SEAM, Mar 6 2008

Matrices & Varieties

I have always had a hegemonic view of analysis. Despite M. Atiyah's claim that math is either geometry or algebra, I hew to a more Freudian view: a field of mathematics is not mature until it has been analyzed.

Today, I want to convince you to annex alg. geometry.

First, show why it is natural

$2^{nd}$, convince you we are tooled up.

$ T is a matrix

(Even algebraists acknowledge the importance of matrices)

Normalize so $\|T\| \leq 1$ (contraction)

Thm ('51, von N): $\|p(T)\| \leq \|p\|_{\mathbb{D}}$.

Case: $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$p(z) = \sum a_n z^n$, $\quad p(T) = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$

$$\|p(T)\| = \sqrt{\frac{|a_1|^2 + 2|a_0|^2 + \sqrt{|a_1|^4 + 4|a_0|^2 |a_1|^2}}{2}}$$

$$\leq 1 \iff |a_1| \leq 1 - |a_0|^2$$

"Schwarz-Pick Lemma"

von Neumann's inequality, in hindsight, is easy to prove: any reasonable approach works, & $\exists$ $\frac{1}{2}$ a dozen different ones.

Not very algebraic though: use same set $\overline{\mathbb{D}}$ for all ctns, & even for nilpotent ones $\sigma(T) = \{0\}$, can't get by $\bar{c}$ smaller set if constant stays 1

(though B&F Delyon:

$W(T) \subseteq \overline{\mathbb{D}} \Rightarrow$ $\quad \to 2$ if $p(0)=0$

$$\|p(T)\| \leq 3 \|p\|_{\overline{\mathbb{D}}} .$$

$ now $T_1$ & $T_2$ are commuting contractions

(2) Supps $T_2 = T_1^3$. or dim space $> 2$, wit always have $T_2 = f(T_1)$, $f \in A$,

Andô (1963): $\|p(T_1, T_2)\| \le \|p\|_{\mathbb{D}^2}$.

D-B Schwarz lemma for $\mathbb{D}^2$, but essentially uses symmetries → Knese

$$\|p(T_1,T_2)\| = \|p(T_1, T_1^3)\| \overset{\text{von N}}{\le} \sup_{z \in \mathbb{D}} |p(z, z^3)|$$

But → (3) Can replace $\mathbb{D}^2$ by $\mathbb{D}^2 \cap \{\text{algebraic set}\}$.

How? Assume

$$T_1 v_j = \lambda^1_j v_j$$

$$T_2 v_j = \lambda^2_j v_j$$

~~&, for notational convenience, all $\lambda^r_j \in \mathbb{R}$.~~

$$\|T_1\| \le 1 \iff I - T_1^* T_1 \geq 0$$

(Reason that op thy on Hilbert spaces is richer than Ban spa)

$$0 \le (1 - \bar{\lambda}^1_i \lambda^1_j) \langle v_j, v_i \rangle = \langle u^1_j, u^1_i \rangle$$

||ly

$$0 \le (1 - \bar{\lambda}^2_i \lambda^2_j) \langle v_j, v_i \rangle = \langle u^2_j, u^2_i \rangle$$

$$(1 - \bar{\lambda}^1_i \lambda^1_j) \langle u^2_j, u^2_i \rangle = (1 - \bar{\lambda}^2_i \lambda^2_j) \langle u^1_j, u^1_i \rangle$$

$$\left\langle \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix}, \begin{pmatrix} u^2_i \\ \lambda^2_i u^1_i \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}, \begin{pmatrix} \lambda^1_i u^2_i \\ u^1_i \end{pmatrix} \right\rangle$$

$$\therefore U : \sum c_j \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix} \longmapsto \sum c_j \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}$$

is isometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u^2_j \\ \lambda^2_j u^1_j \end{pmatrix} = \begin{pmatrix} \lambda^1_j u^2_j \\ u^1_j \end{pmatrix}$$

$$C u^2_j + D \lambda^2_j u^1_j = u^1_j$$

$$u^1_j = (I - \lambda^2_j D)^{-1} C u^2_j$$

$$[A + B\lambda_j (I - \lambda^2_j D)^{-1} C]\, u^2_j = \lambda^1_j u^2_j$$

$$\Psi(\lambda) := A + B(I - \lambda D)^{-1} C$$

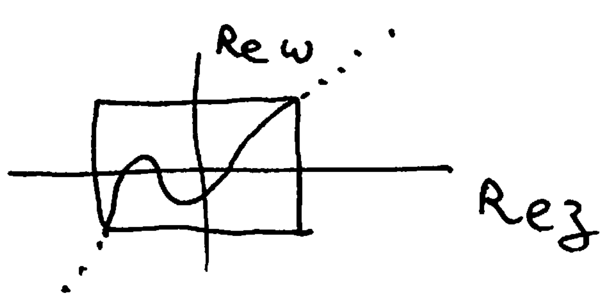
$\Psi$ is analytic, matrix valued

$$I - \Psi(\lambda)^* \Psi(\lambda) = [1 - |\lambda|^2]\, C^* (I - \bar{\lambda} D^*)^{-1} (I - \lambda D)^{-1} C$$

$\Psi$ is unitary valued ~~a.e.~~ on $\mathbb{T}$.

$$V := \{ (z, w) \in \bar{\mathbb{D}}^2 : \det[\Psi(w) - zI] = 0 \}$$

$$(\lambda^1_j, \lambda^2_j) \in V \quad \forall\, j$$



Model $T_2$ as $M^*_{zI}$ on $H^2 \otimes \mathbb{C}^n \Big|_{\vee\{s_{\lambda_j} \otimes u^2_{ij}\}}$

$T_1$ as $M^*_{\Psi}$

Modulo $\lambda_j \mapsto \overline{\lambda_j}$

This is a matrix valued version of $T_1 = \Psi / T_2$

**Def:** A d.v. is a set $V$ st (a) $\exists$ 1-dim alg set $W = Z_p$ & $V = W \cap \overline{\mathbb{D}}^2$ (b) $V \cap \partial(\mathbb{D}^2) = V \cap \mathbb{T}^2$

Thm (Agler-McC, 2005):

(i) $\forall$ commuting contractive matrices $T_1, T_2$, $\exists$ $V$ ∋ a {distinguished variety} ~~= 1 dim alg set $\cap \mathbb{D}^2$~~

st $\|p(T_1, T_2)\| \leq \|p\|_V$.

(ii) Every distinguished variety arises as

$$\{(z,w) \in \overline{\mathbb{D}}^2 : \det(\Psi(w) - zI) = 0\}$$

some rational inner $\Psi$. ← improved by Knese

Q1: When do
$$\mathcal{U} = \begin{array}{cc} \mathbb{C}^m & \mathbb{C}^n \\ \multicolumn{2}{c}{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} \end{array} \quad \& \quad \mathcal{U}' = \begin{array}{cc} \mathbb{C}^m & \mathbb{C}^n \\ \multicolumn{2}{c}{\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}} \end{array}$$
give same d.v?

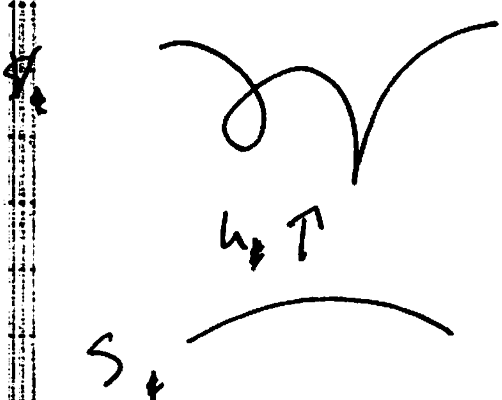
Q2: When are two d.v.'s isomorphic?

Ad 1: Thm (Vegulla '07):
Answered in general ( $\sigma(A) = \sigma(A')$, $\sigma(D) = \sigma(D')$
$\frac{d}{dz^i} \operatorname{tr}[\Psi_{\mathcal{U}}(z)^i] = \frac{d}{dz^i} \operatorname{tr}[\Psi_{\mathcal{U}'}(z)^i]$
$i = 1, \ldots, [\frac{m}{2}]$, $j = 0, 1, \ldots$ )

Case $m = 2, 3$:
$$\sigma(A) = \sigma(A'), \quad \sigma(D) = \sigma(D'),$$
$$\operatorname{tr}(B D^j C) = \operatorname{tr}(B' D'^j C') \qquad 0 \leq j \leq n-1$$

Ad 2: Ask for any hyperbolic algebraic set
:= (alg set) $\cap$ (bdd open set).

Remark: Q seems algebraic-geometric;
but alg. geom. studies global &
local geometry, not hyperbolic geometry
We are tooled up.
Idea is to pass to desingularization




$$A_h = \{ f \in \mathcal{O}(S) : f = F \circ h,\ F \in \mathcal{O}(V) \}.$$

Finite Codimension Subalgebra of $\mathcal{O}(S)$

Thm (Agler-McC):
TFAE:

(i) $\exists \quad \varphi : V_1 \to V_2$

(ii) $\exists \quad \psi : S_1 \to S_2$, st diagram commutes

(iii) $\exists\, \psi : S_1 \to S_2$ st
$A_{h_1}(S_1) = \{ G \circ h_2 \circ \psi : G \in \mathcal{O}(V_2) \}$.

(iv) $\exists\, \psi$ : conn $\mapsto$ conn

Example:
Simple cusp


$h$ 1-1,
$h' = 0$ only at $0$.

$A_h = \{ f \in \mathcal{O}(\mathbb{D}) : \quad f'(0) = 0,$

Certain sums $\sum_{j=2}^{\text{fin}} c_j f^{(j)}(0) = 0 \}$.

skip

$\mathrm{con}(A) = \min\{ k \geq 1 : \exists f \in A, \quad \hat{f}(k+1) \neq 0 \}$.

$\mathrm{ord}(A) = \min\{ k : z^{k+1} \mathcal{O}(\mathbb{D}) \subseteq A \}$.

Cod

ivy cusp simple if $\exists f \in A_h, f''(0) \neq 0$.

A simple (:= con 1)
$\Rightarrow \operatorname{ord} A = 2 \operatorname{cod} A - 1$.

Thm(a) Moduli space of simple cusps of codim
(easy) $n+1$ is $\mathbb{R}^+ \times \mathbb{C}^{n-1}$.

(b) All simple cusps can be realized in $\mathbb{C}^3$.
(less easy)

Q: ¿∃ global embedding obstructions?

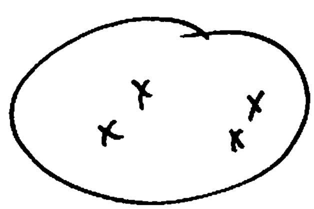
$A \subseteq \mathcal{O}(\mathbb{D})$, finite codim

~~$\exists\, V \in \mathbb{C}^N$~~, Then $\exists\, h : \mathbb{D} \to \mathbb{C}^N$
$V := h(\mathbb{D})$

st $A = \{ F \circ h : F \in \mathcal{O}(V) \}$.

Locally: $A = \{ f'(0) = 0 = f''(0) \}$
Need $N \geq 3$.

Global?
Simplest ex codim 2:



(Narasimhan, '60)

Embeddable?
(Can always do $\bar{c}$ 3 inner fns)

3 matrices:

$\exists\, R > 1$ st

$T_1, T_2, T_3$ commuting ctns $\Rightarrow$

$$\|p(T_1, T_2, T_3)\| \le \|p\|_{(R\,\mathbb{D})^3}.$$

If $T_3 = q(T_1, T_2)$, $\|q\| \le 1$