SEAM, Mar 6 2008

Matrices & Varieties

I have always had a hegemonic view of analysis. Despite M. Atiyah's claim that math is either geometry or algebra, I hew to a more Freudian view: a field of mathematics is not mature until it has been analyzed.

Today, I want to convince you to annex alg. geometry.

First, show why it is natural

2nd, convince you we are tooled up.

$\$$ T is a matrix

(Even algebraists acknowledge the importance of matrices)

Normalize so $\|T\| \leq 1$ (contra)

Tcm ('51, von N): $\|p(T)\| \leq \|p\|_{\mathbb{D}}$.

<u>Case</u>: $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$p(z) = \sum a_n z^n$, $\quad p(T) = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$

$$\|p(T)\| = \sqrt{\frac{|a_1|^2 + 2|a_0|^2 + \sqrt{|a_1|^4 + 4|a_0|^2 |a_1|^2}}{2}}$$

$$\le 1 \iff |a_1| \le 1 - |a_0|^2$$

"Schwarz-Pick Lemma"

von Neumann's inequality, in hindsight, is easy to prove: any reasonable approach works, & $\exists$ $\frac{1}{2}$ a dozen different ones.

Not very algebraic though: use same set $\overline{\mathbb{D}}$ for all ctrns, & even for nilpotent ones $\sigma(T) = \{0\}$, can't get by $\bar{c}$ smaller set if constant stays 1

(though B&F Delyon:

$W(T) \subseteq \overline{\mathbb{D}} \implies$ $\xrightarrow{}$? if $p(0) = 0$

$$\|p(T)\| \le 2^3 \|p\|_{\overline{\mathbb{D}}} .$$

So now T_1 & T_2 are commuting contractions

Andô (1963): $\|p(T_1, T_2)\| \leq \|p\|_{\mathbb{D}^2}$.  ① D-Z Schwarz [illegible] for $\mathbb{D}^2$, but essentially [illegible] → Knese

$\|p(T_1,T_2)\| = \|p(T_1, T_1^3)\| \leq \sup_{z \in \mathbb{D}} |p(z, z^3)|$

② Supps $T_2 = T_1^3$. or dim space > 2, not always have $T_2 = f(T_1)$, $f \in A$,

But → ③ Can replace $\mathbb{D}^2$ by $\mathbb{D}^2 \cap \{\text{algebraic set}\}$.

How? Assume $T_1 v_j = \lambda^1_j v_j$

$T_2 v_j = \lambda^2_j v_j$

~~&, for notational convenience, all $\lambda_j \in \mathbb{R}$.~~

$\|T_1\| \leq 1 \iff I - T_1^* T_1 \geq 0$

(Reason that op thy on Hilbert spaces is richer than Ban spa)

$$0 \leq (1 - \bar\lambda^1_i \lambda^1_j) \langle v_j, v_i \rangle = \langle u^1_j, u^1_i \rangle$$

$\|$ly $$0 \leq (1 - \bar\lambda^2_i \lambda^2_j) \langle v_j, v_i \rangle = \langle u^2_j, u^2_i \rangle$$

$$(1 - \bar\lambda^1_i \lambda^1_j) \langle u^2_j, u^2_i \rangle = (1 - \bar\lambda^2_i \lambda^2_j) \langle u^1_j, u^1_i \rangle$$

$$\left\langle \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix}, \begin{pmatrix} u^2_i \\ \lambda^2_i u^1_i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}, \begin{pmatrix} \lambda^1_i u^2_i \\ u^1_i \end{pmatrix} \right\rangle$$

$$\therefore U : \sum c_j \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix} \longmapsto \sum c_j \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}$$

is isometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix} = \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}$$

$$C u^2_j + D \lambda^2_j u^1_j = u^1_j$$

$$u^1_j = (I - \lambda^2_j D)^{-1} C u^2_j$$

$$[A + B\lambda^2_j (I - \lambda^2_j D)^{-1} C]\, u^2_j = \lambda^1_j u^2_j$$

$$\Psi(\lambda) := A + B(I - \lambda D)^{-1} C$$

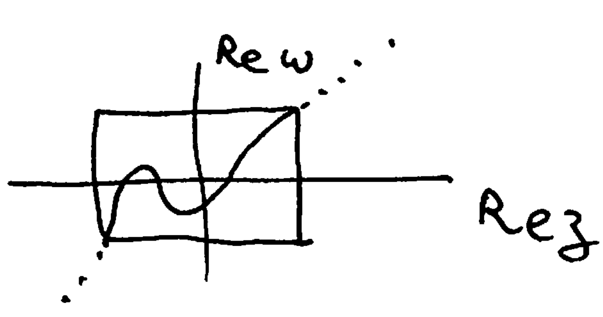
Ψ is analytic, matrix valued

$$I - \Psi(\lambda)^* \Psi(\lambda) = [1 - |\lambda|^2]\, C^*(I - \bar\lambda D^*)^{-1} (I - \lambda D)^{-1} C$$

Ψ is unitary valued ~~a.e.~~ on $\mathbb{T}$.

$$V := \{ (z, w) \in \bar{\mathbb{D}}^2 : \det[\Psi(w) - zI] = 0 \}$$

$$(\lambda^1_j, \lambda^2_j) \in V \quad \forall j$$



Model T_2 as M^*_{zI} on $H^2 \otimes \mathbb{C}^n \big|_{\vee\{s_{\lambda_j} \otimes u_j^2\}}$

T_1 as M^*_{Ψ}

Modulo $\lambda_j \mapsto \overline{\lambda_j}$

This is a matrix valued version of $T_1 = \Psi(T_2)$

Def: A d.v. is a set V st $\overset{(a)}{\exists}$ $\overset{1\text{-dim}}{\text{alg}}$ set $W = Z_p$ & $V = W \cap \overline{\mathbb{D}}^2$ (b) $V \cap \partial(\mathbb{D}^2) = V \cap \mathbb{T}^2$

Thm (Agler-McC, 2005):

(i) $\forall$ commuting $\overset{\text{contractive}}{\wedge}$ matrices T_1, T_2, $\exists$ V ∋

a {distinguished variety} ~~= 1 dim alg set $\cap\ \mathbb{D}^2$~~

st $\|p(T_1, T_2)\| \leq \|p\|_V$.

(ii) Every distinguished variety arises as

$$\{(z,w) \in \overline{\mathbb{D}}^2 : \det(\Psi(w) - zI) = 0\}$$

some rational inner Ψ. ← improved by Knese

Q1: When do $\mathcal{U} = \begin{array}{cc} \mathbb{C}^m & \mathbb{C}^n \\ \end{array} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ & $\mathcal{U}' = \begin{array}{cc} \mathbb{C}^m & \mathbb{C}^n \\ \end{array} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$

give same d.v?

Q2: When are two d.v.'s isomorphic?

Ad 1: Thm (Vegulla '07):

Answered in general ($\sigma(A) = \sigma(A')$, $\sigma(D) = \sigma(D')$

$\oint_{z^j} \operatorname{tr}[\Psi_{\mathcal{U}}(z)^i] = \oint_{z^j} \operatorname{tr}[\Psi_{\mathcal{U}'}(z)^i]$]

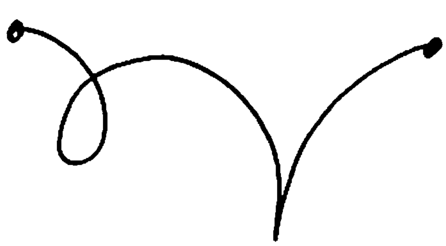
$i = 1, \ldots, [\frac{m}{2}]$, $j = 0, 1, \ldots, n-1$

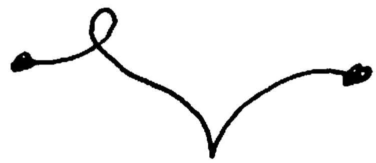
Case $m = 2, 3$:

$\sigma(A) = \sigma(A')$, $\sigma(D) = \sigma(D')$,

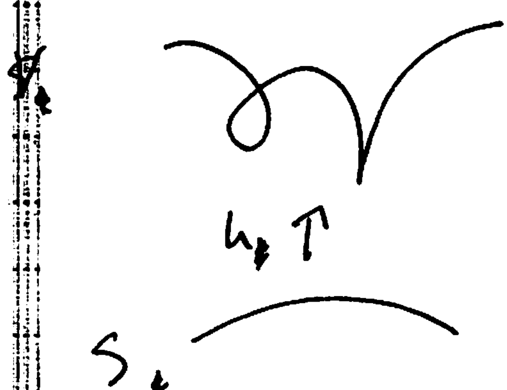
$\operatorname{tr}(B D^j C) = \operatorname{tr}(B' D'^j C')$ $\quad 0 \leq j \leq n-1$

<u>Ad 2:</u> Ask for any hyperbolic algebraic set
$:=$ (alg set) $\cap$ (bdd open set).





Remark: Q seems algebraic-geometric;
but alg. geom. studies global &
local geometry, not hyperbolic geometry
We are tooled up.
Idea is to pass to desingularization



$$A_h = \{ f \in O(S) : f = F \circ h,\ F \in O(V) \}.$$

Finite Codimension Subalgebra of $O(S)$

Thm (Agler-McC):
TFAE:

(i) $\exists\ \varphi : V_1 \to V_2$

(ii) $\exists\ \psi : S_1 \to S_2$, st diagram commutes
[illegible]

(iii) $\exists\ \psi : S_1 \to S_2$ st
$A_{h_1}(S_1) = \{ G \circ h_2 \circ \psi : G \in \mathcal{O}(V_2) \}$.

(iv) $\exists\ \psi$: cond $\mapsto$ cond

Example:
Simple cusp

$\uparrow h$

$\mathbb{D}$

h 1-1,
$h' = 0$ only at 0.

$$A_h = \{ f \in \mathcal{O}(\mathbb{D}) : \quad f'(0) = 0,$$
$$\text{Certain sums } \sum_{j=2}^{\text{fin}} c_j f^{(j)}(0) = 0 \}.$$

skip

$\text{con}(A) = \min\{ k \geq 1 : \exists f \in A,\ \hat{f}(k+1) \neq 0 \}$.

($\text{ord}(A) = \min\{ k : z^{k+1} \mathcal{O}(\mathbb{D}) \subseteq A \}$.)
Cod

i.e. cusp simple if $\exists f \in A_h, f''(0) \neq 0$

A simple (:= cod 1)
$\Rightarrow$ ord $A = 2$ cod $A - 1$.

Thm (a) Moduli space of simple cusps of codim
(easy) $n+1$ is $\mathbb{R}^+ \times \mathbb{C}^{n-1}$.

(b) All simple cusps can be realized in $\mathbb{C}^3$.
(less easy)

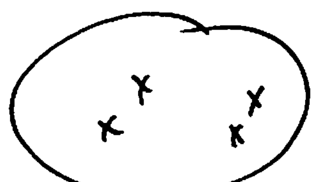
Q: ¿$\exists$ global embedding obstructions?

$A \subseteq \mathcal{O}(\mathbb{D})$, finite codim

~~$\exists\ V \in \mathbb{C}^N$~~ Then $\exists\, h: \mathbb{D} \to \mathbb{C}^N$
$V := h(\mathbb{D})$
st $A = \{F \circ h : F \in \mathcal{O}(V)\}$.

Locally: $A = \{f'(0) = 0 = f''(0)\}$
Need $N \geq 3$.

Global?
Simplest ex codim 2:



(Narasimhan, '60)

Embeddable?
(Can always do $\bar{c}$ 3 inner fns)

3 matrices:

$\exists\, R > 1$ st

T_1, T_2, T_3 commuting ctrs $\Rightarrow$

$$\|p(T_1, T_2, T_3)\| \leq \|p\|_{(R\,\mathbb{D})^3}.$$

If $T_3 = q(T_1, T_2)$, $\|q\| \leq 1$