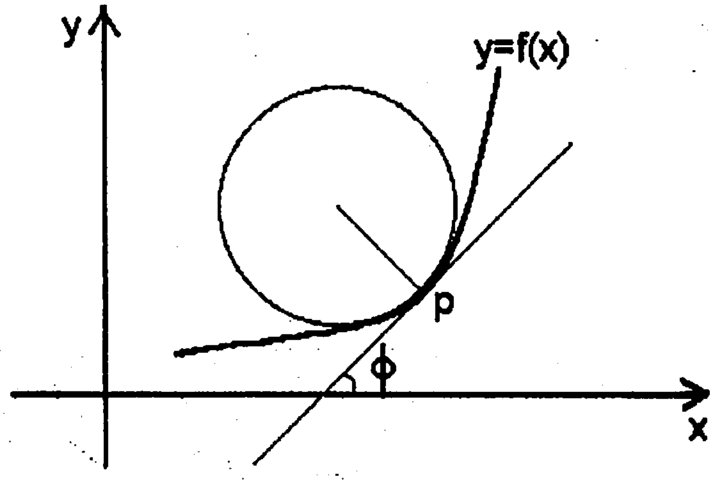
Positive Scalar Curvature and Non-commutative Geometry

Guihua Gong

The University of Puerto Rico, Rio Piedras

(Joint with Guoliang Yu)


y
y=f(x)
p
ϕ
x

To approximate a curve around a point p.

Tangent line: Best choice among lines

Slope of curve = slope of the tangent line.

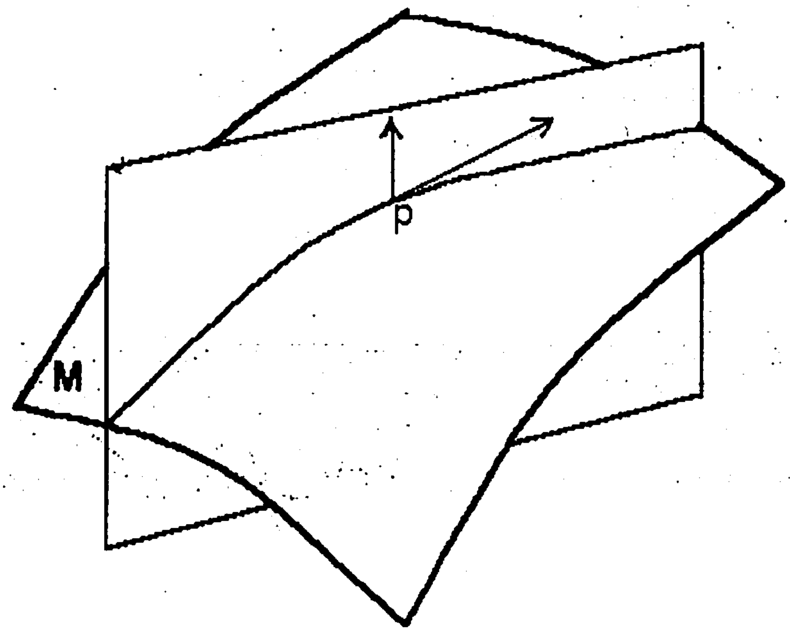
$$\text{slope} = f'(x_0)$$

Osculating circle: Best choice among circles.

Curvature of curve $= \frac{1}{\text{radius of the osculating circle}}$

$$\text{curvature} = \left|\frac{d\phi}{ds}\right|,$$

ϕ is the angle between the tangent line and x-axis, s is parameterized by arc length.


p
M

What is curvature for surface $M \hookrightarrow \mathbb{R}^3$?

Euler: Studied the surface at a point by using the curvature of the curves in which the surface intersects with various normal planes at the point. If we fix a normal direction at the point, we can give the curvature of each curve a sign, positive or negative, depending on whether it curves to or away from the normal direction.

If those curvatures are not all the same, then there are two most important directions (perpendicular to each other), one with the minimum curvature k_{min}, one with maximum curvature k_{max}. The curvature of other directions can be expressed in terms of these two curvatures.

Gauss: $k_{min} \cdot k_{max}$ is important. We define it to be Gauss curvature.

Theorema Egregium. $M \subset I\!R^3, N \subset I\!R^3$ are C^∞-surfaces. If $f : M \to N$ is a C^∞ one to one map satisfying

(*) for any curve $\Gamma \subset M$,

$$\text{arc-length}(\Gamma) = \text{arc-length}(f(\Gamma)),$$

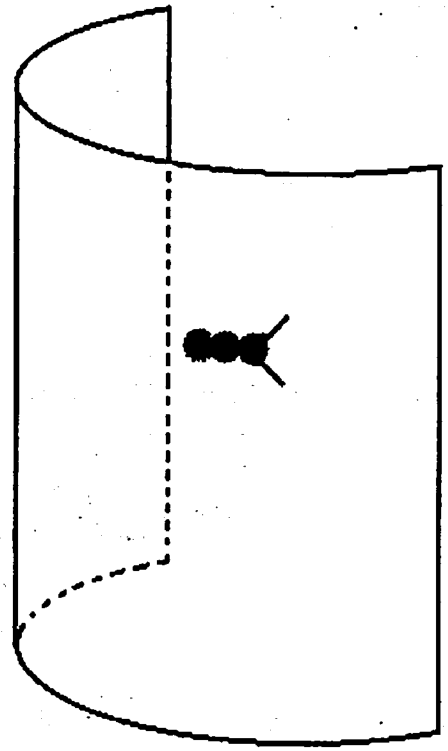
then for any point $p \in M$,

$$\text{G-curvature of } M \text{ at } p = \text{G-curvature of } N \text{ at } f(p)$$

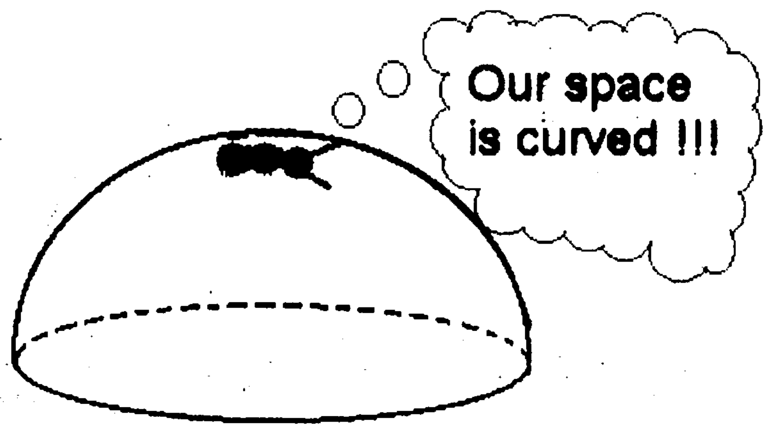
A map satisfying (*) is called an isometric map.

Gauss curvature is invariant under isometric maps.

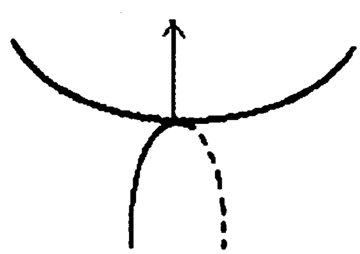
Not curved



Curved positively



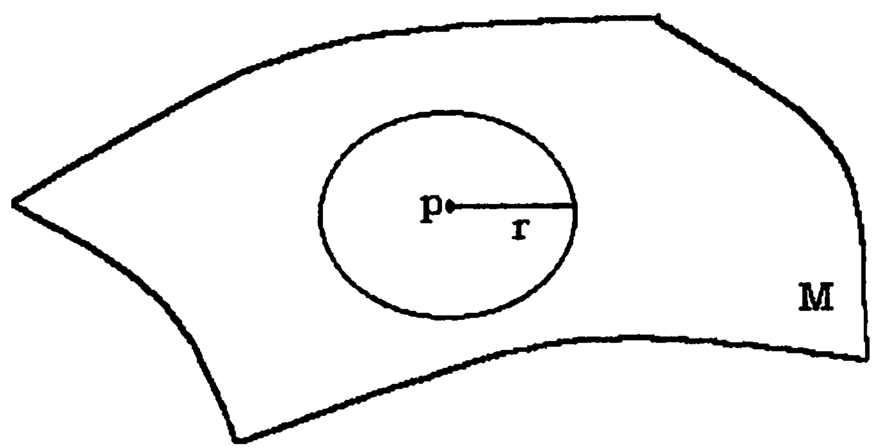
Curved negatively



Gauss curvature could be defined in terms of metric of surface alone.

Notation: For a manifold M, let

$$B_r(M,p) = \{x \in M, \operatorname{dist}(x,p) \leq r\}$$



Let $k(p)$ be Gauss Curvature of M at point $p \in M$

$$\frac{Vol\ B_r(M,p)}{Vol\ B_r(\mathbb{R}^2,0)} = 1 - \frac{k(p)}{24} r^2 + o(r^2).$$

This defines Gauss Curvature in terms of metric of surface alone.

Scalar Curvature $k(p)$: n-manifold M

$$k(p) = 6(n+2) \lim_{r \to 0} \frac{1 - \frac{Vol\ B_r(M,p)}{Vol\ B_r(\mathbb{R}^n,0)}}{r^2}$$

Or equivalently,

$$\frac{Vol\ B_r(M,p)}{Vol\ B_r(\mathbb{R}^n,0)} = 1 - \frac{k(p)}{6(n+2)} r^2 + o(r^2).$$

$k(p) > 0$: **Locally small.**

n-manifold $M \ni p$, $TM_p = I\!R^n$.

For each 2-dimensional subspace $\mathcal{U} \subset TM_p = I\!R^n$, the sectional curvature of $\mathcal{U}$ is defined to be Gauss curvature of $exp(\mathcal{U}) \subset M$, where $exp(\mathcal{U})$ is a 2-dimensional submanifold of M completely determined by $\mathcal{U}$.

Definition. Suppose that $\{e_1, e_2, \cdots, e_n\}$ forms an orthonormal basis for TM_p. Then the **scalar curvature** at p is defined to be

$$k(p) = \sum_{i,j=1}^{n} \text{sectional curvature}(\mathcal{U}(e_i, e_j)).$$

Scalar curvature is an intrinsic invariant of metric structure. It is a local invariant.

S^n has a metric with positive scalar curvature, but T^n can not have a metric with positive scalar curvature.

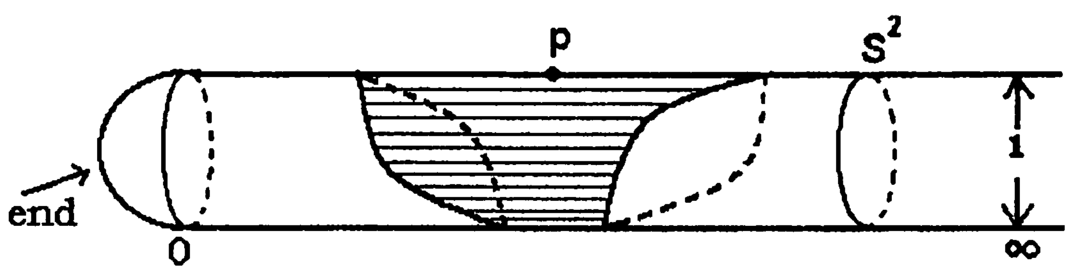
Problem What kind of manifolds permit metrics with positive scalar curvature?

Gromov, Lawson, Schoen, Yau, Rosenberg, Stolz.

<u>Definition.</u> A Riemannian manifold is called uniformly contractible if for any $r > 0$, there exists $r' > r$, such that every ball $B_r(M, p)$ with radius r can be contracted to a point in the ball $B_{r'}(M, p)$ with radius r'.

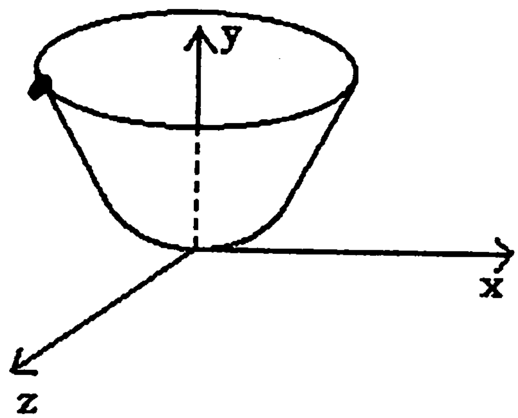
Example 1. $S^2 \times \mathbb{R}^n$ is not contractible.

Example 2. M is obtained by attaching B^3 to the end of $S^2 \times [0, \infty)$: Not uniformly contractible.



For this blue ball of radius 2 to be contracted to a single point, it must go through the end.

Example 3. Rotate $y = x^2$ about y-axis.



Any r-ball can be contracted to a single point inside a ball with radius $r^2 + 1$.

Uniformly contractible

(choosing $r' = r^2 + 1$)

Example 4. Universal covering space of compact $K(\pi, 1)$-manifold is uniformly contractible.

Uniformly contractible: Globally large.

Conjecture. (Gromov) A uniformly contractible complete Riemannian manifold M can not have uniformly positive scalar curvature. (That is, there is no $\epsilon > 0$ such that for any $p \in M, k(p) > \epsilon$.)

Globally large manifolds can't be locally small.

Example 1 and Example 2 have uniformly positive scalar curvatures.

Example 3 does not have uniformly positive scalar curvature.

The above conjecture for the case of Example 4 implies the conjecture (Gromov-Lawson): No $K(\pi, 1)$-manifold can have metric with positive scalar curvature everywhere.

Theorem (Gong-Yu). For a uniformly contractible, complete, Riemannian manifold M, if its volume has sub-exponential growth, i.e.

$$\lim_{r \to \infty} \frac{ln(\sup_{x \in M} \text{volume } B_r(M, x))}{r} = 0,$$

where $B_r(M, x) = \{y : dist(x, y) \leq r\}$, then M can not have uniformly positive scalar curvature.

Before this result, the conjecture was open even for the case of polynomial volume growth.

Conclusion: Globally large +(small) can't be locally small.

Still open: Globally large +(large) can't be locally small

Proof involves Index Theory for open manifolds, Coarse geometry, Noncommutative topology.

Any elliptic differential operator D on a compact manifold M is Fredholm, one can define

$$index(D) = dim(ker D) - dim(ker D^*).$$

Atiyah-Singer Index Theory: The index can be expressed as an integration of certain local invariants.

Heat equation approach:

$$index(D) = Trace(e^{-tD^*D}) - Trace(e^{-tDD^*}).$$

When $t \to 0$, the formula gives local invariant.

If $D^*D\psi = \lambda\psi$, then $e^{-tD^*D}\psi = e^{-t\lambda}\psi$.

If $\lambda > 0$, then $e^{-t\lambda} \to 0$ as $t \to \infty$.

As t gets larger, the operator e^{-tD^*D} (or e^{-tDD^*}) vanishes on the eigenspaces of D^*D (or DD^*) corresponding to large eigenvalues. As $t \to \infty$, both operators vanish on the eigenspaces corresponding to any positive eigenvalues. What is left is the dimension of $ker(D)$ or $ker(D^*)$. This gives Fredholm Index.

Large eigenvalues $\longrightarrow$ high energies $\longrightarrow$ short distance.

Taking index really means "coarsening" the space.

Atiyah-Kasparov-Brown-Douglas-Fillmore: Any elliptic operator D defines an element $[D]$ in K-homology group $K_0(M)$. Taking index means identifying the whole manifold M into a single point on the level of K-homology.

$$\text{Index} : [D] \in K_0(M) \longrightarrow K_0(\{pt\}) \cong K_0(\mathcal{K}).$$

But for non-compact manifold, an elliptic operator D is no longer Fredholm. This means we can not identify the whole manifold to a single point to get any interesting index.

Another way to "coarsen" the space: Identify any bounded set to a single point.

Example: $\mathbb{Z} \approx \mathbb{R} \not\approx \{pt\}$.

$$\mathbb{Z} = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbb{R} = \overline{}$$

Use C^*-algebras: noncommutative quotient spaces.

$$X \longrightarrow C(X) := \{f;\ \ f : X \to \mathbb{C}\}.$$

Noncommutative quotient space (Connes):

$X = \{x, y\}$, $\sim$ identifying x and y

$X/\sim = \{pt\}$, $C(X/\sim) = \mathbb{C}$

Noncommutative quotient:

$C(X) = \mathbb{C} \oplus \mathbb{C} \longrightarrow M_2(\mathbb{C})$ by

$$(a, b) \longmapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$x \sim y$: means $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The partial isometry $v := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has initial space p and final space q.

Put v (and v^*) in: then $p(= v^*v) \sim q(= vv^*)$.

$M_2(\mathbb{C})$—the noncommutative quotient space

$$K_0(M_2(\mathbb{C})) = K_0(\mathbb{C}) = K_0(\mathcal{K})$$

M: complete Riemannian manifold

Noncommutative coarse space $C^*(M)$ (Roe algebra): Closure of the set of all locally compact bounded linear operators $T \in B(L^2(M))$ (or $B(L^2(M,E))$, E is the spinor bundle on which D acts on) with finite propagation.

Locally compact: $f \in C_0(M)$ implies TM_f, M_fT compact, where $M_f g = f \cdot g$, for $g \in L^2(M,E)$.

Finite propagation: There is a $r > 0$ such that if $\text{dist}(supp(f), supp(g)) > r$, then $M_f T M_g = 0$. Or

$$supp(Ts) \subset \{x \in M, \text{dist}(x, supp(s)) \leq r\},$$

$$\forall s \in L^2(M)(\text{or} L^2(M,E)).$$

M compact $\Rightarrow C^*(M) \cong K(H)$

In general, an elliptic operator D on M is not Fredholm, i.e., not invertible modulo $\mathcal{K}$, but it is invertible modulo $C^*(M)$.

Index theory of non-compact manifold: Use $K_*(C^*(M))$ to replace $K_*(\mathcal{K}) = \mathbb{Z}$.

Index: $K_*(M) \longrightarrow K_*(C^*(M))$

Coarse Baum-Connes Conjecture: This index map is an isomorphism, if M is uniformly contractible.

Coarse B-C conjecture can be formulated for general metric spaces. The conjecture for the spaces of finitely generated discrete groups with word length metrics implies **Novikov conjecture**.

Coarse B-C Conjecture implies Gromov's Conjecture

(Rosenberg)

Dirac Operator D: $[D] \neq 0 \in K_*(M)$.

If coarse B-C holds, then

$$index([D]) \neq 0 \in K_*(C^*(M))$$

But if M has uniformly positive scalar curvature $k(p)$, then (Lichnerowicz)

$$D^2 = \nabla^*\nabla + \frac{1}{4}k$$

must be invertible. Hence $index([D]) = 0$.

We prove Coarse B-C Conjecture for the spaces with subexponential volume growth. As a consequence we obtain Gromov's Conjeture for this case.