rational function $p(\psi_2(z), \psi_2(w))$ do not have common factor also. So we obtain

$$g(\psi_2(z), \psi_2(w)) = \frac{(z - w)p_2(z, w)}{q_2(z, w)}p(\psi_2(z), \psi_2(w)).$$

Since $f(z, w)$ is the numerator of the rational function $g(\psi_2(z), \psi_2(w))$, this gives that $f(z, w)$ has at least three factors. This completes the proof.

For $\alpha, \beta \in \mathbb{D}$, define

$$f_{\alpha,\beta}(w, z) = w^2(w - \alpha)(w - \beta)(1 - \bar{\alpha}z)(1 - \bar{\beta}z) - z^2(z - \alpha)(z - \beta)(1 - \bar{\alpha}w)(1 - \bar{\beta}w).$$

It is easy to see that $f_{\alpha,\beta}(w,z)$ is the numerator of $z^2\phi_\alpha(z)\phi_\beta(z) - w^2\phi_\alpha(w)\phi_\beta(w)$. The following lemma gives a criteria when the Blaschke product $z^2\phi_\alpha(z)\phi_\beta(z)$ is decomposable.

Lemma 2.3. *For α and β in $\mathbb{D}$, one of the following holds.*
(1) If both α and β equal zero, then

$$f_{\alpha,\beta}(w,z) = (w-z)(w+z)(w-iz)(w+iz).$$

(2) If α does not equal either β or $-\beta$, then

$$f_{\alpha,\beta}(w,z) = (w-z)p(w,z)$$

for some irreducible polynomial $p(w,z)$.
(3) If α equals either β or $-\beta$ but it does not equal zero, then

$$f_{\alpha,\beta}(w,z) = (w-z)p(w,z)q(w,z)$$

for two irreducible distinct polynomials $p(w,z)$ and $q(w,z)$.

Proof. Clearly, (1) holds.

To prove (2), by the example on page 6 of [13] we may assume that none of α and β equals 0. First observe that $(w-z)$ is a factor of the polynomial $f_{\alpha,\beta}(w,z)$. Taking a long division gives

$$f_{\alpha,\beta}(w,z) \quad = \quad (w-z)g_{\alpha,\beta}(w,z)$$

where

$$\begin{aligned} g_{\alpha,\beta}(w,z) \quad = \quad & (1-\bar\alpha z)(1-\bar\beta z)w^3 + (z-(\alpha+\beta))(1-\bar\alpha z)(1-\bar\beta z)w^2 \\ & +(z-\alpha)(z-\beta)(1-(\bar\alpha+\bar\beta)z)w + z(z-\alpha)(z-\beta). \end{aligned}$$

Next we will show that $g_{\alpha,\beta}(w,z)$ is irreducible. To do this, we assume that $g_{\alpha,\beta}(w,z)$ is reducible to derive a contradiction.

Assuming that $g_{\alpha,\beta}(w,z)$ is reducible, we can factor $g_{\alpha,\beta}(w,z)$ as the product of two polynomials $p(w,z)$ and $q(w,z)$ of z and w with degree of w greater than or equal one. Write

$$\begin{aligned} p(w,z) &= a_1(z)w + a_0(z) \\ q(w,z) &= b_2(z)w^2 + b_1(z)w + b_0(z) \end{aligned}$$

where $a_j(z)$ and $b_j(z)$ are polynomials of z. Since $g_{\alpha,\beta}(w,z)$ equals the product of $p(w,z)$ and $q(w,z)$, taking the product and comparing coefficients of w^k give

$$a_1(z)b_2(z) \quad = \quad (1-\bar\alpha z)(1-\bar\beta z), \tag{2.1}$$

$$a_1(z)b_1(z) + a_0(z)b_2(z) \quad = \quad (z-(\alpha+\beta))(1-\bar\alpha z)(1-\bar\beta z), \tag{2.2}$$

$$a_1(z)b_0(z) + a_0(z)b_1(z) \quad = \quad (z-\alpha)(z-\beta)(1-(\bar\alpha+\bar\beta)z), \tag{2.3}$$

$$a_0(z)b_0(z) \quad = \quad z(z-\alpha)(z-\beta). \tag{2.4}$$

Equation (2.1) gives that either

$$\begin{aligned} a_1(z) &= (1-\bar\alpha z) \text{ or} \\ a_1(z) &= (1-\bar\alpha z)(1-\bar\beta z) \text{ or} \\ a_1(z) &= 1. \end{aligned}$$

In the first case that $a_1(z) = (1 - \bar{\alpha}z)$, (2.1) gives $b_2(z) = (1 - \bar{\beta}z)$. Thus by Equation (2.2), we have

$$a_0(z)(1 - \bar{\beta}z) = (1 - \bar{\alpha}z)[(z - (\alpha + \beta))(1 - \bar{\beta}z) - b_1(z)],$$

to get that $(1-\bar{\alpha}z)$ is a factor of $a_0(z)$, and hence is also a factor of a factor $z(z-\alpha)(z-\beta)$ by (2.4). This implies that α must equal 0. It is a contradiction.

In the second case that $a_1(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)$, we have that $b_2(z) = 1$ to get that either the degree of $b_1(z)$ or the degree of $b_0(z)$ must be one while the degrees of $b_1(z)$ and $b_0(z)$ are at most one. So the degree of $a_0(z)$ is at most two. Also $a_0(z)$ does not equal zero. Equation (2.2) gives

$$(1 - \bar{\alpha}z)(1 - \bar{\beta}z)b_1(z) + a_0(z) = (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z).$$

Thus

$$a_0(z) = c_1(1 - \bar{\alpha}z)(1 - \bar{\beta}z)$$

for some constant c_1. But Equation (2.4) gives

$$c_1(1 - \bar{\alpha}z)(1 - \bar{\beta}z)b_0(z) = z(z - \alpha)(z - \beta).$$

Either $c_1 = 0$ or $(1 - \bar{\alpha}z)(1 - \bar{\beta}z)$ is a factor of $z(z - \alpha)(z - \beta)$. This is impossible.

In the third case that $a_1(z) = 1$, then $b_2(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)$. Since the root w of $f_{\alpha,\beta}(w, z)$ is a nonconstant function of z, the degree of $a_0(z)$ must be one. Thus the degrees of $b_1(z)$ and $b_0(z)$ are at most two. By Equation (2.2) we have

$$(1 - \bar{\alpha}z)(1 - \bar{\beta}z)a_0(z) + b_1(z) = (z - (\alpha + \beta))(1 - \bar{\alpha}z)(1 - \bar{\beta}z),$$

to get

$$b_1(z) = (1 - \bar{\alpha}z)(1 - \bar{\beta}z)[(z - (\alpha + \beta)) - a_0(z)].$$

Since the degree of $b_1(z)$ is at most two, we have

$$\begin{aligned} a_0(z) &= (z - (\alpha + \beta)) - c_0; \\ b_1(z) &= c_0(1 - \bar{\alpha}z)(1 - \bar{\beta}z). \end{aligned}$$

Equations (2.4) and (2.3) give

$$[(z - (\alpha + \beta)) - c_0]b_0(z) = z(z - \alpha)(z - \beta)$$

and

$$\begin{aligned} & b_1(z)[(z - (\alpha + \beta)) - c_0] + b_0(z) \\ = \; & (z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

Multiplying the both sides of the last equality by $[(z - (\alpha + \beta)) - c_0]$ gives

$$\begin{aligned} & b_1(z)[(z - (\alpha + \beta)) - c_0]^2 + z(z - \alpha)(z - \beta) \\ = \; & [(z - (\alpha + \beta)) - c_0](z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

This leads to

$$\begin{aligned} & c_0(1 - \bar{\alpha}z)(1 - \bar{\beta}z)[(z - (\alpha + \beta)) - c_0]^2 + z(z - \alpha)(z - \beta) \\ = \; & [(z - (\alpha + \beta)) - c_0](z - \alpha)(z - \beta)(1 - (\bar{\alpha} + \bar{\beta})z). \end{aligned}$$

If $c_0 \neq 0$, then the above equality gives that $(z-\alpha)(z-\beta)$ is a factor of $[(z-(\alpha+\beta))-c_0]^2$. This is impossible.

If $c_0 = 0$, then we have

$$z(z-\alpha)(z-\beta) = [(z-(\alpha+\beta))](z-\alpha)(z-\beta)(1-(\bar{\alpha}+\bar{\beta})z).$$

to get

$$\bar{\alpha}+\bar{\beta} = 0$$

and hence $\alpha = -\beta$. It is also a contradiction. This completes the proof that $g_{\alpha,\beta}(w,z)$ is irreducible.

To prove (3), we note that if α equals β, an easy computation gives

$$\begin{aligned} f_{\alpha,\beta}(w,z) &= (w-z)[(1-\bar{\alpha}z)w+(z-\alpha)] \\ &\quad \times[w(w-\alpha)(1-\bar{\alpha}z)+z(z-\alpha)(1-\bar{\alpha}w)]. \end{aligned}$$

If $\alpha = -\beta$, we also have

$$f_{\alpha,\beta}(w,z) = (w-z)(w+z)[(1-\bar{\alpha}^2z^2)w^2+(z^2-\alpha^2)].$$

This completes the proof.

Proof of Theorem 2.1. Assume that ϕ is a Blaschke product with the fourth order. By the Bochner Theorem [17], ϕ has a critical point c in the unit disk. Let $\lambda = \phi(c)$ be the critical value of ϕ. Then there are two points α and β in the unit disk such that

$$\phi_\lambda \circ \phi \circ \phi_c(z) = \eta z^2 \phi_\alpha \phi_\beta$$

where η is a unimodule constant. Let ψ be $z^2\phi_\alpha\phi_\beta$. Since $\phi\circ\phi_c$ and ψ are mutually analytic function calculus of each other, both $M_{\phi\circ\phi_c}$ and M_ψ share reducing subspaces.

(1) If ϕ is equivalent to z^4, then ψ must equal a scalar multiple of z^4. By Theorem B in [14], M_ψ has exact four nontrivial minimal reducing subspaces

$$\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$$

where

$$\mathcal{M}_j = \bigvee\{z^n : n \equiv j \bmod 4\}$$

for $j = 1,2,3,4$. The four spaces above are also reducing subspaces for $M_{\phi\circ\phi_c}$. Noting

$$U_c^* M_{\phi\circ\phi_c} U_c = M_\phi,$$

we have that M_ϕ has exact four nontrivial minimal reducing subspaces

$$\{U_c\mathcal{M}_1, U_c\mathcal{M}_2, U_c\mathcal{M}_3, U_c\mathcal{M}_4\}.$$

(2) If ϕ is decomposable but not equivalent to z^4, i.e, $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with degrees two and not both ψ_1 and ψ_2 are scalar multiples of z^2, by Lemmas 2.2 and 2.3, then α equals either β or $-\beta$ but does not equal 0. By Theorem 1.2, the restriction of M_{ψ_2} on $\mathcal{M}_0(\psi_2)$ is unitarily equivalent to the Bergman shift. Thus $\mathcal{M}_0(\psi_2)$ is also a reducing subspace of M_ϕ and the restriction of $M_\phi = M_{\psi_1\circ\psi_2}$ on $M_0(\psi_2)$ is unitarily equivalent to M_{ψ_1} on the Bergman space. By Theorem 1.2 again, there is a unique reducing subspace $\mathcal{M}_0(\psi_1)$ on which the restriction M_{ψ_1} is unitarily equivalent to the Bergman shift. Thus there is a subspace of $\mathcal{M}_0(\psi_2)$ on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift. Theorem 1.2 implies that $\mathcal{M}_0(\phi)$ is contained in $\mathcal{M}_0(\psi_2)$. Therefore $\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)$ is also a minimal reducing subspace of M_ϕ and

$$L_a^2 = \mathcal{M}_0(\phi) \oplus [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By Theorems 3.1 in [16], $\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2)\ominus\mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$ are nontrivial minimal reducing subspaces of M_ϕ. We will show that they are exact nontrivial minimal reducing subspaces of M_ϕ. If this is not true, then there is another minimal reducing subspace Ω of M_ϕ. By Theorem 38 [9], we have

$$\Omega \subset [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \oplus [\mathcal{M}_0(\psi_2)]^\perp.$$

By Theorem 1.3, there is a unitary operator

$$U : [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] \to [\mathcal{M}_0(\psi_2)]^\perp$$

which commutes with both M_ϕ and M_ϕ^*. But

$$dimker M_\phi^* \cap [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)] = 1$$

and

$$dimker M_\phi^* \cap [\mathcal{M}_0(\psi_2)]^\perp = 2.$$

This is a contradiction. Thus $\{\mathcal{M}_0(\phi), [\mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi)], [\mathcal{M}_0(\psi_2)]^\perp\}$ are exact nontrivial minimal reducing subspaces of M_ϕ.

(3) If ϕ is not decomposable, by Lemma 2.3, then ϕ equals $z^3\phi_\alpha$ or $z^2\phi_\alpha\phi_\beta$ for two nonzero points α β in $\mathbb{D}$ and α does not equal β or $-\beta$. The difficult cases will be dealt with in Sections 3 and 4. By Theorems 3.1 and 4.1, M_ϕ has exact two nontrivial minimal reducing subspaces $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

3. Reducing subspaces of $M_{z^3\phi_\alpha}$

In this section we will study reducing subspaces of $M_{z^3\phi_\alpha}$ for a nonzero point $\alpha \in \mathbb{D}$. Recall that $\mathcal{M}_0$ is the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem 1.2.

Theorem 3.1. *Let $\phi = z^3\phi_\alpha$ for a nonzero point $\alpha \in \mathbb{D}$. Then $\phi(\mathcal{B})$ has exact two nontrivial reducing subspaces $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.*

Proof. Let $\mathcal{M}_0$ be the distinguished reducing subspace of $\phi(\mathcal{B})$ as in Theorem 1.2. By Theorem 1.3, we only need to show that $\mathcal{M}_0^\perp$ is a minimal reducing subspace for $\phi(\mathcal{B})$.

Assume that $\mathcal{M}_0^\perp$ is not a minimal reducing subspace for $\phi(\mathcal{B})$. Then by Theorem 3.1 in [16] we may assume

$$\mathcal{H} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a nontrivial reducing subspace for $\phi(\mathcal{B})$, $M_0 = \mathcal{M}_0$ is the distinguished reducing subspace for $\phi(\mathcal{B})$ and

$$\mathcal{M}_0^\perp = M_1 \oplus M_2.$$

Recall that

$$\phi_0 = z^2\phi_\alpha,$$
$$L_0 = span\{1, p_1, p_2, k_\alpha(z)k_\alpha(w)\},$$

and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

We further assume that

$$dim(M_1 \cap L_0) = 1$$

and

$$dim(M_2 \cap L_0) = 2.$$

Take $0 \neq e_1 \in M_1 \cap L_0$, $e_2, e_3 \in M_2 \cap L_0$ such that $\{e_2, e_3\}$ are a basis for $M_2 \cap L_0$, then

$$L_0 = span\{e_0, e_1, e_2, e_3\}$$

By (1.1), we have

$$d^0_{e_j} = we_j(0,w)e_0 - \phi(w)e_j$$

and direct computations show that

$$\begin{aligned}
\langle d^0_{e_j}, p_k\rangle &= \langle we_j(0,w)e_0 - \phi(w)e_j, p_k\rangle \\
&= \langle we_j(0,w)e_0, p_k\rangle \qquad\qquad (by\ T^*_{\phi(w)}p_k = 0) \\
&= \langle we_j(0,w)e_0(w,w), p_k(0,w)\rangle \\
&= \langle we_j(0,w)\phi'(w), w^k\rangle \\
&= \langle w^3 e_j(0,w)(w\phi'_\alpha(w) + 3\phi_\alpha(w)), w^k\rangle \\
&= \langle w^{3-k} e_j(0,w)(w\phi'_\alpha(w) + 3\phi_\alpha(w)), 1\rangle \\
&= 0
\end{aligned}$$

for $0 \leq k \leq 2$, and

$$\begin{aligned}
\langle d^0_{e_j}, k_\alpha(z)k_\alpha(w)\rangle &= \alpha e_j(0,\alpha)e_0(\alpha,\alpha) \\
&= \alpha e_j(0,\alpha)\frac{\alpha^3}{1-|\alpha|^2}.
\end{aligned}$$

This implies that those functions $d^0_{e_j}$ are orthogonal to $\{1, p_1, p_2\}$.

Simple calculations give

$$\langle e_0, p_k\rangle = 0$$

for $0 \leq k \leq 1$,

$$\begin{aligned}
\langle e_0, p_2\rangle &= \langle e_0(0,w), p_2(w,w)\rangle \\
&= \frac{3}{2}\phi_0''(0) \\
&= -3\alpha \neq 0
\end{aligned}$$

and

$$\begin{aligned}
\langle e_0, k_\alpha(z)k_\alpha(w)\rangle &= e_0(\alpha,\alpha) \\
&= \phi'(\alpha) \\
&= \frac{\alpha^3}{1-|\alpha|^2} \neq 0
\end{aligned}$$

By Theorem 1.1, there are numbers μ, λ_j such that

$$d^1_{e_1} = d^0_{e_1} + \mu e_1 + \lambda_1 e_0$$

$$d^1_{e_2} = d^0_{e_2} + \tilde{e_2} + \lambda_2 e_0$$

$$d^1_{e_3} = d^0_{e_3} + \tilde{e_3} + \lambda_3 e_0$$

where $\tilde{e_2}, \tilde{e_3} \in M_2 \cap L_0$.

Now we consider two cases. In each case we will derive a contradiction.

Case 1. $\mu \neq 0$. In this case, we get that e_1 is orthogonal to $\{1, p_1\}$. So $\{1, p_1, e_0, e_1\}$ form an orthogonal basis for L_0.

First we show that $\tilde{e_2} = 0$. If $\tilde{e_2} \neq 0$, then we get that $\{1, p_1, e_0, \tilde{e_2}\}$ are also an orthogonal basis for L_0. Thus

$$\tilde{e_2} = ce_1$$

for some nonzero number c. However, $\tilde{e_2}$ is orthogonal to e_1 since $\tilde{e_2} \in M_2$ and $e_1 \in M_1$. This is a contradiction. Thus

$$d^1_{e_2} = d^0_{e_2} + \lambda_2 e_0.$$

Since both $d^1_{e_2}$ and $d^0_{e_2}$ are orthogonal to p_2 and

$$\langle e_0, p_2 \rangle = -3\alpha \neq 0,$$

we have that $\lambda_2 = 0$ to get that $d^0_{e_2} = d^1_{e_2}$ is orthogonal to L_0. On the other hand,

$$\langle d^0_{e_2}, k_\alpha(z)k_\alpha(w) \rangle = \alpha e_2(0, \alpha) \frac{\alpha^3}{1 - |\alpha|^2}.$$

Thus

$$e_2(0, \alpha) = 0.$$

Similarly we get that

$$e_3(0, \alpha) = 0.$$

Moreover, since e_2 and e_3 are orthogonal to $\{e_0, e_1\}$, write

$$e_2 = c_{11} + c_{12}p_1,$$

$$e_3 = c_{21} + c_{22}p_1.$$

Thus we have

$$e_2(0, \alpha) = c_{11} + c_{12}\alpha = 0,$$
$$e_3(0, \alpha) = c_{21} + c_{22}\alpha = 0,$$

to get that e_2 and e_3 are linearly dependent. This leads to a contradiction in this case.

Case 2. $\mu = 0$. In this case we have

$$d^1_{e_1} = d^0_{e_1} + \lambda_1 e_0.$$

Similarly to the proof in **Case 1** we get that $\lambda_1 = 0$,

$$d^1_{e_1} = d^0_{e_1} \perp L_0 \tag{3.1}$$

and

$$e_1(0, \alpha) = 0.$$

Theorem 2.2 in [16] gives that at least one $\tilde{e}_j$, say $\tilde{e}_2$ does not equal 0. Assume that $\tilde{e}_2 \neq 0$, write

$$\tilde{e_2} = d^1_{e_2} - d^0_{e_2} - \lambda_2 e_0.$$

Note that we have shown above that both $d^0_{e_2}$ and e_0 are orthogonal to both 1 and p_1. Thus

$$\tilde{e}_2 \perp \{1, p_1\}$$

and

$$L_0 = span\{1, p_1, e_0, \tilde{e}_2\}.$$

Since e_1 is orthogonal to $\{e_0, \tilde{e}_2\}$ we have

$$e_1 = c_1 + c_2 p_1.$$

Noting that $e_1(0, \alpha) = c_1 + c_2\alpha = 0$ we get

$$e_1 = c_2(-\alpha + p_1).$$

Without loss of generality we assume that

$$e_1 = -\alpha + p_1. \tag{3.2}$$

Letting e be in $M_2 \cap L_0$ such that e is a nonzero function orthogonal to $\tilde{e}_2$, we have that e is orthogonal to $\{e_0, \tilde{e}_2\}$. Thus e must be in the subspace $span\{1, p_1\}$. So there are two constants b_1 and b_2 such that

$$e = b_1 + b_2 p_1.$$

Noting

$$\begin{aligned} 0 &= \langle e, e_1 \rangle \\ &= -b_1\bar{\alpha} + 2b_2 \end{aligned}$$

we have

$$e = \frac{b_1}{2}(2 + \bar{\alpha}p_1).$$

Hence we may assume that

$$e = 2 + \bar{\alpha}p_1. \tag{3.3}$$

By Theorem 1.1 we have

$$d_e^1 = d_e^0 + \tilde{e} + \lambda e_0$$

for some number λ and $\tilde{e} \in M_2 \cap L_0$. Thus

$$\begin{aligned} 0 &= \langle d_{e_1}^1, d_e^1 \rangle \\ &= \langle d_{e_1}^1, d_e^0 + \tilde{e} + \lambda e_0 \rangle \\ &= \langle d_{e_1}^1, d_e^0 \rangle \\ &= \langle d_{e_1}^0, d_e^0 \rangle \qquad (by\ (3.1)). \end{aligned}$$

However, a simple computation gives

$$\begin{aligned} \langle d_{e_1}^0, d_e^0 \rangle &= \langle d_{e_1}^0, we(0,w)e_0 - \phi(w)e \rangle \\ &= \langle d_{e_1}^0, we(0,w)e_0 \rangle \qquad (by\ T_{\phi(w)}^* d_{e_1}^0 = 0) \\ &= \langle we_1(0,w)e_0 - \phi(w)e_1, we(0,w)e_0 \rangle \\ &= \langle we_1(0,w)e_0, we(0,w)e_0 \rangle - \langle \phi(w)e_1, we(0,w)e_0 \rangle. \end{aligned}$$

We need to calculate two terms in the right hand of the above equality. By (3.2) and (3.3), the first term becomes

$$\begin{aligned} & \langle we_1(0,w)e_0, we(0,w)e_0 \rangle \\ = & \langle w(-\alpha + w)e_0, w(2 + \bar{\alpha}w)e_0 \rangle \\ = & \langle (-\alpha + w)e_0, (2 + \bar{\alpha}w)e_0 \rangle \\ = & \langle -\alpha e_0, 2e_0 \rangle + \langle we_0, 2e_0 \rangle + \langle -\alpha e_0, \bar{\alpha}we_0 \rangle + \langle we_0, \bar{\alpha}we_0 \rangle \\ = & -\alpha\langle e_0, e_0 \rangle + 2\langle we_0, e_0 \rangle - \alpha^2\langle e_0, we_0 \rangle. \end{aligned}$$

The first term in right hand of the last equality is

$$\begin{aligned}\langle e_0, e_0\rangle &= \langle e_0(w,w), e_0(0,w)\rangle \\ &= \langle w\phi_0^{'} + \phi_0, \phi_0\rangle \\ &= \langle w(2w\phi_\alpha + w^2\phi_\alpha^{'}), w^2\phi_\alpha\rangle + \langle \phi_0, \phi_0\rangle. \\ &= 2 + \langle w\phi_\alpha^{'}, \phi_\alpha\rangle + 1 \\ &= 4.\end{aligned}$$

The last equality follows from

$$\begin{aligned}\phi_\alpha &= -\frac{1}{\bar{\alpha}} + \frac{\frac{1}{\bar{\alpha}} - \alpha}{1-\bar{\alpha}w} \\ &= -\frac{1}{\bar{\alpha}} + (\frac{1}{\bar{\alpha}} - \alpha)K_\alpha(w).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\langle we_0, e_0\rangle &= \langle we_0(w,w), e_0(0,w)\rangle \\ &= \langle w(w\phi_0^{'} + \phi_0), \phi_0\rangle \\ &= \alpha.\end{aligned}$$

This gives

$$\begin{aligned}\langle we_1(0,w)e_0, we(0,w)e_0\rangle &= \langle e_1(0,w)e_0, e(0,w)e_0\rangle \\ &= \langle (-\alpha + w)e_0, (2+\bar{\alpha}w)e_0\rangle \\ &= -2\alpha\langle e_0, e_0\rangle - \alpha^2\langle e_0, we_0\rangle \\ &\quad +2\langle we_0, e_0\rangle + \alpha\langle we_0, we_0\rangle \\ &= -8\alpha - \alpha|\alpha|^2 + 2\alpha + 4\alpha \\ &= -2\alpha - \alpha|\alpha|^2\end{aligned}$$

A simple calculation gives that the second term becomes

$$\begin{aligned}&\langle \phi(w)e_1, we(0,w)e_0\rangle \\ =\ &\langle \phi_0(w)e_1, (2+\bar{\alpha}w)e_0\rangle \\ =\ &\langle \phi_0(w)e_1, 2e_0\rangle + \langle \phi_0(w)e_1, \bar{\alpha}we_0\rangle \\ =\ &2\langle \phi_0(w)e_1(w,w), e_0(0,w)\rangle + \alpha\langle \phi_0(w)e_1(w,w), we_0(0,w)\rangle \\ =\ &2\langle e_1(w,w), 1\rangle + \alpha\langle e_1(w,w), w\rangle \\ =\ &2\langle -\alpha + 2w, 1\rangle + \alpha\langle -\alpha + 2w, w\rangle = -2\alpha + 2\alpha = 0.\end{aligned}$$

Thus we conclude

$$\begin{aligned}\langle d_{e_1}^0, d_e^0\rangle &= \langle we_1(0,w)e_0, we(0,w)e_0\rangle - \langle \phi(w)e_1, we(0,w)e_0\rangle \\ &= -2\alpha - \alpha|\alpha|^2 \\ &= -\alpha(2+|\alpha|^2) \neq 0\end{aligned}$$

to get a contradiction in this case. This completes the proof.

4. REDUCING SUBSPACES FOR $M_{z^2\phi_\alpha\phi_\beta}$

In this section we will classify minimal reducing subspaces of $M_{z^2\phi_\alpha\phi_\beta}$ for two nonzero points α and β in $\mathbb{D}$ and with $\alpha \neq \beta$.

Theorem 4.1. *Let ϕ be the Blaschke product $z^2\phi_\alpha\phi_\beta$ for two nonzero points α and β in $\mathbb{D}$. If α does not equal either β or $-\beta$, then $\phi(\boldsymbol{\mathcal{B}})$ has exact two nontrivial reducing subspaces $\{\mathcal{M}_0, \mathcal{M}_0^\perp\}$.*

Proof. By Theorem 27 in [9], if $\mathcal{N}$ is a nontrivial minimal reducing subspace of $\phi(\boldsymbol{\mathcal{B}})$ which is not equal to $\boldsymbol{\mathcal{M}}_0$ then $\boldsymbol{\mathcal{N}}$ is a subspace of $\boldsymbol{\mathcal{M}}_0^\perp$, so we only need to show that $\boldsymbol{\mathcal{M}}_0^\perp$ is a minimal reducing subspace for $\phi(\boldsymbol{\mathcal{B}})$ unless $\alpha = -\beta$.

Assume that $\boldsymbol{\mathcal{M}}_0^\perp$ is not a minimal reducing subspace for $\phi(\boldsymbol{\mathcal{B}})$. By Theorem 3.1 in [16], we may assume

$$\boldsymbol{\mathcal{H}} = \bigoplus_{i=0}^{2} M_i$$

such that each M_i is a reducing subspace for $\phi(\boldsymbol{\mathcal{B}})$, $M_0 = \boldsymbol{\mathcal{M}}_0$ is the distinguished reducing subspace for $\phi(\boldsymbol{\mathcal{B}})$ and

$$M_1 \oplus M_2 = \boldsymbol{\mathcal{M}}_0^\perp.$$

Recall that

$$\phi_0 = z\phi_\alpha\phi_\beta,$$

$$L_0 = span\{1, p_1, e_\alpha, e_\beta\},$$

with $e_\alpha = k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w)$ and

$$L_0 = (L_0 \cap M_0) \oplus (L_0 \cap M_1) \oplus (L_0 \cap M_2).$$

So we further assume that the dimension of $M_1 \cap L_0$ is one and the dimension of $M_2 \cap L_0$ is two. Take a nonzero element e_1 in $M_1 \cap L_0$, then by Theorem 1.1, there are numbers μ_1, λ_1 such that

$$d_{e_1}^1 = d_{e_1}^0 + \mu_1 e_1 + \lambda_1 e_0. \tag{4.1}$$

We only need to consider two possibilities, μ_1 is zero or nonzero.

If μ_1 is zero, then (4.1) becomes

$$d_{e_1}^1 = d_{e_1}^0 + \lambda_1 e_0. \tag{4.2}$$

In this case, simple calculations give

$$\begin{aligned}
\langle d_{e_1}^0, p_1\rangle &= \langle we_1(0,w)e_0(z,w) - w\phi_0(w)e_1(z,w), p_1(z,w)\rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(z,w)\rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), p_1(0,w)\rangle \\
&= \langle we_1(0,w)e_0(w,w) - w\phi_0(w)e_1(w,w), w\rangle \\
&= \langle e_1(0,w)e_0(w,w) - \phi_0(w)e_1(w,w), 1\rangle \\
&= e_1(0,0)e_0(0,0) - \phi_0(0)e_1(0,0) = 0,
\end{aligned}$$

and

$$\begin{aligned}\langle e_0, p_1\rangle &= \langle e_0(z,w), p_1(z,w)\rangle \\ &= \langle e_0(z,w), p_1(w,w)\rangle \\ &= \langle e_0(0,w), 2w\rangle \\ &= \langle \phi_0(w), 2w\rangle \\ &= 2\langle w\phi_\alpha(w)\phi_\beta(w), w\rangle \\ &= 2\phi_\alpha(0)\phi_\beta(0) = 2\alpha\beta \neq 0.\end{aligned}$$

Noting that $d^1_{e_1}$ is orthogonal to L_0, by (4.2) we have that $\lambda_1 = 0$, and hence

$$d^0_{e_1} = d^1_{e_1} \perp L_0.$$

So

$$\langle d^0_{e_1}, e_\alpha\rangle = 0 = \langle d^0_{e_1}, e_\beta\rangle.$$

On the other hand,

$$\begin{aligned}\langle d^0_{e_1}, e_\alpha\rangle &= \alpha e_1(0,\alpha)e_0(\alpha,\alpha) - \alpha\phi_0(\alpha)e_1(\alpha,\alpha) \\ &= \alpha e_1(0,\alpha)e_0(\alpha,\alpha)\end{aligned}$$

and

$$\begin{aligned}\langle d^0_{e_1}, e_\beta\rangle &= \beta e_1(0,\beta)e_0(\beta,\beta) - \beta\phi_0(\beta)e_1(\beta,\beta) \\ &= \beta e_1(0,\beta)e_0(\beta,\beta).\end{aligned}$$

Consequently

$$e_1(0,\alpha) = e_1(0,\beta) = 0. \tag{4.3}$$

Observe that e_0, e_1 and 1 are linearly independent. If this is not so, then $1 = ae_0 + be_1$ for some numbers a, b. But $e_1(0,\alpha) = 0$ and $e_0(0,\alpha) = 0$. This forces that $1 = 0$ and leads to a contradiction.

By Theorem 1.1, we can take an element $e \in M_2 \cap L_0$ such that

$$d^1_e = d^0_e + e_2 + \mu e_0$$

with $e_2 \neq 0$ and $e_2 \in M_2 \cap L_0$. Thus we have that e_2 is orthogonal to 1 and so e_2 is in $\{1, e_0, e_1\}^\perp$ and $\{1, e_0, e_1, e_2\}$ form a basis for L_0. Moreover for any $f \in M_2 \cap L_0$,

$$d^1_f = d^0_f + g + \lambda e_0$$

for some number λ and $g \in M_2 \cap L_0$. If g does not equal 0 then g is orthogonal to 1. Thus g is in $\{1, e_0, e_1\}^\perp$ and hence

$$g = ce_2$$

for some number c. Therefore taking a nonzero element $e_3 \in M_2 \cap L_0$ which is orthogonal to e_2, we have

$$d^1_{e_2} = d^0_{e_2} + \mu_2 e_2 + \lambda_2 e_0,$$

$$d^1_{e_3} = d^0_{e_3} + \mu_3 e_2 + \lambda_3 e_0,$$

and $\{e_0, e_1, e_2, e_3\}$ is an orthogonal basis for L_0.

If $\mu_2 = 0$, then by the same reason as before we get

$$\begin{aligned} \lambda_2 &= 0, \\ d^0_{e_2} &= d^1_{e_2} \perp L_0 \\ e_2(0,\alpha) &= e_2(0,\beta) \\ &= 0. \end{aligned}$$

So using

$$p_1 \in L_0 = span\{1, e_0, e_1, e_2\}$$

we have

$$\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta,$$

which contradicts our assumption that $\alpha \neq \beta$. Hence $\mu_2 \neq 0$.

Observe that 1 is in $L_0 = span\{e_0, e_1, e_2, e_3\}$ and orthogonal to both e_0 and e_2. Thus

$$1 = c_1 e_1 + c_3 e_3$$

for some numbers c_1 and c_3. So

$$\begin{aligned} 1 \quad &= c_1 e_1(0,\alpha) + c_3 e_3(0,\alpha) \\ &= c_1 e_1(0,\beta) + c_3 e_3(0,\beta). \end{aligned}$$

By (4.3), we have

$$1 = c_3 e_3(0,\alpha) = c_3 e_3(0,\beta),$$

to obtain that $c_3 \neq 0$ and

$$e_3(0,\alpha) = e_3(0,\beta) = 1/c_3.$$

If $\mu_3 = 0$, then by the same reason as before we get $e_3(0,\alpha) = e_3(0,\beta) = 0$. Hence $\mu_3 \neq 0$.

Now by the linearality of $d^1_{(\cdot)}$ and $d^0_{(\cdot)}$ we have

$$d^1_{\mu_3 e_2 - \mu_2 e_3} = d^0_{\mu_3 e_2 - \mu_2 e_3} + (\mu_3\lambda_2 - \mu_2\lambda_3)e_0.$$

By the same reason as before we get

$$\mu_3\lambda_2 - \mu_2\lambda_3 = 0$$

and

$$d^0_{\mu_3 e_2 - \mu_2 e_3} = d^1_{\mu_3 e_2 - \mu_2 e_3} \perp L_0$$

and therefore

$$\begin{aligned} \mu_3 e_2(0,\alpha) - \mu_2 e_3(0,\alpha) &= \mu_3 e_2(0,\beta) - \mu_2 e_3(0,\beta) \\ &= 0. \end{aligned}$$

So we get

$$e_2(0,\alpha) = \mu_2/\mu_3 c_3 = e_2(0,\beta).$$

Hence

$$p_1 \in L_0 = span\{1, e_0, e_1, e_2\}.$$

This implies that

$$\alpha = p_1(0,\alpha) = p_1(0,\beta) = \beta$$

which again contradicts our assumption that $\alpha \neq \beta$.

Another case is that μ_1 is not equal to 0. In this case, (4.1) can be rewritten as

$$e_1 = \frac{1}{\mu_1} d^1_{e_1} - \frac{1}{\mu_1} d^0_{e_1} - \frac{\lambda_1}{\mu_1} e_0,$$

and we have that e_1 is orthogonal to 1 since $d^1_{e_1}$, $d^0_{e_1}$ and e_0 are orthogonal to 1. Thus 1 is in $M_2 \cap L_0$.

By Theorem 1.1, there is an element $e \in M_2 \cap L_0$ and a number λ_0 such that

$$d^1_1 = d^0_1 + e + \lambda_0 e_0. \tag{4.4}$$

If $e = 0$ then $\lambda_0 = 0$, and hence $d^0_1 \perp L_0$ and

$$1 = 1(0, \alpha) = 1(0, \beta).$$

So $e \neq 0$.

Since d^1_1 is in $L_0^\perp$, d^1_1 is orthogonal to 1. Noting that d^0_1 and e_0 are orthogonal to 1, we have that $e \perp 1$. Hence we get an orthogonal basis $\{e_0, e_1, 1, e\}$ of L_0.

Claim.

$$e(0, \alpha) - e(0, \beta) = 0.$$

Proof of the claim. Using Theorem 1.1 again, we have that

$$d^1_e = d^0_e + g + \lambda e_0$$

for some $g \in L_0 \cap M_2$. If $g \neq 0$, we have that $g \perp 1$ since d^1_e, d^0_e, and e_0 are orthogonal to 1. Thus we have that $g = \mu e$ for some number μ to obtain

$$d^1_e = d^0_e + \mu e + \lambda e_0.$$

Furthermore by the linearality of $d^1_{(\cdot)}$ and $d^0_{(\cdot)}$ we have that

$$d^1_{e-\mu 1} = d^0_{e-\mu 1} + (\lambda - \mu\lambda_0) e_0.$$

By the same reason (namely $d^1_{e-\mu 1} \perp L_0$, $d^0_{e-\mu 1} \perp 1$ and $\langle e_0, 1\rangle \neq 0$) we have that

$$\lambda - \mu\lambda_0 = 0,$$

$$d^0_{e-\mu 1} = d^1_{e-\mu 1} \perp L_0$$

and

$$(e - \mu 1)(0, \alpha) = (e - \mu 1)(0, \beta) = 0.$$

Hence we have

$$e(0, \alpha) - e(0, \beta) = \mu - \mu = 0,$$

to complete the proof of the claim.

Let us find the value of λ_0 in (4.4) which will be used to make the coefficients symmetric with respect to α and β. To do this, we first state a technical lemma which will be used in several other places in the sequel.

Lemma 4.2. *If g is in $H^2(\mathbb{T})$, then*

$$\langle w g \phi_0', \phi_0\rangle = g(0) + g(\alpha) + g(\beta).$$

Proof. Since ϕ_0 equals $z\phi_\alpha\phi_\beta$, simple calculations give

$$\begin{aligned}
\langle wg\phi_0^{'}, \phi_0\rangle &= \langle wg(w\phi_\alpha\phi_\beta)^{'}, w\phi_\alpha\phi_\beta\rangle \\
&= \langle g(w\phi_\alpha\phi_\beta)^{'}, \phi_\alpha\phi_\beta\rangle \\
&= \langle g(\phi_\alpha\phi_\beta + w\phi_\alpha^{'}\phi_\beta + w\phi_\alpha\phi_\beta^{'}), \phi_\alpha\phi_\beta\rangle \\
&= \langle g, 1\rangle + \langle wg\phi_\alpha^{'}, \phi_\alpha\rangle + \langle wg\phi_\beta^{'}, \phi_\beta\rangle \\
&= g(0) + \langle wg\phi_\alpha^{'}, \phi_\alpha\rangle + \langle wg\phi_\beta^{'}, \phi_\beta\rangle
\end{aligned}$$

Writing ϕ_α as

$$\begin{aligned}
\phi_\alpha &= -\frac{1}{\bar{\alpha}} + \frac{\frac{1}{\bar{\alpha}} - \alpha}{1 - \bar{\alpha}w} \\
&= -\frac{1}{\bar{\alpha}} + \frac{1 - |\alpha|^2}{\bar{\alpha}} k_\alpha(w),
\end{aligned}$$

we have

$$\begin{aligned}
\langle wg\phi_\alpha^{'}, \phi_\alpha\rangle &= \frac{1 - |\alpha|^2}{\alpha}(wg\phi_\alpha^{'})(\alpha) \\
&= g(\alpha).
\end{aligned}$$

The first equality follows from $\langle wg\phi_\alpha^{'}, 1\rangle$ equals 0 and the second equality follows from

$$\phi_\alpha^{'}(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

By the symmetry of α and β, similar computations lead to

$$\langle wg\phi_\beta^{'}, \phi_\beta\rangle = g(\beta)$$

and the proof is finished.

We state the values of λ_0 and $\langle e_0, e_0\rangle$ as a lemma.

Lemma 4.3.

$$\lambda_0 = -\frac{\alpha + \beta}{4} \tag{4.5}$$

$$\langle e_0, e_0\rangle = 4 \tag{4.6}$$

Proof. Since d_1^1 is orthogonal to L_0, e_0 is in L_0, and e is orthogonal to e_0, (4.4) gives

$$\begin{aligned}
0 &= \langle d_1^1, e_0\rangle \\
&= \langle d_1^0 + e + \lambda_0 e_0, e_0\rangle \\
&= \langle d_1^0, e_0\rangle + \lambda_0\langle e_0, e_0\rangle.
\end{aligned}$$

We need to compute $\langle d_1^0, e_0\rangle$ and $\langle e_0, e_0\rangle$ respectively.

$$\begin{aligned}\langle d_1^0, e_0\rangle &= \langle -\phi(w) + we_0, e_0\rangle \\ &= \langle we_0, e_0\rangle \\ &= \langle we_0(w,w), e_0(0,w)\rangle \\ &= \langle w(w\phi_0^{'} + \phi_0), \phi_0\rangle \\ &= \langle w^2\phi_0^{'}, \phi_0\rangle + \langle w\phi_0, \phi_0\rangle \\ &= \langle w^2\phi_0^{'}, \phi_0\rangle \\ &= \alpha + \beta.\end{aligned}$$

The last equality follows from Lemma 4.2 with $g = w$.

$$\begin{aligned}\langle e_0, e_0\rangle &= \langle e_0(w,w), e_0(0,w)\rangle \\ &= \langle w\phi_0^{'} + \phi_0, \phi_0\rangle \\ &= \langle w\phi_0^{'}, \phi_0\rangle + \langle \phi_0, \phi_0\rangle \\ &= \langle w\phi_0^{'}, \phi_0\rangle + 1 \\ &= 4,\end{aligned}$$

where the last equality follows from Lemma 4.2 with $g = 1$. Hence

$$\alpha + \beta + 4\lambda_0 = 0$$

and

$$\lambda_0 = -\frac{\alpha+\beta}{4}.$$

Let P_{L_0} denote the projection of $H^2(\mathbb{T}^2)$ onto L_0. The element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ has the property that for any $g \in L_0$,

$$\begin{aligned}\langle g, P_{L_0}(k_\alpha(w) - k_\beta(w))\rangle &= \langle g, k_\alpha(w) - k_\beta(w)\rangle \\ &= g(0,\alpha) - g(0,\beta).\end{aligned}$$

Thus $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to g for $g \in L_0$ with

$$g(0,\alpha) = g(0,\beta).$$

So $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is orthogonal to $e_0, 1, e$. On the other hand,

$$\begin{aligned}\langle p_1, P_{L_0}(k_\alpha(w) - k_\beta(w))\rangle &= \alpha - \beta \\ &\neq 0.\end{aligned}$$

This gives that the element $P_{L_0}(k_\alpha(w) - k_\beta(w))$ is a nonzero element. Therefore there exists a nonzero number b such that

$$P_{L_0}(k_\alpha(w) - k_\beta(w)) = be_1.$$

Without loss of generality we assume that

$$e_1 = P_{L_0}(k_\alpha(w) - k_\beta(w)).$$

Observe that

$$\begin{aligned} p_1(\phi(z),\phi(w))e_1 + d^1_{e_1} &\in M_1, \\ p_1(\phi(z),\phi(w)) + d^1_1 &\in M_2, \\ M_1 &\perp M_2, \end{aligned}$$

to get

$$\langle p_1(\phi(z),\phi(w))e_1 + d^1_{e_1}, p_1(\phi(z),\phi(w)) + d^1_1\rangle = 0.$$

Thus we have

$$\begin{aligned} 0 &= \langle p_1(\phi(z),\phi(w))e_1 + d^1_{e_1}, p_1(\phi(z),\phi(w)) + d^1_1\rangle \\ &= \langle (\phi(z)+\phi(w))e_1, \phi(z)+\phi(w)\rangle + \langle d^1_{e_1}, d^1_1\rangle \\ &= \langle d^1_{e_1}, d^1_1\rangle. \end{aligned} \tag{4.7}$$

The second equality follows from

$$d^1_{e_1}, d^1_1 \in kerT^*_{\phi(z)} \cap kerT^*_{\phi(z)}.$$

The last equality follows from

$$e_1 \perp 1$$

and

$$e_1, 1 \in kerT^*_{\phi(z)} \cap kerT^*_{\phi(z)}.$$

Substituting (4.4) into Equation (4.7), we have

$$\begin{aligned} 0 &= \langle d^1_{e_1}, d^0_1 + e + \lambda_0 e_0\rangle \\ &= \langle d^1_{e_1}, d^0_1\rangle \\ &= \langle d^1_{e_1}, -\phi(w) + we_0\rangle \\ &= \langle d^1_{e_1}, we_0\rangle \\ &= \langle d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0, we_0\rangle \\ &= \langle d^0_{e_1}, we_0\rangle + \mu_1\langle e_1, we_0\rangle + \lambda_1\langle e_0, we_0\rangle. \end{aligned}$$

The second equation comes from that $d^1_{e_1}$ is orthogonal to L_0 and both e and e_0 are in L_0. The third equation follows from the definition of d^0_1 and the forth equation follows from that $d^1_{e_1}$ is in $kerT^*_{\phi(z)} \cap kerT^*_{\phi(w)}$. We need to calculate $\langle d^0_{e_1}, we_0\rangle$, $\langle e_1, we_0\rangle$, and $\langle e_0, we_0\rangle$ separately.

To get $\langle d^0_{e_1}, we_0\rangle$, by the definition of $d^0_{e_1}$, we have

$$\begin{aligned} \langle d^0_{e_1}, we_0\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, we_0\rangle \\ &= \langle -\phi(w)e_1, we_0\rangle + \langle we_1(0,w)e_0, we_0\rangle \end{aligned}$$

Thus we need to compute $\langle -\phi(w)e_1, we_0\rangle$ and $\langle we_1(0,w)e_0, we_0\rangle$ one by one. The equality

$$\langle -\phi(w)e_1, we_0\rangle = 0$$

follows from the following computations.

$$\begin{aligned}\langle -\phi(w)e_1, we_0\rangle &= \langle -w\phi_0(w)e_1, we_0\rangle\\ &= -\langle \phi_0(w)e_1, e_0\rangle\\ &= -\langle \phi_0(w)e_1(w,w), e_0(0,w)\rangle\\ &= -\langle \phi_0(w)e_1(w,w), \phi_0(w)\rangle\\ &= -\langle e_1(w,w), 1\rangle\\ &= -\langle e_1, 1\rangle\\ &= 0.\end{aligned}$$

To get $\langle we_1(0,w)e_0, we_0\rangle$, we continue as follows.

$$\begin{aligned}\langle we_1(0,w)e_0, we_0\rangle &= \langle e_1(0,w)e_0, e_0\rangle\\ &= \langle e_1(0,w)e_0(w,w), e_0(0,w)\rangle\\ &= \langle e_1(0,w)e_0(w,w), \phi_0(w)\rangle\\ &= \langle e_1(0,w)(\phi_0(w)+w\phi_0^{'}(w)), \phi_0(w)\rangle\\ &= \langle e_1(0,w)\phi_0(w), \phi_0(w)\rangle + \langle e_1(0,w)w\phi_0^{'}(w), \phi_0(w)\rangle\\ &= \langle e_1(0,w), 1\rangle + \langle e_1(0,w)w\phi_0^{'}(w), \phi_0(w)\rangle\\ &= e_1(0,0) + \langle e_1(0,w)w\phi_0^{'}(w), \phi_0(w)\rangle\\ &= \langle e_1, 1\rangle + \langle e_1(0,w)w\phi_0^{'}(w), \phi_0(w)\rangle\\ &= \langle e_1(0,w)w\phi_0^{'}(w), \phi_0(w)\rangle\\ &= e_1(0,\alpha) + e_1(0,\beta).\end{aligned}$$

The last equality follows from Lemma 4.2 and

$$e_1(0,0) = \langle e_1, 1\rangle = 0.$$

Hence

$$\langle d^0_{e_1}, we_0\rangle = e_1(0,\alpha) + e_1(0,\beta)$$

Recall that

$$d_1^1 = d_1^0 + e + \lambda_0 e_0$$

is orthogonal to L_0 and e_1 is orthogonal to both e, and e_0. Thus

$$\begin{aligned}0 &= \langle e_1, d_1^0 + e + \lambda_0 e_0\rangle\\ &= \langle e_1, -\phi(w) + we_0\rangle\\ &= \langle e_1, we_0\rangle.\end{aligned}$$

From the computation of $\langle d_1^0, e_0\rangle$ in the proof of Lemma 4.3 we have showed that

$$\langle we_0, e_0\rangle = \alpha + \beta.$$

Therefore we have that

$$e_1(0,\alpha) + e_1(0,\beta) + \lambda_1(\bar{\alpha} + \bar{\beta}) = 0. \tag{4.8}$$

On the other hand,

$$\begin{aligned} 0 &= \langle d^1_{e_1}, e_0\rangle \\ &= \langle d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0, e_0\rangle \\ &= \langle d^0_{e_1}, e_0\rangle + 4\lambda_1 \end{aligned}$$

and

$$\begin{aligned} \langle d^0_{e_1}, e_0\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, e_0\rangle \\ &= \langle we_1(0,w)e_0, e_0\rangle \\ &= \langle we_1(0,w)e_0(w,w), e_0(0,w)\rangle \\ &= \langle we_1(0,w)(\phi_0(w) + w\phi_0^{'}), \phi_0(w)\rangle \\ &= \langle w^2 e_1(0,w)\phi_0^{'}, \phi_0(w)\rangle \\ &= \alpha e_1(0,\alpha) + \beta e_1(0,\beta). \end{aligned}$$

The last equality follows from Lemma 4.2 with $g = we_1(0,w)$. Thus

$$\alpha e_1(0,\alpha) + \beta e_1(0,\beta) + 4\lambda_1 = 0.$$

So

$$\lambda_1 = -\frac{\alpha}{4}e_1(0,\alpha) - \frac{\beta}{4}e_1(0,\beta). \tag{4.9}$$

Substituting (4.9) into (4.8), we have

$$[1 - \frac{\alpha(\bar\alpha + \bar\beta)}{4}]e_1(0,\alpha) + [1 - \frac{\beta(\bar\alpha + \bar\beta)}{4}]e_1(0,\beta) = 0.$$

Recall that

$$\lambda_0 = -\frac{\alpha+\beta}{4},$$

to get

$$(1 + \bar\lambda_0\alpha)e_1(0,\alpha) + (1 + \bar\lambda_0\beta)e_1(0,\beta) = 0. \tag{4.10}$$

We are going to draw another equation about $e_1(0,\alpha)$ and $e_1(0,\beta)$ from the property that $d^1_{e_1}$ is orthogonal to L_0. To do this, recall that

$$\begin{aligned} e_1 &= P_{L_0}(k_\alpha(w) - k_\beta(w)) \in M_1 \cap L_0, \\ d^1_{e_1} &= d^0_{e_1} + \mu_1 e_1 + \lambda_1 e_0 \perp L_0, \\ L_0 &= span\{1, p_1, e_\alpha, e_\beta\}, \\ e_\alpha &= k_\alpha(z)k_\alpha(w), e_\beta = k_\beta(z)k_\beta(w). \end{aligned}$$

Thus $d^1_{e_1}$ is orthogonal to p_1, e_α and e_β.

Since $d^1_{e_1}$ is orthogonal to p_1 we have

$$\langle d^0_{e_1}, p_1\rangle + \mu_1\langle e_1, p_1\rangle + \lambda_1\langle e_0, p_1\rangle = 0.$$

Noting

$$\begin{aligned}\langle d_{e_1}^0, p_1\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, p_1\rangle\\ &= \langle we_1(0,w)e_0, p_1\rangle\\ &= \langle we_1(0,w)e_0(w,w), w\rangle\\ &= \langle e_1(0,w)e_0(w,w), 1\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle e_1, p_1\rangle &= \langle P_{L_0}(K_\alpha(w) - K_\beta(w)), p_1\rangle\\ &= \langle K_\alpha(w) - K_\beta(w), p_1\rangle\\ &= \bar\alpha - \bar\beta,\end{aligned}$$

and

$$\begin{aligned}\langle e_0, p_1\rangle &= \langle e_0(0,w), p_1(w,w)\rangle\\ &= \langle \phi_0(w), 2w\rangle\\ &= \langle w\phi_\alpha\phi_\beta, 2w\rangle\\ &= 2\langle \phi_\alpha\phi_\beta, 1\rangle\\ &= 2\phi_\alpha(0)\phi_\beta(0)\\ &= 2\alpha\beta,\end{aligned}$$

we have

$$(\bar\alpha - \bar\beta)\mu_1 + 2\alpha\beta\lambda_1 = 0,$$

to obtain

$$\lambda_1 = -\mu_1\frac{\bar\alpha - \bar\beta}{2\alpha\beta}. \tag{4.11}$$

Since $d_{e_1}^1 \perp e_\alpha$, we have

$$\langle d_{e_1}^0, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle + \lambda_1\langle e_0, e_\alpha\rangle = 0,$$

to get

$$\langle d_{e_1}^0, e_\alpha\rangle + \mu_1\langle e_1, e_\alpha\rangle - \mu_1\frac{\bar\alpha - \bar\beta}{2\alpha\beta}\langle e_0, e_\alpha\rangle = 0. \tag{4.12}$$

We need to calculate $\langle d_{e_1}^0, e_\alpha\rangle$, $\langle e_1, e_\alpha\rangle$ and $\langle e_0, e_\alpha\rangle$. Simple calculations show that

$$\begin{aligned}\langle d_{e_1}^0, e_\alpha\rangle &= \langle -\phi(w)e_1 + we_1(0,w)e_0, e_\alpha\rangle\\ &= \langle we_1(0,w)e_0, e_\alpha\rangle\\ &= \alpha e_1(0,\alpha)e_0(\alpha,\alpha),\end{aligned}$$

$$\begin{aligned}\langle e_1, e_\alpha\rangle &= e_1(\alpha,\alpha)\\ &= \langle P_{L_0}(k_\alpha(w) - k_\beta(w)), e_\alpha\rangle\\ &= \langle k_\alpha(w) - k_\beta(w), e_\alpha\rangle\\ &= \frac{1}{1-|\alpha|^2} - \frac{1}{1-\alpha\bar\beta}\end{aligned}$$

$$= \quad \frac{\alpha(\bar{\alpha}-\bar{\beta})}{(1-|\alpha|^2)(1-\alpha\bar{\beta})}, \tag{4.13}$$

and

$$\langle e_0, e_\alpha\rangle = e_0(\alpha,\alpha) \quad = \quad \alpha\phi_0^{'}(\alpha)+\phi_0(\alpha)$$

$$= \quad \alpha^2\frac{1}{1-|\alpha|^2}\frac{\alpha-\beta}{1-\alpha\bar{\beta}}. \tag{4.14}$$

Thus (4.13) and (4.14) give

$$\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)} = \frac{\bar{\alpha}-\bar{\beta}}{\alpha(\alpha-\beta)}.$$

Substituting the above equality in Equation (4.12) leads to

$$\alpha e_1(0,\alpha)e_0(\alpha,\alpha)+\mu_1 e_1(\alpha,\alpha)-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}e_0(\alpha,\alpha)=0.$$

Dividing the both sides of the above equality by $e_0(\alpha,\alpha)$ gives

$$\alpha e_1(0,\alpha)+\mu_1\frac{e_1(\alpha,\alpha)}{e_0(\alpha,\alpha)}-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}=0.$$

Hence we have

$$\alpha e_1(0,\alpha)+\mu_1\frac{\bar{\alpha}-\bar{\beta}}{\alpha(\alpha-\beta)}-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}=0,$$

to obtain

$$\alpha e_1(0,\alpha)+(\beta+\lambda_0)\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)}=0. \tag{4.15}$$

Similarly, since $d^1_{e_1}$ is orthogonal to e_β, we have

$$\langle d^0_{e_1}, e_\beta\rangle+\mu_1\langle e_1, e_\beta\rangle+\lambda_1\langle e_0, e_\beta\rangle=0,$$

to obtain

$$\langle d^0_{e_1}, e_\beta\rangle+\mu_1\langle e_1, e_\beta\rangle-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}\langle e_0, e_\beta\rangle=0. \tag{4.16}$$

We need to calculate $\langle d^0_{e_1}, e_\beta\rangle$, $\langle e_1, e_\beta\rangle$ and $\langle e_0, e_\beta\rangle$. Simple calculations as above show that

$$\begin{aligned}
\langle d^0_{e_1}, e_\beta\rangle &= \langle -\phi(w)e_1+we_1(0,w)e_0, e_\beta\rangle\\
&= \langle we_1(0,w)e_0, e_\beta\rangle\\
&= \beta e_1(0,\beta)e_0(\beta,\beta),\\
\langle e_1, e_\beta\rangle &= e_1(\beta,\beta)\\
&= \langle P_{L_0}(k_\alpha(w)-k_\beta(w)), e_\beta\rangle\\
&= \langle k_\alpha(w)-k_\beta(w), e_\beta\rangle\\
&= \frac{1}{1-\bar{\alpha}\beta}-\frac{1}{1-|\beta|^2}
\end{aligned}$$

$$= \frac{\beta(\bar{\alpha}-\bar{\beta})}{(1-\bar{\alpha}\beta)(1-|\beta|^2)} \tag{4.17}$$

$$\langle e_0, e_\beta\rangle = e_0(\beta,\beta) = \beta\phi_0'(\beta)+\phi_0(\beta)$$

$$= \beta^2\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\frac{1}{1-|\beta|^2} \tag{4.18}$$

Combining (4.17) with (4.18) gives

$$\frac{e_1(\beta,\beta)}{e_0(\beta,\beta)} = -\frac{\bar{\alpha}-\bar{\beta}}{\beta(\alpha-\beta)}.$$

Substituting the above equality in (4.16) gives

$$\beta e_1(0,\beta)e_0(\beta,\beta)+\mu_1 e_1(\beta,\beta)-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}e_0(\beta,\beta)=0.$$

Dividing both sides of the above equality by $e_0(\beta,\beta)$ gives

$$\beta e_1(0,\beta)+\mu_1\frac{e_1(\beta,\beta)}{e_0(\beta,\beta)}-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}=0$$

Hence we have

$$\beta e_1(0,\beta)-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{\beta(\alpha-\beta)}-\mu_1\frac{\bar{\alpha}-\bar{\beta}}{2\alpha\beta}=0,$$

to get

$$\beta e_1(0,\beta)-(\alpha+\lambda_0)\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)}=0. \tag{4.19}$$

Eliminating $\frac{2\mu_1(\bar{\alpha}-\bar{\beta})}{\alpha\beta(\alpha-\beta)}$ from (4.15) and (4.19) gives

$$\alpha(\alpha+\lambda_0)e_1(0,\alpha)+\beta(\beta+\lambda_0)e_1(0,\beta)=0. \tag{4.20}$$

Now combining (4.10) and (4.20), we have the following linear system of equations about $e_1(0,\alpha)$ and $e_1(0,\beta)$

$$\begin{aligned}(1+\bar{\lambda}_0\alpha)e_1(0,\alpha)+(1+\bar{\lambda}_0\beta)e_1(0,\beta)&=0\\ \alpha(\alpha+\lambda_0)e_1(0,\alpha)+\beta(\beta+\lambda_0)e_1(0,\beta)&=0.\end{aligned} \tag{4.21}$$

If

$$e_1(0,\alpha)=e_1(0,\beta)=0,$$

then p_1 is in $L_0 = span\{e_0, e_1, 1, e\}$. But noting

$$e_0(0,\alpha)=e_0(0,\beta)$$

and

$$e(0,\alpha)=e(0,\beta)$$

we have

$$p_1(0,\alpha)=p_1(0,\beta),$$

which contradicts the assumption that $\alpha \neq \beta$. So at least one of $e_1(0,\alpha)$ and $e_1(0,\beta)$ is nonzero. Then the determinant of the coefficient matrix of System (4.21) has to be zero. This implies

$$\begin{vmatrix} 1+\bar\lambda_0\alpha & 1+\bar\lambda_0\beta \\ \alpha(\alpha+\lambda_0) & \beta(\beta+\lambda_0) \end{vmatrix} = 0$$

Making elementary row reductions on the above the determinant, we get

$$\begin{vmatrix} (\alpha-\beta)\bar\lambda_0 & 1+\bar\lambda_0\beta \\ (\alpha-\beta)(\alpha+\beta+\lambda_0) & \beta(\beta+\lambda_0) \end{vmatrix} = 0.$$

Since

$$\alpha+\beta = -4\lambda_0$$

and

$$\alpha-\beta \neq 0,$$

we have

$$\begin{vmatrix} \bar\lambda_0 & 1+\bar\lambda_0\beta \\ -3\lambda_0 & \beta(\beta+\lambda_0) \end{vmatrix} = 0.$$

Expanding this determinant we have

$$\begin{aligned} 0 &= \bar\lambda_0(\beta^2+\beta\lambda_0) + 3\lambda_0(1+\bar\lambda_0\beta) \\ &= \bar\lambda_0(\beta^2+\beta\lambda_0+3\beta\lambda_0) + 3\lambda_0 \\ &= \bar\lambda_0(\beta^2+4\beta\lambda_0) + 3\lambda_0 \\ &= \bar\lambda_0(-\alpha\beta) + 3\lambda_0 \end{aligned}$$

Taking absolute value on both sides of the above equation, we have

$$\begin{aligned} 0 &= |\bar\lambda_0(-\alpha\beta) + 3\lambda_0| \\ &\geq |\lambda_0|(3-|\alpha\beta|) \\ &\geq 2|\lambda_0|, \end{aligned}$$

to get

$$\lambda_0 = 0.$$

This implies

$$\alpha+\beta = 0,$$

to complete the proof.

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