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Products of Bergman space Toeplitz operators on the polydisk

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Abstract Motivated by recent works of Ahern and Čučković on the disk, we study the generalized zero product problem for Toeplitz operators acting on the Bergman space of the polydisk. First, we extend the results to the polydisk. Next, we study the generalized compact product problem. Our results are new even on the disk. As a consequence on higher dimensional polydisks, we show that the generalized zero and compact product properties are the same for Toeplitz operators in a certain case.

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1 Introduction

Let D be the unit disk in the complex plane. For a fixed positive integer n, the unit polydisk D^n is the cartesian product of n copies of D. Let $L^p = L^p(D^n)$

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denote the usual Lebesgue space with respect to the volume measure $V = V_n$ on D^n normalized to have total mass 1. The Bergman space A^2 is then space of all L^2-holomorphic functions on D^n. Due to the mean value property of holomorphic functions, the space A^2 is a closed subspace of L^2, and thus is a Hilbert space. The Bergman projection P is defined to be the Hilbert space orthogonal projection from L^2 onto A^2. For a function $u \in L^\infty$, the *Toeplitz operator* T_u *with symbol* u is defined by

$$T_u f = P(uf)$$

for $f \in A^2$. It is clear that $T_u : A^2 \to A^2$ is a bounded linear operator.

In their recent paper Ahern and Čučković [2] studied the "generalized zero product" problem for Toeplitz operators acting on the Bergman space of the unit disk. Namely, they studied the problem of when the product of two Toeplitz operators is another Toeplitz operator. More explicitly, they considered bounded harmonic symbols u, v and bounded C^2-symbol σ with bounded invariant Laplacian and proved that *if* $T_u T_v = T_\sigma$, *then either* u *is co-holomorphic or* v *is holomorphic, and in either case* $\sigma = uv$. More recently, Ahern [1] has shown that, for bounded harmonic symbols u, v and L^∞-symbol σ, $T_u T_v = T_\sigma$ *if and only if either* u *is co-holomorphic or* v *is holomorphic, and* $\sigma = uv$.

The results of Ahern and Čučković ([1,2]) quite naturally suggest further studies at least in two directions. One is to investigate the same generalized zero product problem on higher dimensional domains and the other is to investigate the analogous "generalized compact product" problem. In this paper we take the polydisk as our domain and study the generalized zero and compact product problems.

Working on higher dimensional polydisks, we naturally consider pluriharmonic symbols in place of harmonic ones. Let $C^2 = C^2(D^n)$ for simplicity. Recall that a complex-valued function $u \in C^2$ is said to be *pluriharmonic* if its restriction to an arbitrary complex line that intersects D^n is harmonic as a function of single complex variable. It turns out that every pluriharmonic function on D^n can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and a co-holomorphic function. Note that every pluriharmonic u satisfies

$$\partial_j \overline{\partial}_i u = 0, \quad i, j = 1, 2, \ldots, n$$

where ∂_j denotes the complex partial differentiation with respect to jth variable. See Chapt. 2 of [9] for details.

For the generalized zero product problem, our main result is Theorem 1 below. For $n = 1$, the theorem is just a restatement of the result of Ahern mentioned above. While our method is basically adapted from [1], substantial amount of extra work is necessary for the setting of higher dimensional polydisks.

Theorem 1 *Let* $u, v \in L^\infty$ *be pluriharmonic symbols and assume* $\sigma \in L^\infty$. *Then the following statements are equivalent:*

(a) $T_u T_v = T_\sigma$.
(b) $\sigma = uv$ *and either* $\partial_j u = 0$ *or* $\partial_j \bar{v} = 0$ *for each* j.

When some additional assumptions are imposed on σ in Theorem 1, one might be able to derive a more concrete characterization. In [2] the case of harmonic σ is considered and it is deduced that characterization reduces to a complete triviality. As a higher dimensional analogue, we consider n-harmonic symbols σ. Recall that a function $u \in C^2$ is called *n-harmonic* as in [12] if u is harmonic in each variable separately. More explicitly, u is n-harmonic if

$$\partial_j \overline{\partial}_j u = 0, \quad j = 1, 2, \ldots, n.$$

For a more concrete description when the symbol σ is n-harmonic, see Theorem 5. It turns out that the characterization is symmetric in u and v. Also, our result shows that the characterization for $n \geq 2$ does not reduce to such a triviality as in [2].

We need more terminology and notation. We let $C_0 = C_0(D^n)$ denote the class of all continuous functions ψ on D^n such that $\psi(a) \to 0$ as $a \to \partial D^n$ where ∂D^n denotes the topological boundary of D^n. Also, we let $\mathscr{A} \subset L^\infty$ denote the algebra of functions that are uniformly continuous with respect to the pseudohyperbolic distance; see Sect. 2 for definition.

For the generalized compact product problem, our result is Theorem 2 below. It is new even on the disk. Moreover, the case $\sigma = uv$ recovers the main result on the semi-commutator of two Toeplitz operators on the disk [16] and the analogous result on the polydisk [4]. In case σ is n-harmonic, we also have a symmetric characterization as in Theorem 7. Meanwhile, in case symbols are continuous up to the boundary, the characterization for $n = 1$ can be much simplified as in Corollary 3.

Theorem 2 *Let* $u, v \in L^\infty$ *be pluriharmonic symbols and assume* $\sigma \in L^\infty$. *Then the following two statements are equivalent:*

(a) $T_u T_v - T_\sigma$ *is compact.*
(b) $T_u T_v - T_{uv}$ *and* $T_{uv-\sigma}$ *are both compact.*

If, in addition, $\sigma \in \mathscr{A}$, *then the above conditions are also equivalent to*

(c) $T_u T_v - T_{uv}$ *is compact and* $\sigma - uv \in C_0$.

The differences of the function theory between on the polydisks and on the disk have shown that the theory of Toeplitz operators on the polydisks is quite different, especially when pluriharmonic symbols are considered, from that on the disk as is shown in [4,6,8,15]. The next theorem provides another evidence for such differences.

Theorem 3 $(n \geq 2)$ *Let* $u, v \in L^\infty$ *be pluriharmonic symbols and assume that* $\sigma \in L^\infty$ *is an n-harmonic symbol. Then the following statements are equivalent:*

(a) $T_u T_v = T_\sigma$.
(b) $T_u T_v - T_\sigma$ *is compact.*

Of course, one cannot expect the same on the disk. For example, take $u = \overline{v} = z$ and $\sigma = 1$. Then, $T_z T_{\overline{z}} - I$ is compact by Theorem 2, but $T_z T_{\overline{z}} \neq I$ where $I = T_1$ denotes the identity operator. One cannot expect the same for general $\sigma \in \mathscr{A}$, either. For example, consider the case $u = v = 0$. In case the symbol σ is in addition pluriharmonic, there is a more concrete description which in turn yields the remarkably simple characterization for zero products of Toeplitz operators with pluriharmonic symbols (see Theorem 9 and Corollary 5).

This paper is organized as follows. In Sect. 2, we briefly review basic facts and some recent results on the Berezin transform which is the main tool for our proofs. In Sect. 3, we prove Theorem 1 and observe its consequences. In Sect. 4, we prove a strengthened version of Theorem 2 and observe its consequences. Finally, in Sect. 5, we prove Theorem 3 and observe its consequences.

2 Back ground: Berezin transform

One of the main tools in the theory of Toeplitz operators is the Berezin transform. We briefly review basic facts and some recent results related to the Berezin transform. Throughout the section we let $a \in D^n$ denote an arbitrary point, unless otherwise specified.

Since every point evaluation is a bounded linear functional on A^2, there corresponds to every $a \in D^n$ a unique function $K_a \in A^2$ which has following reproducing property:

$$f(a) = \langle f, K_a \rangle, \quad f \in A^2 \tag{2.1}$$

where the notation $\langle\ ,\ \rangle$ denotes the inner product in L^2. The function K_a is the well-known Bergman kernel and its explicit formula is given by

$$K_a(z) = \prod_{j=1}^{n} \frac{1}{(1 - \overline{a}_j z_j)^2}, \quad z \in D^n.$$

Here and elsewhere z_j denotes the jth component of z. We let k_a denote the normalized kernel, namely,

$$k_a(z) = \prod_{j=1}^{n} \frac{1 - |a_j|^2}{(1 - \overline{a}_j z_j)^2}, \quad z \in D^n. \tag{2.2}$$

We let $\varphi_a(z) = \big(\phi_{a_1}(z_1), \ldots, \phi_{a_n}(z_n)\big)$ where each ϕ_{a_j} is the usual Möbius map on D given by

$$\phi_{a_j}(z_j) = \frac{a_j - z_j}{1 - \overline{a}_j z_j}, \quad z_j \in D.$$

The map φ_a is an automorphism on D^n such that $\varphi_a \circ \varphi_a = id$. It has real Jacobian $\prod_{j=1}^{n} |\phi'_{a_j}(z_j)|^2$, which by (2.2) is equal to $|k_a(z)|^2$, so we have the following change of variable formula:

$$\int_{D^n} h\big(\varphi_a(z)\big)|k_a(z)|^2 \,\mathrm{dV}(z) = \int_{D^n} h(w)\,\mathrm{dV}(w) \tag{2.3}$$

for every $h \in L^1$.

Let $\mathfrak{L}(A^2)$ be the algebra of bounded linear operators on A^2. Recall that the *Berezin transform* of $S \in \mathfrak{L}(A^2)$ is the function $B[S]$ on D^n defined by

$$B[S](a) = \langle Sk_a, k_a\rangle.$$

It is easily seen that $B[S]$ is a continuous function on D^n. It has been recently proved by Coburn [5] that $B[S]$ is actually Lipschitz ρ-continuous where ρ denotes the *pseudohyperbolic* distance on D^n defined by $\rho(z,w) = \max_{1\le j\le n} |\phi_{z_j}(w_j)|$ for $z, w \in D^n$. More precisely, there is a constant $C_n > 0$ such that

$$|B[S](z) - B[S](w)| \le C_n ||S|| \rho(z,w) \tag{2.4}$$

for all $z, w \in D^n$.

For $u \in L^\infty$, we simply let $Bu = B[T_u]$. Note that we have

$$Bu(a) = \langle uk_a, k_a\rangle = \int_{D^n} u|k_a|^2 \,\mathrm{dV}, \tag{2.5}$$

so that by (2.3)

$$Bu(a) = \int_{D^n} (u \circ \varphi_a)\,\mathrm{dV}. \tag{2.6}$$

This integral representation provides some useful information. First, it allows us to extend the notion of the Berezin transform to functions $u \in L^1$. Moreover, the mean value property yields $Bu = u$ for n-harmonic functions $u \in L^1$. In particular, we see from (2.4) that bounded n-harmonic functions are all contained in $\mathscr{A}$. Next, it is easily seen from (2.6) that the Berezin transform is automorphism invariant; $B(u \circ \varphi_a) = (Bu) \circ \varphi_a$. Finally, for $u \in C(\overline{D^n})$, we have $Bu \in C(\overline{D^n})$. Moreover, if $u \in C_0$, then $Bu \in C_0$, because $\varphi_a(z) \to \partial D^n$ as $a \to \partial D^n$ for each $z \in D^n$.

The Berezin transform turns out to provide a compactness criterion for certain type of operators. Consider operators which are finite sums of finite products of Toeplitz operators with bounded symbols. Thus, such an operator S is of the form

$$S = \sum_{i=1}^{M} T_{u_{i1}} \cdots T_{u_{iN_i}} \tag{2.7}$$

where each $u_{ij} \in L^\infty$. The compactness of operators of this form is characterized by the boundary vanishing property of the Berezin transform as in the next lemma. See Axler-Zheng [3] on the disk and Englis [7] on the polydisk.

Lemma 1 *Let S be as in* (2.7). *Then S is compact if and only if $B[S] \in C_0$. In particular, if $u \in C_0$, then T_u is compact.*

We now introduce quite recent results about the Berezin transform. The same notation B will be used for the Berezin transform on D and D^n. Dimensions involved in B should be clear from the context. First, we need a couple of one-variable facts describing certain ranges of the Berezin transform on D. The following is taken from Theorem 1 of [1].

Lemma 2 *Let f and g be nonconstant holomorphic functions on D and assume that $f\overline{g} = B\tau$ for some $\tau \in L^1(D)$. Then there are nonconstant holomorphic polynomials p, q with* $\deg(pq) \leq 3$ *and a point $b \in D$ such that $f = p \circ \phi_b$ and $g = q \circ \phi_b$. Hence $p\overline{q} = B(\tau \circ \phi_b)$.*

Moreover, we have the following.

Lemma 3 *Let p and q be nonconstant holomorphic polynomials with* $\deg(pq) \leq 3$ *and assume that $p\overline{q} = B\tau$ for some $\tau \in L^1(D)$. Then $\tau(\lambda) = \eta(\lambda) + c_1 \log|\lambda|^2 + c_2\lambda^{-1} + c_3(\overline{\lambda})^{-1}$ for some constants c_i, not all 0, and $\eta \in L^\infty(D)$.*

Proof The Lemma is implicit in the proof of Corollary 1 of [1]. Also, one may directly verify the lemma by using the straightforward identities $\lambda\overline{\lambda} = B(1 + \log|\zeta|^2)$ and $\lambda\overline{\lambda}^2 = B(2\overline{\zeta} - \zeta^{-1})$. □

Next, we need recent results on higher order Berezin transforms. In [14] Suárez first introduced and studied higher order Berezin transforms for operators on the disk. Nam and Zheng [11] have recently extended such notions to the polydisk. All the results about higher order Berezin transforms mentioned below are taken from [11], unless otherwise specified, to which we refer for the proofs or details and further results.

We introduce more notation first. We define a linear operator U_a on A^2 by

$$U_a\psi = (\psi \circ \varphi_a)k_a$$

for $\psi \in A^2$. It follows from (2.3) that each U_a is an isometry on A^2. It is easily verified that $U_a k_a = (k_a \circ \varphi_a)k_a = 1$. It follows that $U_a U_a = I$ and thus $U_a^{-1} = U_a$. Now, being an invertible linear isometry, U_a is unitary. We set

$$S_a = U_a S U_a$$

for $S \in \mathfrak{L}(A^2)$. For $f, g \in A^2$, we let $f \otimes g$ denote the operator $h \mapsto \langle h, g\rangle f$ on A^2. It is easily seen that $f \otimes g$ is a trace class operator whose trace is given by $\operatorname{tr}[f \otimes g] = \langle f, g\rangle$. Using this notation, note that we have

$$B[S](a) = \langle Sk_a, k_a\rangle = \langle S_a 1, 1\rangle = \operatorname{tr}[S_a(1 \otimes 1)].$$

Now, fix an integer $m \geq 0$. Nam and Zheng [11] adopted this more insightful formula to define the *m-Berezin transform* of an operator $S \in \mathfrak{L}(A^2)$. More explicitly, they have given a definition as follows:

$$B_m[S](a) = (m+1)^n \operatorname{tr}\left[S_a\left(\sum_{|\alpha|\leq mn} c_{m\alpha}\left(\prod_{j=1}^n \frac{1}{\alpha_j+1}\right)\frac{z^\alpha}{\|z^\alpha\|} \otimes \frac{z^\alpha}{\|z^\alpha\|}\right)\right]$$

where $\|\ \|$ denotes the L^2-norm and $c_{m\alpha} = (-1)^{|\alpha|}\binom{m}{\alpha_1}\cdots\binom{m}{\alpha_n}$. Here, we use the conventional multi-index notation. That is, given an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we let $|\alpha| = \sum_{j=0}^n \alpha_j$ and $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. They then deduce the formula

$$B_m[S](a) = (m+1)^n \sum_{|\alpha|\leq mn} c_{m\alpha}\left\langle S(z^\alpha k_a^m), z^\alpha k_a^m\right\rangle \tag{2.8}$$

where

$$k_a^m(z) = \prod_{j=1}^n \left(\frac{\sqrt{1-|a_j|^2}}{1-\overline{a}_j z_j}\right)^{m+2}, \quad z \in D^n.$$

It is this formula which was used by Suárez [14] for the definition of B_m on the disk. It is now clear from (2.8) that if $S_j \to S$ in the weak operator topology of $\mathfrak{L}(A^2)$, then $B_m[S_j] \to B_m[S]$ pointwise on D^n. Clearly, $B_0 = B$.

For $u \in L^\infty$, we let $B_m u = B_m[T_u]$ as in the case $m = 0$. It turns out that there is also an integral representation for $B_m u$:

$$B_m u(a) = \int_{D^n} (u \circ \varphi_a)\, d\nu_m \tag{2.9}$$

where $d\nu_m(z) = (m+1)^n \prod_{j=1}^n (1-|z_j|^2)^m\, dV(z)$. For a positive $u \in L^\infty$, it is clear from (2.6) and (2.9) that if Bu vanishes on the boundary, then so does $B_m u$. With regard to such a boundary vanishing property, Nam and Zheng proved that

$$B[S] \in C_0 \implies B_m[S] \in C_0, \tag{2.10}$$

which is not at all clear for general S.

It is well known that

$$(T_u)_a = T_{u\circ\varphi_a}. \tag{2.11}$$

So, as is the case for $m = 0$, it follows from (2.9) that $B_m(u \circ \varphi_a) = (B_m u) \circ \varphi_a$, or said differently, $B_m[(T_u)_a] = B_m[T_u] \circ \varphi_a$. More generally, the m-Berezin transform is automorphism invariant in the sense that

$$B_m[S_a] = B_m[S] \circ \varphi_a. \tag{2.12}$$

Finally, one of the main properties of B_m is the approximate identity property as in the following.

Lemma 4 *Let $S \in \mathfrak{L}(A^2)$. If there is some $p > 3$ such that*

$$\sup_{a\in D^n, m\geq 0} \|T_{B_m[S_a]}1\|_{L^p} < \infty \quad \textit{and} \quad \sup_{a\in D^n, m\geq 0} \|T^*_{B_m[S_a]}1\|_{L^p} < \infty, \tag{2.13}$$

then $T_{B_m S} \to S$ as $m \to \infty$ in the norm topology of $\mathfrak{L}(A^2)$.

3 Zero products

In this section we prove Theorem 1 and then derive Theorem 5 as a consequence. Before proceeding to the proof, we recall some well known facts. The notation $H(D^n)$ denotes the class of all functions holomorphic on D^n.

Given $0 < p < \infty$, let $H^p(D^n)$ be the well-known Hardy space on D^n. For a pluriharmonic function $u = f + \overline{g} \in L^\infty$ where $f, g \in H(D^n)$, it is well known that $f, g \in H^p(D^n)$ for all $0 < p < \infty$ by the L^p-boundedness of the Cauchy projection. Hence, in particular, we have $f, g \in A^2$.

By the reproducing property (2.1), the Bergman projection P can be represented by

$$P\psi(a) = \int_{D^n} \psi \overline{K}_a \,\mathrm{dV}, \quad a \in D^n$$

for functions $\psi \in L^2$. It follows that P naturally extends via the above formula to an integral operator from L^1 into $H(D^n)$. Moreover, we have $Pf = f$ and

$$P(\overline{f}k_a) = \overline{f(a)}k_a, \quad a \in D^n \tag{3.1}$$

for functions $f \in L^1 \cap H(D^n)$. It is well known that $P : L^p \to L^p \cap H(D^n)$ is bounded for $1 < p < \infty$. See, for example, Theorem 4.2.3 of [17] for details on the disk; the same proof works on D^n.

The special case $\sigma = uv$ of Theorem 1 has been already noticed by many authors (see [4,6,16]).

Lemma 5 *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$. Then $T_u T_v = T_{uv}$ if and only if $f\overline{k}$ is n-harmonic.*

Now, we are ready to prove Theorem 1. We actually prove the following version of Theorem 1.

Theorem 4 *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$ and assume $\sigma \in L^\infty$. Then the following statements are equivalent:*

(a) $T_u T_v = T_\sigma$.
(b) *$\sigma = uv$ and $f\overline{k}$ is n-harmonic.*

Proof As is mentioned in the introduction, the theorem for $n = 1$ is just a restatement of the result of Ahern(Corollary 1 of [1]). So, let $n \geq 2$.

We first prove the implication (a) $\Longrightarrow$ (b). So, assume (a). Having n-harmonicity of $f\overline{k}$, we have $T_\sigma = T_u T_v = T_{uv}$ by Lemma 5 and hence $\sigma = uv$. We now prove n-harmonicity of $f\overline{k}$. We need to show $\partial_j \overline{\partial}_j (f\overline{k}) = 0$ for each j. By symmetry, it is sufficient to prove only for $j = 1$. Since u and v are bounded, functions f, g, h and k are all in A^2, as is mentioned above. Fix an arbitrary point $a \in D^n$. By (3.1), we have

$$T_{h+\overline{k}} k_a = P[(h + \overline{k}) k_a] = [h + \overline{k(a)}] k_a.$$

Therefore, we have

$$T_{f+\overline{g}} T_{h+\overline{k}} k_a = fhk_a + f\overline{k(a)} k_a + \overline{g(a)}\,\overline{k(a)} k_a + P(h\overline{g} k_a).$$

It follows from (2.5) that

$$\begin{aligned} B[T_u T_v](a) &= \langle T_{f+\overline{g}} T_{h+\overline{k}} k_a, k_a \rangle \\ &= f(a)h(a) + f(a)\overline{k(a)} + \overline{g(a)}\,\overline{k(a)} + B(h\overline{g})(a) \\ &= B(fh + \overline{g}\overline{k} + h\overline{g})(a) + f(a)\overline{k(a)}. \end{aligned}$$

Since $T_u T_v = T_\sigma$ by assumption, we have $B[T_u T_v] = B[T_\sigma] = B\sigma$ and thus

$$f(a)\overline{k(a)} = B\psi(a), \quad a \in D^n \tag{3.2}$$

where $\psi = \sigma - fh - \overline{g}\,\overline{k} - h\overline{g}$. Now, inserting $a = (\lambda, 0)$, $\lambda \in D$, into (3.2), we have

$$f(\lambda, 0)\overline{k(\lambda, 0)} = \int\limits_D \frac{(1 - |\lambda|^2)^2}{|1 - \lambda\overline{\zeta}|^4} \tau(\zeta)\, \mathrm{dV}_1(\zeta) = B\tau(\lambda) \tag{3.3}$$

where

$$\tau(\zeta) = \int\limits_{D^{n-1}} \psi(\zeta, z)\, \mathrm{dV}_{n-1}(z), \quad \zeta \in D.$$

Since $\sigma \in L^\infty$ and $fh, \overline{g}\,\overline{k}, h\overline{g} \in L^1(D^n)$, we see $\tau \in L^1(D)$.

Since $f \in H^2(D^n)$, we have $f(\zeta,\cdot) \in H^2(D^{n-1})$ for each $\zeta \in D$ by Lemma 3.3 of [4]. In particular, we have $f(\zeta,\cdot) \in A^2(D^{n-1})$ for each $\zeta \in D$. The same is true for g,h,k and hence $(fh)(\zeta,\cdot), (gk)(\zeta,\cdot)$ are all in $A^1(D^{n-1})$ for each $\zeta \in D$. Thus, an application of the mean value property yields

$$\tau(\zeta) = \tau_1(\zeta) - \tau_2(\zeta) - \tau_3(\zeta), \quad \zeta \in D$$

where

$$\begin{aligned}
\tau_1(\zeta) &= \int_{D^{n-1}} \sigma(\zeta,z)\, \mathrm{dV}_{n-1}(z), \\
\tau_2(\zeta) &= \int_{D^{n-1}} h(\zeta,z)\overline{g(\zeta,z)}\, \mathrm{dV}_{n-1}(z), \\
\tau_3(\zeta) &= f(\zeta,0)h(\zeta,0) + \overline{g(\zeta,0)}\,\overline{k(\zeta,0)}.
\end{aligned}$$

Clearly, we have $\tau_1 \in L^\infty(D)$ and τ_3 is continuous on D. Note that, given a compact subset $K \subset D$, we have

$$\sup_{\zeta\in K} |h(\zeta,z)g(\zeta,z)| \le C \int_D |h(\lambda,z)g(\lambda,z)|\, \mathrm{dV}_1(\lambda), \quad z \in D^{n-1} \tag{3.4}$$

for some constant C independent of z. This is easily seen by submean value property of subharmonic functions $|h(\cdot,z)g(\cdot,z)|$. Also, note that the function $z \mapsto \int_D |h(\lambda,z)g(\lambda,z)|\, \mathrm{dV}_1(\lambda)$ is integrable on D^{n-1}. Thus, by the Lebesgue dominated convergence theorem, we see that τ_2 is continuous on D. Combining all these observations together, we see that τ is essentially bounded on each compact subset of D. So, by Lemmas 2 and 3, we conclude that either $f(\lambda,0)$ or $k(\lambda,0)$ is a constant function of $\lambda \in D$. In other words,

$$\text{either} \quad \partial_1 f(\lambda,0) = 0 \quad \text{or} \quad \partial_1 k(\lambda,0) = 0 \tag{3.5}$$

for $\lambda \in D$.

Next, we consider arbitrary points. So, fix $z = (z_1,\dots,z_n) \in D^n$. Since $T_uT_v = T_\sigma$ by assumption, we have $T_{u\circ\varphi_z}T_{v\circ\varphi_z} = T_{\sigma\circ\varphi_z}$ by (2.11). Thus, replacing u, v and σ, respectively, by $u \circ \varphi_{(0,z_2,\dots,z_n)}$, $v \circ \varphi_{(0,z_2,\dots,z_n)}$ and $\sigma \circ \varphi_{(0,z_2,\dots,z_n)}$ in the above argument, we see from (3.5) that either

$$0 = \partial_1(f \circ \varphi_{(0,z_2,\dots,z_n)})(-z_1,0) = -\partial_1 f(z)$$

or

$$0 = \partial_1(k \circ \varphi_{(0,z_2,\dots,z_n)})(-z_1,0) = -\partial_1 k(z).$$

Consequently, we have either $\partial_1 f = 0$ or $\partial_1 k = 0$ on D^n. In other words, we have $\partial_1\overline{\partial}_1(f\overline{k}) = 0$, as desired.

Next, we prove the implication (b) $\Longrightarrow$ (a). So, assume (b). Since $f\overline{k}$ is n-harmonic, we have $T_uT_v = T_{uv}$ by Lemma 5. Now, since $\sigma = uv$ by assumption, we have (a). The proof is complete. □

In what follows, for $J_1 \subset J = \{z_1, \dots, z_n\}$, we write $H(J_1)$ for the set of all holomorphic functions on D^n independent of variables $z_j \notin J_1$. In particular, $H(\emptyset)$ consists of all constant functions. Also, note $H(J) = H(D^n)$. For n-harmonic symbols σ, we have the following symmetric characterization. For pluriharmonic symbols σ, much more turns out to hold for $n \geq 2$. See Theorem 9 below.

Theorem 5 *Let $u, v \in L^\infty$ be pluriharmonic symbols and assume that $\sigma \in L^\infty$ is an n-harmonic symbol. Then the following statements are equivalent:*

(a) $T_uT_v = T_\sigma$.
(b) $T_vT_u = T_\sigma$.
(c) *$\sigma = uv$ and there are sets $J_1, J_2 \subset J$, functions $f \in H(J_1)$, $g \in H(J_2)$, $h \in H(J \setminus J_2)$, $k \in H(J \setminus J_1)$ such that $u = f + \overline{g}$ and $v = h + \overline{k}$.*

In case $n = 1$, the above is already noticed in [2]. Note that the second part of condition (c) above reduces to trivial cases for $n = 1$. Namely, it simply means that one of the following conditions holds: (i) u is constant, (ii) v is constant, (iii) u, v are both holomorphic, (iv) u, v are both co-holomorphic. Thus, what seems interesting here for $n \geq 2$ is that there are something more in addition to such trivial cases.

Proof As is mentioned before, the theorem is already noticed for $n = 1$ in [2]. So, let $n \geq 2$. Note that we only need to prove the equivalence (a) $\Longleftrightarrow$ (c) by symmetry. The implication (c) $\Longrightarrow$ (a) follows from Theorem 4. It remains to prove the implication (a) $\Longrightarrow$ (c). So, assume (a) and write $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$. Then, by Theorem 4 and assumption, $\sigma = uv = fh + f\overline{k} + h\overline{g} + \overline{gk}$ is n-harmonic. Since $fh + \overline{gk}$ is already n-harmonic and $f\overline{k}$ is n-harmonic by Theorem 4, it follows that $h\overline{g}$ is also n-harmonic. Now, (c) follows from n-harmonicity of $f\overline{k}$ and $h\overline{g}$. The proof is complete. □

4 Compact products

In this section we first prove Theorem 2 and then derive Theorem 7 as a consequence. As applications, we will observe some consequences in case symbols are continuous up to the boundary. We also recall some well known facts.

We let $\widetilde{\Delta}_j$ denote n-Laplacians defined by

$$\widetilde{\Delta}_j\sigma(z) = (1 - |z_j|^2)^2\partial_j\overline{\partial}_j\sigma(z), \quad z \in D^n,\ j = 1, \dots, n.$$

We remark in passing that n-Laplacians commute with automorphisms. That is, we have

$$\widetilde{\Delta}_j(\sigma \circ \varphi_a) = (\widetilde{\Delta}_j \sigma) \circ \varphi_a, \quad j = 1, \ldots, n$$

for $\sigma \in C^2$ and $a \in D^n$. Note that u is n-harmonic if and only if u is annihilated by all $\widetilde{\Delta}_j$. Thus, we will say that u is *boundary n-harmonic* if $\widetilde{\Delta}_j u \in C_0$ for each j.

The following characterization of compact semi-commutators has been known (see [4]). Also, see [16] on the disk.

Lemma 6 *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$. Then $T_u T_v - T_{uv}$ is compact if and only if $f\overline{k}$ is boundary n-harmonic.*

Our proof will rely on some auxiliary function class related to the maximal ideal space of $\mathscr{A}$. Recall that the maximal ideal space $\mathfrak{M}$ of $\mathscr{A}$ is defined to be the set of all nonzero multiplicative linear functionals on $\mathscr{A}$. As is well known, we have $\mathscr{A} \subset C(\mathfrak{M})$ via the Gelfand transform. We will use the same notation for a function $u \in \mathscr{A}$ and its continuous extension u on the whole $\mathfrak{M}$.

Identifying $z \in D^n$ with the multiplicative evaluation functional $f \mapsto f(z)$, we can regard D^n as a subset of $\mathfrak{M}$. So, given $z \in D^n$, we can think of φ_z as a map from D^n to $\mathfrak{M}$. In other words, $\varphi_z \in \mathfrak{M}^{D^n}$. Equipped with product topology, the function space $\mathfrak{M}^{D^n}$ is compact by Tychonoff's theorem. Now, we let

$$\Phi = (\text{closure of } D^n) \setminus D^n$$

where the closure is taken in $\mathfrak{M}$. Let $\mathfrak{m} \in \Phi$ be given and choose a net $\{z_\alpha\}$ in D^n such that $z_\alpha \to \mathfrak{m}$. By compactness the net $\{\varphi_{z_\alpha}\}$ in $\mathfrak{M}^{D^n}$ contains a convergent subnet $\{\varphi_{z_{\alpha_\beta}}\}$. That is, there is a function $\varphi \in \mathfrak{M}^{D^n}$ such that $u \circ \varphi_{z_{\alpha_\mathfrak{m}}} \to u \circ \varphi$ pointwise on D^n for every $u \in \mathscr{A}$. It is noticed on the disk in [14] that such φ does not depend on the net. The same continues to hold on the polydisk by the same proof. The map φ will be denoted by $\varphi_\mathfrak{m}$.

The following lemma comes from [14] of the disk case. The proof is the same as the disk case and included here for completeness.

Lemma 7 *Let $\{z_\alpha\}$ be a net in D^n converging to $\mathfrak{m} \in \Phi$. Then the following statements hold*:

(a) *$\varphi_\mathfrak{m}$ is a continuous map from D^n into $\mathfrak{M}$.*
(b) *$u \circ \varphi_\mathfrak{m} \in \mathscr{A}$ for every $u \in \mathscr{A}$.*
(c) *$u \circ \varphi_{z_\alpha} \to u \circ \varphi_\mathfrak{m}$ uniformly on compact subsets of D^n for every $u \in \mathscr{A}$.*

Proof Suppose $w \in D^n$ and $u \in \mathscr{A}$. Given $\epsilon > 0$ there is $\delta > 0$ such that $|u(z) - u(z')| < \epsilon$ if $\rho(z, z') < \delta$. Take w' such that $\rho(w', w) < \delta$. Since ρ is automorphism invariant, we have $\rho\big(\varphi_{z_\alpha}(w'), \varphi_{z_\alpha}(w)\big) = \rho(w', w) < \delta$. Thus we have

$$\begin{aligned}
&|u \circ \varphi_{\mathfrak{m}}(w') - u \circ \varphi_{\mathfrak{m}}(w)| \\
&\quad \le |u \circ \varphi_{\mathfrak{m}}(w') - u \circ \varphi_{z_\alpha}(w')| \\
&\qquad + |u \circ \varphi_{z_\alpha}(w') - u \circ \varphi_{z_\alpha}(w)| + |u \circ \varphi_{z_\alpha}(w) - u \circ \varphi_{\mathfrak{m}}(w)| \\
&\quad \le |u \circ \varphi_{\mathfrak{m}}(w') - u \circ \varphi_{z_\alpha}(w')| + |u \circ \varphi_{z_\alpha}(w) - u \circ \varphi_{\mathfrak{m}}(w)| + \epsilon
\end{aligned}$$

for every α. Taking the limit $z_\alpha \to \mathfrak{m}$, we have $|u \circ \varphi_{\mathfrak{m}}(w') - u \circ \varphi_{\mathfrak{m}}(w)| \le \epsilon$ when $\rho(w', w) < \delta$. This proves (a) and (b).

To prove (c), suppose that it is not true. Then there are $u \in \mathscr{A}$, $0 < r < 1$ and $\epsilon > 0$ such that $|u \circ \varphi_{z_\alpha}(\xi_\alpha) - u \circ \varphi_{\mathfrak{m}}(\xi_\alpha)| > \epsilon$ for some points $\xi_\alpha \in rD^n$. We can also assume that $\xi_\alpha \to \xi$. Since $(u \circ \varphi_{z_\alpha})(\xi) \to (u \circ \varphi_{\mathfrak{m}})(\xi)$, this contradicts the uniform ρ-continuity of u. The proof is complete. □

For $u \in L^\infty$ pluriharmonic(n-harmonic), Lemma 7 gives that $u \circ \varphi_{\mathfrak{m}} \in L^\infty$ is also pluriharmonic(n-harmonic) for $\mathfrak{m} \in \Phi$. We will use, often without further comments, these additional consequences of Lemma 7.

Proposition 1 *Let $u, v \in \mathscr{A}$ and $\sigma \in L^\infty$. Assume that $T_u T_v - T_\sigma$ is compact. Then, to each $\mathfrak{m} \in \Phi$, there corresponds a function $\sigma_{\mathfrak{m}} \in L^\infty$ such that $T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} = T_{\sigma_{\mathfrak{m}}}$. If, in addition, $\sigma \in \mathscr{A}$, then $\sigma_{\mathfrak{m}} = \sigma \circ \varphi_{\mathfrak{m}}$.*

Let S be an operator of the form (2.7) with $u_{ij} \in \mathscr{A}$. Then the proof below actually shows that the above proposition remains true with S in place of $T_u T_v$.

Proof Let $\mathfrak{m} \in \Phi$ be given and choose a net $\{z_\alpha\}$ in D^n converging to $\mathfrak{m}$. Lemma 7 gives that $T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} \to T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}}$ in the weak operator topology. Thus, for $f, g \in A^2$, we have

$$\langle T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} f, g\rangle = \lim_\alpha \langle T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} f, g\rangle. \tag{4.1}$$

Let $m \ge 0$ be an arbitrary integer. It follows from (4.1) and (2.8) that

$$B_m[T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}}](z) = \lim_\alpha B_m[T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}}](z), \quad z \in D^n. \tag{4.2}$$

Put $K = T_u T_v - T_\sigma$. Then K is a compact operator by assumption. Noting that

$$T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} = T_{\sigma\circ\varphi_{z_\alpha}} + K_{z_\alpha},$$

we have by (2.6) and (2.12)

$$\begin{aligned}
B_m[T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}}](z) &= B_m(\sigma \circ \varphi_{z_\alpha})(z) + B_m[K_{z_\alpha}](z) \\
&= \int_{D^n} (\sigma \circ \varphi_{z_\alpha} \circ \varphi_z)\, d\nu_m + B_m[K]\big(\phi_{z_\alpha}(z)\big)
\end{aligned}$$

so that

$$\big|B_m[T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}}](z)| \le \|\sigma\|_{L^\infty} + |B_m[K]\big(\phi_{z_\alpha}(z)\big)\big|$$

for each z_α and $z \in D^n$. Since K is compact, we have $B[K] \in C_0$ by Lemma 1 and thus $B_m[K] \in C_0$ by (2.10). Note $\phi_{z_\alpha}(z) \to \partial D^n$ and thus $B_m[K]\big(\phi_{z_\alpha}(z)\big) \to 0$ as $z_\alpha \to \mathfrak{m}$. So, taking the limit $z_\alpha \to \mathfrak{m}$ (with m and z fixed), we therefore obtain by (4.2)

$$|B_m[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}](z)| \le \|\sigma\|_{L^\infty}, \quad z \in D^n \tag{4.3}$$

and this is true for all $m \ge 0$.

Now, by (4.3), we have a subsequence $\{m_k\}$ such that $B_{m_k}[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}]$ converges in the weak-star topology of L^∞ to some function $\sigma_\mathfrak{m}$, with $\|\sigma_\mathfrak{m}\|_{L^\infty} \le \|\sigma\|_{L^\infty}$, such that

$$\begin{aligned}\langle T_{B_{m_k}[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}]}f, g\rangle &= \int\limits_{D^n} B_{m_k}[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}]f\bar{g}\,\mathrm{dV}\\ &\to \int\limits_{D^n} \sigma_\mathfrak{m} f\bar{g}\,\mathrm{dV}\\ &= \langle T_{\sigma_\mathfrak{m}}f, g\rangle\end{aligned}$$

for each $f, g \in A^2$. Hence, on one hand, we have $T_{B_{m_k}[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}]} \to T_{\sigma_\mathfrak{m}}$ in the weak operator topology. The estimates (2.13) with $S = T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}$ are easily verified by (4.3), because the Bergman projection P is L^p-bounded for $1 < p < \infty$. Thus, on the other hand, we see $T_{B_{m_k}[T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}]} \to T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}}$ in the norm topology by Lemma 4. So we conclude that $T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}} = T_{\sigma_\mathfrak{m}}$, as desired. This completes the proof of the first part.

For the second part, assume $\sigma \in \mathscr{A}$. Then we also have $T_{\sigma\circ\varphi_{z_\alpha}} \to T_{\sigma\circ\varphi_\mathfrak{m}}$ in the weak operator topology by Lemma 7. So, for each $a \in D^n$, we have by (2.11)

$$\begin{aligned}(T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}} - T_{\sigma\circ\varphi_\mathfrak{m}})k_a &= \lim_\alpha (T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} - T_\sigma \circ \varphi_{z_\alpha})k_a\\ &= \lim_\alpha U_{z_\alpha}(T_u T_v - T_\sigma)U_{z_\alpha}k_a\\ &= 0\end{aligned}$$

where the last equality holds by the weak convergence of $\{U_{z_\alpha}k_a\}$ to 0 and the compactness of $T_uT_v - T_\sigma$. Since $\{k_a\}$ spans a dense sunset of A^2, it follows that $T_{u\circ\varphi_\mathfrak{m}} T_{v\circ\varphi_\mathfrak{m}} = T_{\sigma\circ\varphi_\mathfrak{m}}$. Accordingly, we have $\sigma_\mathfrak{m} = \sigma \circ \varphi_\mathfrak{m}$ by what we've just proved above. The proof is complete. □

We are finally ready to prove Theorem 2. In fact we prove a strengthened version of the theorem.

Theorem 6 *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$ and assume $\sigma \in L^\infty$. Then the following three statements are equivalent*:

(a) $T_u T_v - T_\sigma$ *is compact.*
(b) $T_u T_v - T_{uv}$ *and* $T_{uv-\sigma}$ *are both compact.*
(c) $f\overline{k}$ *is boundary n-harmonic and* $B(\sigma - uv) \in C_0$.

If, in addition, $\sigma \in \mathscr{A}$, *then the above conditions are also equivalent to either one of the following statements:*

(d) $T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} = T_{\sigma\circ\varphi_{\mathfrak{m}}}$ *for each* $\mathfrak{m} \in \Phi$.
(e) $T_u T_v - T_{uv}$ *is compact and* $\sigma - uv \in C_0$.

Proof The equivalence (b) $\iff$ (c) holds by Lemmas 1 and 6. The implication (b) $\Longrightarrow$ (a) is trivial. We now prove the converse implication (a) $\Longrightarrow$ (b). Suppose that $T_u T_v - T_\sigma$ is compact. First we show that

$$\lim_{z\to\partial D^n} B[T_u T_v - T_{uv}](z) = 0, \quad z \in D^n. \tag{4.4}$$

Having this, we deduce from Lemma 1 that $T_u T_v - T_{uv}$ is compact and hence $T_{uv-\sigma}$ is compact.

It remains to prove (4.4). Suppose not. Then there is a subnet $\{z_\alpha\} \subset D^n$ converging to some $\mathfrak{m} \in \Phi$ such that

$$\limsup_\alpha |B[T_u T_v - T_{uv}](z_\alpha)| > 0. \tag{4.5}$$

By Proposition 1, there is a function $\sigma_{\mathfrak{m}} \in L^\infty$ such that $T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} = T_{\sigma_{\mathfrak{m}}}$. By Theorem 4, we have that $\sigma_{\mathfrak{m}} = (uv)\circ\varphi_{\mathfrak{m}}$ and thus $T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} = T_{(uv)\circ\varphi_{\mathfrak{m}}}$. Recall that $T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} - T_{(uv)\circ\varphi_{z_\alpha}} \to T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} - T_{(uv)\circ\varphi_{\mathfrak{m}}}$ in the weak operator topology by Lemma 7. It follows from (2.11) and (2.12) that

$$\begin{aligned} 0 &= B[T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} - T_{(uv)\circ\varphi_{\mathfrak{m}}}](0) \\ &= \lim_\alpha B[T_{u\circ\varphi_{z_\alpha}} T_{v\circ\varphi_{z_\alpha}} - T_{(uv)\circ\varphi_{z_\alpha}}](0) \\ &= \lim_\alpha B[(T_u T_v - T_{uv})_{z_\alpha}](0) \\ &= \lim_\alpha B[T_u T_v - T_{uv}](z_\alpha), \end{aligned}$$

which contradicts (4.5). This completes the proof of the first part of the theorem.

Next, we further assume $\sigma \in \mathscr{A}$. The implications (e) $\Longrightarrow$ (b) $\Longrightarrow$ (d) hold by Lemma 1 and Proposition 1. It remains to prove the implication (d) $\Longrightarrow$ (e). So, assume (d). Fix an arbitrary $\mathfrak{m} \in \Phi$ and let $\{z_\alpha\}$ be any net in D^n such that $z_\alpha \to \mathfrak{m}$. Since $\sigma \in \mathscr{A}$, we also have $\sigma \circ \varphi_{\mathfrak{m}} \in \mathscr{A}$ by Lemma 7. Now, since $T_{u\circ\varphi_{\mathfrak{m}}} T_{v\circ\varphi_{\mathfrak{m}}} = T_{\sigma\circ\varphi_{\mathfrak{m}}}$ by assumption, we have $\sigma \circ \varphi_{\mathfrak{m}} = (uv) \circ \varphi_{\mathfrak{m}}$ by Theorem 4. Note that $\varphi_{\mathfrak{m}}(0) = \lim \varphi_{z_\alpha}(0) = \lim z_\alpha = \mathfrak{m}$. Thus, we have

$$(\sigma - uv)(z_\alpha) \to \mathfrak{m}(\sigma - uv) = (\sigma - uv) \circ \varphi_{\mathfrak{m}}(0) = 0. \tag{4.6}$$

Since $\mathfrak{m} \in \Phi$ is arbitrary, one may repeat a similar argument as above to conclude from (4.6) that $\sigma - uv \in C_0$, as desired. Also, one may repeat the proof of (a) $\Longrightarrow$ (b) to verify that $T_uT_v - T_{uv}$ is compact. (One may also repeat the proof of Theorem 1.3 in [4] to derive the boundary n-harmonicity of $f\overline{k}$.) The proof is complete. $\square$

Having seen Theorem 6, one may ask whether the hypothesis $\sigma \in \mathscr{A}$ is essential for the second part of the theorem. The answer is yes and it is most easily seen simply by considering the case $u = v = 0$ which yields the following characterization.

Corollary 1 *Let $\sigma \in \mathscr{A}$. Then T_σ is compact if and only if $B\sigma \in C_0$ if and only if $\sigma \in C_0$.*

The study in the direction of this corollary has been of some independent interest. Stroethoff [13] first studied the compactness of Toeplitz operators with symbols in $\mathscr{A}$ on the disk (actually on the ball) and obtained that T_σ is compact if and only if $B\sigma \in C_0$. Later Axler and Zheng [3] showed that this is true for bounded symbols on the unit disk and Englis [7] showed that this is true on the polydisk. Let $\mathscr{U}$ be the C^*-algebra in L^∞ generated by H^∞, the space of all bounded holomorphic functions on D^n. Clearly, $\mathscr{U} \subset \mathscr{A}$. For symbols in $\mathscr{U}$ on D, the above corollary has been long known by McDonald and Sundberg [10].

Now, we remark yet another consequence of the above corollary and the maximum principle.

Corollary 2 *Let $f, g \in H^\infty$. If $B(f\overline{g}) \in C_0$, then either $f = 0$ or $g = 0$.*

In the case of disk, it turns out that continuity up to the boundary already implies boundary harmonicity of $f\overline{k}$ in the condition (c) of Theorem 6.

Corollary 3 *Let $u, v \in C(\overline{D})$ be harmonic symbols and assume $\sigma \in L^\infty(D)$. Then $T_uT_v - T_\sigma$ is compact if and only if $T_{uv-\sigma}$ is compact.*

Proof We first recall a well-known fact that the Bergman projection P maps $C(\overline{D})$ into $\mathcal{B}_0(D)$, the little Bloch space on D which consists of functions $\psi \in H(D)$ such that

$$\lim_{|\lambda|\to 1} (1 - |\lambda|^2)|\psi'(\lambda)| = 0.$$

See, for example, Theorem 5.2.5 of [17]. Now, let $u = f + \overline{g}$ and $v = h + \overline{k}$ for some $f, g, h, k \in H(D)$. Then, we have $P(u) = f + \overline{g(0)} \in \mathcal{B}_0(D)$ and thus $f \in \mathcal{B}_0(D)$. Applying the same reasoning to $\overline{v}$, we obtain $k \in \mathcal{B}_0(D)$. In particular, $f\overline{k}$ is boundary harmonic. Now, the corollary follows from Theorem 6. The proof is complete. $\square$

Suppose $n \geq 2$ and consider a function $f \in H(D^n)$ such that $(1-|z_j|^2)\partial_j f \in C_0$ for each j. Fix z_1, let $(z_2, \ldots, z_n) \to \partial D^{n-1}$ and then apply the maximum principle. The result is $\partial_1 f = 0$. Similarly, $\partial_j f = 0$ for each j and thus f is constant. This shows that the proof of Corollary 3 does not extend to $n \geq 2$. In fact, the corollary itself is false for $n \geq 2$ if harmonicity hypothesis is replaced by pluriharmonicity. We will see a counter-example after Theorem 8.

As another consequence, we have the following symmetric characterization when σ is n-harmonic, as in the case of generalized zero products (Theorem 5). Recall that bounded n-harmonic functions are all contained in $\mathscr{A}$. As is seen in Theorem 9 below, Theorem 7 can be much improved for higher dimensional polydisks when σ is in addition pluriharmonic.

Theorem 7 *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$ and assume that $\sigma \in L^\infty$ is an n-harmonic symbol. Then the following statements are equivalent:*

(a) $T_u T_v - T_\sigma$ *is compact.*
(b) $T_v T_u - T_\sigma$ *is compact.*
(c) $f\overline{k}$, $h\overline{g}$ *are both boundary n-harmonic and* $\sigma - uv \in C_0$.
(d) $f\overline{k}$, $h\overline{g}$ *are both boundary n-harmonic and* $B(\sigma - uv) \in C_0$.
(e) $T_u T_v - T_{uv}$, $T_v T_u - T_{uv}$, $T_{uv-\sigma}$ *are all compact.*

Proof Since $u, v \in L^\infty$ are pluriharmonic, so are $u \circ \varphi_{\mathfrak{m}}$ and $v \circ \varphi_{\mathfrak{m}}$ for each $\mathfrak{m} \in \Phi$. Similarly, since $\sigma \in L^\infty$ is n-harmonic, so is $\sigma \circ \varphi_{\mathfrak{m}}$ for each $\mathfrak{m} \in \Phi$. Thus, the theorem follows from Theorems 6 and 5. $\square$

In the special case where $\sigma = 0$ and each of u and v is holomorphic or co-holomorphic, Theorem 7, together with the maximum principle, yields a remarkably simple characterization of compact products.

Corollary 4 *Let $u, v \in H^\infty \cup \overline{H^\infty}$. Then the following statements are equivalent:*

(a) $T_u T_v = 0$.
(b) $T_u T_v$ *is compact.*
(c) *Either* $u = 0$ *or* $v = 0$.

5 The case $n \geq 2$

In the theory of Toeplitz operators on higher dimensional polydisks, it has been known that the zero and compact properties often coincide, especially when pluriharmonic symbols are considered, as one can see from results in [4,6,8, 15]. For example, we have the following characterization of compact semi-commutators (see [6] and [4]).

Lemma 8 $(n \geq 2)$ *Let $u, v \in L^\infty$ be pluriharmonic symbols. Then $T_u T_v = T_{uv}$ if and only if $T_u T_v - T_{uv}$ is compact.*

Here, we prove Theorem 3, which is another result showing such a phenomenon. We restate the theorem combined with Theorems 5 and 7 for convenience.

Theorem 8 $(n \geq 2)$ *Let $u, v \in L^\infty$ be pluriharmonic symbols with $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$ and assume that $\sigma \in L^\infty$ is an n-harmonic symbol. Then the following statements are equivalent*:

(a) $T_u T_v = T_\sigma$.
(b) $T_v T_u = T_\sigma$.
(c) $T_u T_v - T_\sigma$ *is compact.*
(d) $T_v T_u - T_\sigma$ *is compact.*
(e) $f\overline{k}$, $h\overline{g}$ *are both n-harmonic and* $\sigma = uv$.

Proof We already have the equivalences (a) $\iff$ (b) and (c) $\iff$ (d) by Theorems 5 and 7. Now, since the implication (a) $\Longrightarrow$ (c) is trivial and the implication (e) $\Longrightarrow$ (a) holds by Theorem 4, we only need to prove the implication (c) $\Longrightarrow$ (e). So, assume that $T_u T_v - T_\sigma$ is compact. We then see that $f\overline{k}$ and $h\overline{g}$ are both boundary n-harmonic by Theorem 7 and thus they are actually n-harmonic by Lemmas 8 and 5. This yields the first part of (e). Having seen that $f\overline{k}$ and $h\overline{g}$ are both n-harmonic, we also see that uv is n-harmonic. Now, $\sigma - uv$ is n-harmonic and vanishes on the boundary by Theorem 7. So, we have $\sigma = uv$ by the maximum principle. The proof is complete. □

The proof above depends on Lemmas 8 and 5. Here, we provide a direct proof of the implication (c) $\Longrightarrow$ (e) in order to illustrate how the assumption $n \geq 2$ plays its role.

Proof (Another proof) Suppose that $T_u T_v - T_\sigma$ is compact. We continue using notation introduced in the proof above. Also, we introduce temporary notation $\Phi_n = \Phi$ in order to avoid confusion with dimensions. By Theorems 6 and 1 we have

$$\partial_j(u \circ \varphi_{\mathfrak{m}}) = 0 \quad \text{or} \quad \overline{\partial}_j(v \circ \varphi_{\mathfrak{m}}) = 0, \quad j = 1, \ldots, n \tag{5.1}$$

for each $\mathfrak{m} \in \Phi_n$. So, for the first part of (e), it is sufficient to show $\partial_2 f = 0$ under the hypothesis that $\partial_2(u \circ \varphi_{\mathfrak{m}}) = 0$ for each $\mathfrak{m} \in \Phi_n$.

Fix $w \in D^n$ and consider the function $F(\lambda) = \partial_2 f(\lambda, w_2, \ldots, w_n)$ for $\lambda \in D$. It is sufficient to show that F vanishes everywhere on D. We claim that all the radial limits of F exist and are zero. Note that $F \in H^2(D)$ by Lemma 3.3 of [4]. So, using the claim, we conclude $F = 0$ on D, as desired.

Now, we prove the claim. Suppose not. Then there exist a point $\xi \in \partial D$ and a sequence $\{r_\ell\}$ of radii such that $r_\ell \to 1$ and

$$F(r_\ell \xi) = \partial_2 f(r_\ell \xi, w_2, \ldots, w_n) \nrightarrow 0 \tag{5.2}$$

as $\ell \to \infty$. By compactness of $\mathfrak{M}_1$ we may take a subsequence (still called $\{r_\ell\}$) of $\{r_\ell\}$ such that $r_\ell \xi \to \mathfrak{m}_1 \in \Phi_1$ for some $\mathfrak{m}_1$. Now, we extend $\mathfrak{m}_1$ to $\mathfrak{m} = (\mathfrak{m}_1, 0, \ldots, 0) \in \Phi_n$ in the canonical way. This means that $\mathfrak{m} = \lim(r_\ell \xi, 0, \ldots, 0)$. Note that $u \circ \varphi_{\mathfrak{m}}(w) = u\big(\phi_{\mathfrak{m}_1}(w_1), w_2, \ldots, w_n\big)$ by Lemma 7. Also, by Lemma 7, we have

$$
\begin{aligned}
0 &= \partial_2(u \circ \varphi_{\mathrm{m}})(0, w_2, \ldots, w_n) \\
&= \lim \partial_2 \left(u \circ \varphi_{(r_\ell \xi, 0, \ldots, 0)}\right)(0, w_2, \ldots, w_n) \\
&= \lim (\partial_2 f) \circ \varphi_{(r_\ell \xi, 0, \ldots, 0)}(0, w_2, \ldots, w_n) \\
&= \lim \partial_2 f(r_\ell \xi, w_2, \ldots, w_n),
\end{aligned}
$$

which is a contradiction to (5.2). This completes the proof of the claim. The proof of $\sigma = uv$ is the same as the proof above. The proof is complete. □

We now observe several consequences of Theorem 8. First, we give an example demonstrating that Corollary 3, when harmonicity hypothesis is replaced by pluriharmonicity, cannot be extended to $n \geq 2$.

Example Let $n = 2$ for simplicity. Consider functions

$$
\begin{aligned}
u(z_1, z_2) &= (1 - z_1)(1 - z_2) + (1 - \overline{z}_1)(1 - \overline{z}_2), \\
v(z_1, z_2) &= (1 + z_1)(1 + z_2) + (1 + \overline{z}_1)(1 + \overline{z}_2), \\
\sigma(z_1, z_2) &= (1 - z_1^2)(1 - z_2^2) + (1 - \overline{z}_1^2)(1 - \overline{z}_2^2) + 2(z_1 - \overline{z}_1)(z_2 - \overline{z}_2).
\end{aligned}
$$

Clearly, u, v are pluriharmonic on D^2 and σ is 2-harmonic. A straightforward calculation yields

$$
uv - \sigma = 2(1 - |z_1|^2)(1 - |z_2|^2)
$$

so that $uv - \sigma \in C_0$. Thus, $T_{uv-\sigma}$ is compact. However, $T_u T_v - T_\sigma$ is not compact by Theorem 8.

Next, recall that the condition (e) of Theorem 8 goes back to the condition (c) of Theorem 5. For pluriharmonic σ, we have a more concrete description as in the next theorem.

Theorem 9 ($n \geq 2$) *Let $u, v, \sigma \in L^\infty$ be pluriharmonic symbols. Then the following statements are equivalent:*

(a) $T_u T_v = T_\sigma$.
(b) $T_v T_u = T_\sigma$.
(c) $T_u T_v - T_\sigma$ *is compact.*
(d) $T_v T_u - T_\sigma$ *is compact.*
(e) $\sigma = uv$ *and one of the following conditions holds;*
  (i) u *is constant.*
  (ii) v *is constant.*
  (iii) $u, v \in H^\infty$.
  (iv) $u, v \in \overline{H^\infty}$.
  (v) *There are constants α, β, a set $J_1 \subset J$ and functions $f \in H(J_1)$, $g \in H(J \setminus J_1)$ such that $u = f + \overline{g}$ and $v = \alpha(f - \overline{g}) + \beta$.*

Proof By symmetry and Theorem 8 we only need to prove the equivalence (a) $\Longleftrightarrow$ (e). Note that the implication (e) $\Longrightarrow$ (a) follows from Theorem 4. Thus, it remains to prove the implication (a) $\Longrightarrow$ (e).

Assume (a) and let $u = f + \overline{g}$, $v = h + \overline{k}$ for some $f, g, h, k \in H(D^n)$. By Theorem 5 we have $f \in H(J_1)$, $g \in H(J \setminus J_2)$, $h \in H(J_2)$, $k \in H(J \setminus J_1)$ for some $J_1, J_2 \subset J$. To avoid trivial cases, we may further assume that functions f, g, h, k are all nonconstant. Moreover, $\sigma = uv$ is pluriharmonic. Thus, since $fh + \overline{gk}$ is already pluriharmonic, we see that $f\overline{k} + h\overline{g}$ is also pluriharmonic so that, for any indices i and j, we have $(\partial_i f)(\overline{\partial_j k}) \neq 0$ if and only if $(\partial_i h)(\overline{\partial_j g}) \neq 0$. So, we may take $J_1 = J_2$. We may further assume $J_1 = \{z_1, \ldots, z_r\}$ for simplicity. Now, let $1 \leq i \leq r$ and $r + 1 \leq j \leq n$. Then, we have

$$\partial_i f(z)\overline{\partial_j k(w)} = -\partial_i h(z)\overline{\partial_j g(w)} \tag{5.3}$$

for all $z \in D^r$, $w \in D^{n-r}$. Since f, g, h, k are all nonconstant, this yields indices i_0, j_0 and points z_0, w_0 such that $\partial_{i_0} f(z_0)$, $\partial_{j_0} g(w_0)$, $\partial_{i_0} h(z_0)$, $\partial_{j_0} k(w_0)$ are all nonzero. Now, putting

$$\alpha = -\overline{\left\{\frac{\partial_{j_0} k(w_0)}{\partial_{j_0} g(w_0)}\right\}} = \frac{\partial_{i_0} h(z_0)}{\partial_{i_0} f(z_0)},$$

we see from (5.3) that

$$\partial_i h = \alpha \partial_i f, \quad \overline{\partial_j k} = -\alpha(\overline{\partial_j g})$$

for all $1 \leq i \leq r$ and $r + 1 \leq j \leq n$. It follows that $h - \alpha f$ and $\overline{k} + \alpha\overline{g}$ are both constant. Note that we may assume $h = \alpha f$ by modifying h if necessary. So, we conclude (e), as desired. The proof is complete. □

Theorem 9 has some immediate consequences which might be of some independent interest. We consider three special cases, as is done in [2]. First, taking $\sigma = 0$ in Theorem 9, we see that the zero and compact product properties are the same for Toeplitz operators with bounded pluriharmonic symbols on higher dimensional polydisks.

Corollary 5 ($n \geq 2$) *Let $u, v \in L^\infty$ be pluriharmonic symbols. Then the following statements are equivalent:*

(a) $T_u T_v = 0$.
(b) $T_u T_v$ *is compact.*
(c) *Either* $u = 0$ *or* $v = 0$.

Next, consider the case $\sigma = 1$ in Theorem 9 and suppose that condition (v) of (e) in Theorem 9 holds with $\alpha \neq 0$. Then, we have $uv = \alpha f^2 + \beta f - \alpha\overline{g}^2 + \beta\overline{g} = 1$ and thus both $\alpha f^2 + \beta f$ and $\overline{\alpha} g^2 - \overline{\beta} g$ are constant. Now, since $\alpha f^2 + \beta f$ is constant, we have $0 = \partial_j[\alpha f^2 + \beta f] = (\partial_j f)(2\alpha f + \beta)$ and thus either $\partial_j f = 0$ or $2\alpha f + \beta = 0$ for each j. It follows that f is constant. Similarly, g is also constant. So, u and v are both constant. Thus, we see that a Toeplitz operator with bounded pluriharmonic symbol can have an (essential) inverse of the same type only in the obvious cases, as in the following corollary.

Corollary 6 ($n \geq 2$) *Let $u, v \in L^\infty$ be pluriharmonic symbols. Then the following statements are equivalent:*

(a) $T_u T_v = I$.
(b) $T_v T_u = I$.
(c) $T_u T_v - I$ *is compact.*
(d) $T_v T_u - I$ *is compact.*
(e) $uv = 1$ *and either* $u, v \in H^\infty$ *or* $u, v \in \overline{H^\infty}$.

As is pointed out in the introduction, the fact that $T_z T_{\bar{z}} - I$ is compact shows that Theorem 3 cannot be extended to $n = 1$. The same example shows that Corollary 6 cannot be extended to $n = 1$. Next, take nonzero functions u, v harmonic on D and continuous on $\overline{D}$ such that they, when restricted to ∂D, are supported on disjoint sets. Then, $T_u T_v$ is compact, because $uv = 0$ on ∂D. This shows that Corollary 5 cannot be extended to $n = 1$, either.

Finally, by taking $u = v = \sigma$ in Theorem 9, one can see that there is no nontrivial (essentially) idempotent Toeplitz operator with bounded pluriharmonic symbol, as in the following corollary. The equivalence (a) $\Longleftrightarrow$ (c) in the following corollary still holds for $n = 1$, as is noticed in [2].

Corollary 7 ($n \geq 2$) *Let $u \in L^\infty$ be a pluriharmonic symbol. Then the following statements are equivalent:*

(a) $T_u^2 = T_u$.
(b) $T_u^2 - T_u$ *is compact.*
(c) *Either* $u = 0$ *or* $u = 1$.

When the symbols are confined to those that are continuous up to the boundary, the above corollary remains true for $n = 1$ by Corollary 3.

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