

The Distribution Function Inequality and Products of Toeplitz Operators and Hankel Operators

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In this paper we study Hankel operators and Toeplitz operators through a distribution function inequality on the Lusin area integral function and the Littlewood–Paley theory. A sufficient condition and a necessary condition are obtained for the boundedness of the product of two Hankel operators. They lead to a way to approach Sarason’s conjecture on products of Toeplitz operators and shed light on the compactness of the product of Hankel operators. An elementary necessary and sufficient condition for the product of two Toeplitz operators to be a compact perturbation of a Toeplitz operator is obtained. Moreover, a necessary condition is given for the product of Hankel operators to be in the commutator ideal of the algebra generated by the Toeplitz operators with symbols in a Sarason algebra. © 1996 Academic Press, Inc.

INTRODUCTION

Let D be the open unit disk in the complex plane and ∂D the unit circle. Let $d\sigma(w)$ be the normalized Lebesgue measure on the unit circle. The Hardy space H^2 is the subspace of $L^2(\partial D, d\sigma)$ which is spanned by \mathcal{P}_+ , the space of analytic polynomials. So there is an orthogonal projection P from L^2 onto the Hardy space H^2 , the so-called Hardy projection. Let f be in L^2 . The Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined by $T_f p = P(fp)$, and $H_f p = (1 - P)(fp)$, for all p in \mathcal{P}_+ . Obviously they are densely defined on the Hardy space H^2 .

A central problem in the theory of Toeplitz operators and Hankel operators is to establish relationships between the fundamental properties of those operators and analytic and geometric properties of their symbols. In this paper we will focus on two such basic properties. One is boundedness of the product of two Toeplitz operators or two Hankel operators. The other one is compactness of the product.

The map $\zeta: f \rightarrow T_f$ is a contractive *-linear mapping from L^∞ into the bounded operators on H^2 . But it does not (fortunately) preserve products.

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Without this apparent defect, the theory of H_f and T_f would be much less interesting. On the other hand, Douglas [8] showed that ξ is actually an isometric cross section for a *-homomorphism from the Toeplitz algebra onto L^∞ . In special cases, ξ is multiplicative. Brown and Halmos [2] showed that $T_f T_g = T_{fg}$ if and only if either \bar{f} or g is in H^∞ .

The discovery of such multiplicative properties as ξ possesses has provided one key to analysis of H_f and T_f . A weak form of multiplicativity is suggested by the fact that two Toeplitz operators with symbols in $C(\partial D)$ commute with each other modulo the compact operators [5]. This leads to a hard problem: for which $f, g \in L^\infty$ is the semi-commutator $T_{fg} - T_f T_g$ (which equals $H_{\bar{f}}^* H_g$) compact? This problem also arose from studying the Fredholm theory of Toeplitz operators by Douglas, Sarason and many other people in the 1970s. The problem was solved by the combined efforts of Axler, Chang and Sarason [1] and Volberg [17] more than ten years ago. Their beautiful result is that $T_{fg} - T_f T_g$ is compact if and only if $H^\infty[\bar{f}] \cap H^\infty[g] \subset H^\infty + C(\partial D)$; here $H^\infty[g]$ denotes the closed subalgebra of L^∞ generated by H^∞ and g . Since then several other papers have studied this problem and found additional equivalent conditions; see [10], [13] and [18]. Sarason [15] asked for more comprehensible conditions. In this paper we will obtain an elementary characterization of the compactness of the product of two Hankel operators. One of main results in this paper is that $T_{fg} - T_f T_g$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \|H_{\bar{f}} k_z\|_2 \|H_g k_z\|_2 = 0; \quad (*)$$

here k_z denotes the normalized reproducing kernel in H^2 for point evaluation at z .

Little is known about when the product of Hankel operators is bounded. Recently Sarason [14] found a class of examples for which the product $T_g T_{\bar{h}}$ is bounded for two outer functions g and h in H^2 , from his study of de Branges spaces. He posed the problem of characterizing pairs of outer functions f and g in H^2 of the unit disk such that the operator $T_f T_g$ is bounded on H^2 . He made the following conjecture.

Sarason's Conjecture. Let g and h be outer functions in H^2 . The product $T_g T_{\bar{h}}$ is bounded if and only if

$$\sup_{z \in D} |g|^2(z) |h|^2(z) < \infty. \quad (**)$$

Here we follow the convention of identifying functions on the unit circle with their harmonic extensions, defined via Poisson's formula, into the unit disk D .

The Sarason conjecture is related to a famous open problem: When is the Hilbert transform bounded from a weighted $L^2(v)$ to another weighted $L^2(w)$? We will address this problem in another paper.

Treil showed that if the product $T_g T_{\bar{h}}$ is bounded, then the condition (***) holds in Sarason's Conjecture. Conversely, we will show that the condition (***) with 2 replaced by $2 + \varepsilon$ implies that $T_g T_{\bar{h}}$ is bounded.

Because of the identity $H_{\bar{f}}^* H_{\bar{g}} = T_{f\bar{g}} - T_f T_{\bar{g}}$ the problem of determining when $T_f T_{\bar{g}}$ is bounded reduces to the problem of determining when $H_{\bar{f}}^* H_{\bar{g}}$ is bounded. On the product of Hankel operators we make the following conjecture.

Conjecture ()*. Let f and g be in L^2 . Then the product $H_{\bar{f}}^* H_{\bar{g}}$ is bounded if and only if

$$\sup_{z \in D} \|H_{\bar{f}} k_z\|_2 \|H_{\bar{g}} k_z\|_2 < \infty. \tag{†}$$

Let f_+ and f_- denote $P(f)$ and $(1 - P)f$, respectively for $f \in L^2$. Then the condition (†) is equivalent to

$$\sup_{z \in D} \|f_+ \circ \phi_z - f_+(z)\|_2 \|g_- \circ \phi_z - g_-(z)\|_2 < \infty. \tag{***}$$

We will show that the condition (†) is necessary for $H_{\bar{f}}^* H_{\bar{g}}$ to be bounded and that the condition (***) is sufficient if the 2 is replaced by $2 + \varepsilon$.

This paper is arranged as follows. In Section 1 a necessary condition is obtained for the boundedness of the product of two Hankel operators. We will present an elementary condition for the compactness for the product in Section 2 and show that the condition is also sufficient in Section 8. Using the condition we will give another proof of Volberg's part of the Axler–Chang–Sarason–Volberg Theorem in Section 3. In Section 4 the result in Section 3 is extended to more general Douglas algebras. The distribution function inequality is established in Section 5. A sufficient condition is obtained for the boundedness of the product of Hankel operators in Section 6. The result in Section 6 leads to a sufficient condition for the boundedness of the product of Toeplitz operators in Section 7. The letter C will denote a positive constant, possibly different on each occurrence.

1. A NECESSARY CONDITION FOR BOUNDEDNESS

In this section a necessary condition is obtained for the boundedness of the product of two Hankel operators. First we introduce an antiunitary operator V on L^2 by defining $(Vh)(w) = e^{-i\theta} \overline{h(w)}$. The operator enjoys

many nice properties such as $V^{-1}(1 - P)V = P$ and $V = V^{-1}$. These properties lead easily to the relation $V^{-1}H_fV = H_f^*$.

Let x and y be two vectors in L^2 . $x \otimes y$ is the operator of rank one defined by

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

Observe that the norm of the operator $x \otimes y$ is $\|x\|_2 \|y\|_2$.

For z in D , let k_z be the normalized reproducing kernel $(1 - |z|)^{1/2} (1 - \bar{z}w)$ for point evaluation at z , and ϕ_z the Möbius map on the unit disk,

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

ϕ_z can also be viewed as a function on the unit circle. The product $T_{\phi_z}T_{\bar{\phi}_z}$ is the orthogonal projection onto $H^2 \ominus \{k_z\}$. Thus $1 - T_{\phi_z}T_{\bar{\phi}_z}$ is the operator $k_z \otimes k_z$ of rank one. This leads to the following Lemma.

LEMMA 1. *Let f and g be in L^2 , and z in D . Then $H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}$, is a bounded operator with norm $\|H_fk_z\|_2 \|H_gk_z\|_2$.*

Proof. By the following identity due to Treil:

$$T_{\phi_z}^*H_f^*H_gT_{\phi_z} = H_f^*(1 - P)\bar{\phi}_z(1 - P)\phi_zH_g,$$

we have

$$H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z} = H_f^*[1 - (1 - P)\bar{\phi}_z(1 - P)\phi_z]H_g.$$

On the other hand, one easily verifies that

$$1 - (1 - P)\bar{\phi}_z(1 - P)\phi_z = V(1 - T_{\phi_z}T_{\bar{\phi}_z})V^* = (Vk_z) \otimes (Vk_z).$$

Thus

$$H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z} = (H_f^*Vk_z) \otimes (H_g^*Vk_z).$$

Since $V^{-1}H_fV = H_f^*$, we obtain

$$H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z} = V[(H_fk_z) \otimes (H_gk_z)]V^*.$$

Because V is antiunitary, we conclude that

$$\|H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}\| = \|H_fk_z\|_2 \|H_gk_z\|_2.$$

This completes the proof of Lemma 1.

We thank the referee for pointing out the above proof to simplify our original proof. Using Lemma 1, we present a proof of the result of Brown–Halmos [2].

COROLLARY. *Let f and g be in L^2 . If $H_f^*H_g$ is zero, then either f or g is in H^2 .*

Proof. Assuming $H_f^*H_g$ is zero, $H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}$ is zero. But by Lemma 1, the norm of $H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}$ is $\|H_fk_z\|_2 \|H_gk_z\|_2$. So either $\|H_fk_z\|_2$ or $\|H_gk_z\|_2$ is zero. In particular, either $\|H_f1\|_2$ or $\|H_g1\|_2$ is zero. This implies that either f or g is in H^2 .

The following Theorem gives a necessary condition for the boundedness of the product of two Hankel operators.

THEOREM 1. *If $H_f^*H_g$ is bounded on H^2 , then*

$$\sup_{z \in D} \|H_fk_z\|_2 \|H_gk_z\|_2 < \infty.$$

Proof. If $H_f^*H_g$ is bounded on H^2 , let M be the norm of $H_f^*H_g$. Let z be a fixed point in D . Then $H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}$ is also bounded. Its norm is not greater than $2M$. On the other hand, by Lemma 1, the norm of $H_f^*H_g - T_{\phi_z}^*H_f^*H_gT_{\phi_z}$ is $\|H_fk_z\|_2 \|H_gk_z\|_2$. So

$$\sup_{z \in D} \|H_fk_z\|_2 \|H_gk_z\|_2 \leq 2M,$$

which completes the proof of the theorem.

2. A NECESSARY CONDITION FOR COMPACTNESS

Let $[T_f, T_g)$ be the semi-commutator $T_fT_g - T_{fg}$. Thus $[T_f, T_g) = H_f^*H_g$. If f is in $C(\partial D)$, by a result of Coburn [5], we see that both H_f and $H_{\bar{f}}$ are compact and so $\lim_{z \rightarrow \partial D} \|H_fk_z\| = 0$ and $\lim_{z \rightarrow \partial D} \|H_{\bar{f}}k_z\| = 0$ since k_z weakly converges to zero as z goes to ∂D . The following lemma gives a nice property of compact operators.

LEMMA 2. *Let K be a compact operator on H^2 . Then*

$$\lim_{z \rightarrow \partial D} \|K - T_{\phi_z}^*KT_{\phi_z}\| = 0.$$

Proof. By a result in [5], the commutator ideal of the Toeplitz algebra $\mathcal{F}(C(\partial D))$ generated by the Toeplitz operators with symbols in $C(\partial D)$ equals the ideal of compact operators. One easily verifies that the operators

$T_h[T_{f_1}, T_{g_1}][T_{f_2}, T_{g_2}] \cdots [T_{f_n}, T_{g_n}]$, with h, f_i and g_i in $C(\partial D)$ for $i = 1, \dots, n$ and any $n > 0$, span the commutator ideal. So it suffices to show

$$\lim_{z \rightarrow \partial D} \|K - T_{\phi_z}^* K T_{\phi_z}\| = 0$$

for $K = T_h[T_f, T_g]$. Because $1 - T_{\phi_z} T_{\phi_z}^*$ is $k_z \otimes k_z$ and $T_{\phi_z}^* T_h T_{\phi_z} = T_h$,

$$\begin{aligned} & \|T_h[T_f, T_g] - T_{\phi_z}^* T_h[T_f, T_g] T_{\phi_z}\| \\ &= \|T_h[T_f, T_g] - T_{\phi_z}^* T_h(T_{\phi_z} T_{\phi_z}^* + k_z \otimes k_z)[T_f, T_g] T_{\phi_z}\| \\ &\leq \|T_h\| \|[T_f, T_g] - T_{\phi_z}^*[T_f, T_g] T_{\phi_z}\| + \|T_h\| \|[T_f, T_g]^* k_z\|_2. \end{aligned}$$

By Lemma 1, the first term goes to zero as $z \rightarrow \partial D$ while the second term goes zero too as $z \rightarrow \partial D$ by the compactness of $[T_f, T_g]$. So

$$\lim_{z \rightarrow \partial D} \|T_h[T_f, T_g] - T_{\phi_z}^* T_h[T_f, T_g] T_{\phi_z}\| = 0,$$

which completes the proof of the lemma.

THEOREM 2. *Let f and g be in L^2 . If $H_f^* H_g$ is compact, then*

$$\lim_{z \rightarrow \partial D} \|H_f k_z\|_2 \|H_g k_z\|_2 = 0.$$

Proof. Let $H_f^* H_g$ be compact. By Lemma 2 we have

$$\lim_{z \rightarrow \partial D} \|H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z}\| = 0.$$

On the other hand, by Lemma 1,

$$\|H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z}\| = \|H_f k_z\|_2 \|H_g k_z\|_2.$$

This completes the proof of the theorem.

3. MORE ON NECESSARY CONDITIONS FOR COMPACTNESS

In this section we will present a new proof of Volberg's part of the Axler–Chang–Sarason–Volberg theorem using the elementary condition in Theorem 2. The following lemma is the key.

Without loss of generality we may assume that $\|f\|_\infty < 1$. As in [17], there is a unimodular function u in $f + H^\infty$ such that T_u is invertible.

LEMMA 3. If T_u is invertible, then there is a constant $C_u > 0$ such that

$$\|H_u k_z\|_2^2 \leq (1 - |u(z)|^2) \leq C_u \|H_u k_z\|_2^2$$

for all z in D .

Proof. The first inequality does not need the hypothesis of invertibility and is evident. An elementary computation yields

$$\begin{aligned} \|H_u k_z\|_2^2 &= \int |(1 - P)(u k_z)|^2 \\ &= \int |(1 - P)((u - u(z)) k_z)|^2 \leq \int |(u - u(z))|^2 |k_z|^2 d\sigma \\ &= 1 - |u(z)|^2 \end{aligned}$$

since u is unimodular. The hard part is to prove the second inequality. Now we turn to the proof. First, we have

$$H_u k_z = H_{u_-} k_z = u_- k_z - T_{u_-} k_z = (u_- - \overline{u_-(z)}) k_z,$$

and

$$\begin{aligned} T_{\bar{u}} k_z &= T_{\bar{u}_-} k_z + T_{\bar{u}_+} k_z \\ &= \overline{u_-} k_z + \overline{u_+(z)} k_z = (\overline{u_-} - \overline{u_-(z)}) k_z + \overline{u_+(z)} k_z, \end{aligned}$$

from which the equality

$$\|H_u k_z\|_2 = \|T_{\bar{u} - \overline{u(z)}} k_z\|_2$$

is immediate. On the other hand,

$$\begin{aligned} \|T_{\bar{u} - \overline{u(z)}} k_z\|_2^2 &= \|P((\bar{u} - \overline{u(z)}) k_z)\|_2^2 = \|P[\bar{u}(1 - \overline{u(z)} u) k_z]\|_2^2 \\ &\geq 1/2 \|T_{\bar{u}} P[(1 - \overline{u(z)} u) k_z]\|_2^2 \\ &\quad - \|P\{\bar{u}(1 - P)[(1 - \overline{u(z)} u) k_z]\}\|_2^2 \\ &= 1/2 \|T_{\bar{u}} P[(1 - \overline{u(z)} u) k_z]\|_2^2 - \|H_u^* H_{1 - \overline{u(z)} u} k_z\|_2^2. \end{aligned}$$

Since $T_{\bar{u}}$ is invertible, there is a constant $K > 0$ such that

$$\|T_{\bar{u}} P[(1 - \overline{u(z)} u) k_z]\|_2^2 \geq K \|P[(1 - \overline{u(z)} u) k_z]\|_2^2 = K \|T_{1 - \overline{u(z)} u} k_z\|_2^2.$$

One easily verifies

$$\begin{aligned} \|T_{1-\overline{u(z)}u}k_z\|_2^2 &= \|(1-\overline{u(z)}u)k_z\|_2^2 - \|H_{1-\overline{u(z)}u}k_z\|_2^2 \\ &= 1 - |u(z)|^2 - |u(z)|^2 \|H_u k_z\|_2^2. \end{aligned}$$

Combining the above inequality and equation yields

$$\begin{aligned} \|H_u k_z\|_2^2 &\geq K/2(1 - |u(z)|^2 - |u(z)|^2 \|H_u k_z\|_2^2) - \|H_u^* H_{1-\overline{u(z)}u} k_z\|_2^2 \\ &\geq K/2(1 - |u(z)|^2 - |u(z)|^2 \|H_u k_z\|_2^2) - |u(z)|^2 \|H_u k_z\|_2^2 \\ &= K/2(1 - |u(z)|^2) - (K/2 + 1) |u(z)|^2 \|H_u k_z\|_2^2. \end{aligned}$$

The second inequality in Lemma 3 follows immediately from the above inequality.

This completes the proof of Lemma 2.

THEOREM 3. *Let u and v be two unimodular functions such that T_u and T_v are invertible. For any $z \in D$,*

$$\|H_u^* H_v - T_{\phi_z}^* H_u^* H_v T_{\phi_z}\|^2 \geq \frac{(1 - |u(z)|^2)(1 - |v(z)|^2)}{C_u C_v},$$

where C_u is the constant in Lemma 3.

Proof. By Lemma 1 we have

$$\|H_u^* H_v - T_{\phi_z}^* H_u^* H_v T_{\phi_z}\|^2 = \|H_u k_z\|_2^2 \|H_v k_z\|_2^2.$$

Lemma 3 yields

$$\|H_u^* H_v - T_{\phi_z}^* H_u^* H_v T_{\phi_z}\|^2 \geq \frac{(1 - |u(z)|^2)(1 - |v(z)|^2)}{C_u C_v}.$$

This completes the proof of Theorem 3.

Now we are ready to present a new proof of Volberg's part of the Axler–Chang–Sarason–Volberg Theorem. Before doing so we recall concepts about function algebras. As Douglas algebras play a prominent role in various problems on Toeplitz operators and Hankel operators, we need some properties of them. A Douglas algebra is, by definition, a closed subalgebra of L^∞ which contains H^∞ . It is a consequence of the Gleason–Whitney Theorem that the maximal ideal space $\mathcal{M}(B)$ of a Douglas algebra B is naturally imbedded in $\mathcal{M}(H^\infty)$, the maximal ideal space of H^∞ . A subset of $\mathcal{M}(L^\infty)$ will be called a support set if it is the (closed) support of the representing measure for a functional in $\mathcal{M}(H^\infty + C(\partial D))$.

LEMMA 4. Let f and g be in L^∞ . let m be a point in $\mathcal{M}(H^\infty + C(\partial D))$ and S the support set of m . If there is a net $\{z_\alpha\}$ in D converging to m such that

$$\lim_{z_\alpha \rightarrow m} \|H_f k_{z_\alpha}\|_2 \|H_g k_{z_\alpha}\|_2 = 0$$

then either $f|_S$ or $g|_S$ is in $H^\infty|_S$.

Proof. Without loss of generality we may assume that $\|f\|_\infty < 1$ and $\|g\|_\infty < 1$. Then it is well-known that there are unimodular functions u in $f + H^\infty$ and v in $g + H^\infty$ such that both T_u and T_v are invertible. Since $H_f = H_u$ and $H_g = H_v$, we have

$$\lim_{z_\alpha \rightarrow m} \|H_u k_{z_\alpha}\|_2 \|H_v k_{z_\alpha}\|_2 = 0.$$

But Theorem 3 implies

$$\|H_u k_{z_\alpha}\|_2 \|H_v k_{z_\alpha}\|_2^2 \geq \frac{(1 - |u(z_\alpha)|^2)(1 - |v(z_\alpha)|^2)}{C_u C_v}$$

for some constants C_u and C_v . Therefore

$$\lim_{z_\alpha \rightarrow m} (1 - |u(z_\alpha)|^2)(1 - |v(z_\alpha)|^2) = 0.$$

As u and v are continuous on the maximal ideal space of H^∞ , we see

$$(1 - |u(m)|^2)(1 - |v(m)|^2) = 0.$$

So either $1 - |u(m)|^2 = 0$ or $1 - |v(m)|^2 = 0$. We may assume that $(1 - |u(m)|^2) = 0$. Thus $|u(m)| = 1$. Let $d\mu$ be the representing measure of m . Then $u(m) = \int_S u \, d\mu$. Because u is unimodular, we obtain that u is constant on S , so $f|_S$ is in $H^\infty|_S$. This completes the proof of the lemma.

Let f be in L^∞ . The Douglas algebra generated by the function f in L^∞ will be denoted by $H^\infty[f]$. The following theorem is Volberg's part of the Axler–Chang–Sarason–Volberg theorem. Our proof seems new.

THEOREM 4. Let f and g be in L^∞ . If $H_f^* H_g$ is compact, then $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C(\partial D)$.

Proof. From Lemma 2 [1], it suffices to show that for each support set S , either $f|_S$ or $g|_S$ is in $H^\infty|_S$. The set S is the support set for a point m in $\mathcal{M}(H^\infty + C(\partial D))$. The corona theorem tells us that there is a net $\{z_\alpha\}$

in D converging to m . On the other hand, by Theorem 3 the compactness of $H_f^* H_g$ implies

$$\lim_{z_\alpha \rightarrow m} \|H_f k_{z_\alpha}\|_2 \|H_g k_{z_\alpha}\|_2 = 0.$$

The theorem follows from Lemma 4.

4. COMMUTATOR IDEALS OF TOEPLITZ ALGEBRAS

In this section we extend results to general Douglas algebras. Let B be a Douglas algebra. The Chang-Marshall theorem [5] tells us that every Douglas algebra is generated as a closed algebra over H^∞ by a family of complex conjugates of Blaschke products. An important part of Chang’s proof is the study of a certain mean oscillation condition connected with a Douglas algebra. As in [16], a statement such as “ $\psi(z) \rightarrow 0$ as $z \rightarrow \mathcal{M}(B)$ ” has the following obvious interpretation: *given $\varepsilon > 0$ there is a Blaschke product $b \in B^{-1}$ and $\mu \in (0, 1)$ such that $|\psi(z)| < \varepsilon$ whenever $|b(z)| > \mu$.* This is equivalent to the statement: “*Whenever a net $\{z_\alpha\}$ in D converges to a point in $\mathcal{M}(B)$, $\psi(z_\alpha) \rightarrow 0$.*”

Define $VMO_B = \{f \in BMO : |f - f(z)|^2(z) \rightarrow 0 \text{ as } z \rightarrow \mathcal{M}(B)\}$. There are several characterizations on VMO_B . ([4], [16]). we are interested in an operator theoretic characterization of VMO_B , which is easily gotten from [4] and [16]. we need appropriate concepts from operator theory. Let C_B be the Sarason algebra, the C^* -algebra generated by the inner functions that are invertible in B . Chang [3] proved that $B = H^\infty + C_B$ and $VMO_B = C_B + \tilde{C}_B$. It is easy to check that

$$\lim_{z \rightarrow \mathcal{M}(B)} \max\{\|H_f k_z\|_2, \|H_{\bar{f}} k_z\|_2\} = 0,$$

for $f \in VMO_B$.

For A a subset of L^∞ , let $\mathcal{F}(A)$ denote the closed subalgebra of the algebra of bounded operators on H^2 generated by $\{T_f : f \in A\}$ and $\mathcal{J}(A)$ the commutator ideal of the algebra $\mathcal{F}(A)$.

PROPOSITION. *Let f be in BMO and B a Douglas algebra. Then both $H_f^* H_f$ and $H_{\bar{f}}^* H_{\bar{f}}$ are in $\mathcal{J}(C_B)$ if and only if f is in VMO_B .*

To prove the Proposition we need the following Lemma.

LEMMA 5. *Let C_B be a Sarason algebra. If K is in the commutator ideal $\mathcal{J}(C_B)$, then*

$$\lim_{z \rightarrow \mathcal{M}(B)} \|K - T_{\phi_z}^* K T_{\phi_z}\| = 0.$$

Proof. Since the commutator ideal $\mathcal{J}(C_B)$ is generated by those elements $T_h[T_f, T_g)$ for $f, g,$ and h in $C_B,$ we need only to show

$$\lim_{z \rightarrow \mathcal{M}(B)} \|T_h[T_f, T_g) - T_{\phi_z}^* T_h[T_f, T_g) T_{\phi_z}\| = 0.$$

Using the same method in the proof of Lemma 1, we have

$$\begin{aligned} & \|T_h[T_f, T_g) - T_{\phi_z}^* T_h[T_f, T_g) T_{\phi_z}\| \\ & \leq \|T_h\| \|H_{\bar{f}} k_z\| \|H_g k_z\| + \|T_h\| \|[T_f, T_g)^* k_z\|. \end{aligned}$$

Because f and g are in $C_B,$ $\|H_{\bar{f}} k_z\|,$ $\|H_g k_z\|,$ and $\|[T_f, T_g)^* k_z\|$ go to zero as $z \rightarrow \mathcal{M}(B).$ Thus

$$\lim_{z \rightarrow \mathcal{M}(B)} \|T_h[T_f, T_g) - T_{\phi_z}^* T_h[T_f, T_g) T_{\phi_z}\| = 0,$$

which completes the lemma.

Proof of Proposition. If both $H_f^* H_f$ and $H_{\bar{f}}^* H_{\bar{f}}$ are in $\mathcal{J}(C_B),$ by Lemma 5, we have

$$\lim_{z \rightarrow \mathcal{M}(B)} \|H_f^* H_f - T_{\phi_z}^* H_f^* H_f T_{\phi_z}\| = 0$$

and

$$\lim_{z \rightarrow \mathcal{M}(B)} \|H_{\bar{f}}^* H_{\bar{f}} - T_{\phi_z}^* H_{\bar{f}}^* H_{\bar{f}} T_{\phi_z}\| = 0.$$

It follows from Lemma 1 that $\lim_{z \rightarrow \mathcal{M}(B)} \|H_f k_z\|_2 = 0$ and $\lim_{z \rightarrow \mathcal{M}(B)} \|H_{\bar{f}} k_z\|_2 = 0.$ So f is in $VMO_B.$

The other direction follows from the equality $VMO_B = C_B + \tilde{C}_B$ [4].

The following theorem is the extension of Theorem 4. It was proved in [5] that $\mathcal{J}(C_B)$ is the ideal of compact operators if the Douglas algebra B is the minimal Douglas algebra $H^\infty + C(\partial D).$

THEOREM 5. *Let f and g be in $L^\infty.$ If $H_f^* H_g$ is in the commutator ideal $\mathcal{J}(C_B),$ then $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C_B.$*

Proof. First we show that for each support set S for a point in $\mathcal{M}(B),$ either $f|_S$ or $g|_S$ is in $H^\infty|_S.$ Since $H_f^* H_g$ is in the commutator ideal $\mathcal{J}(C_B),$ from Lemma 5 it follows that

$$\lim_{z \rightarrow \mathcal{M}(B)} \|H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z}\| = 0.$$

On the other hand, by Lemma 1 we have

$$\|H_f^* H_g - T_{\phi_z}^* H_f^* H_g T_{\phi_z}\| = \|H_f k_z\|_2 \|H_g k_z\|_2.$$

So

$$\lim_{z \rightarrow \mathcal{M}(B)} \|H_f k_z\|_2 \|H_g k_z\|_2 = 0.$$

Let S be a support set for a point m in $\mathcal{M}(B)$. Then there is a net $\{z_\alpha\}$ in D converging to m . Thus

$$\lim_{z_\alpha \rightarrow m} \|H_f k_{z_\alpha}\|_2 \|H_g k_{z_\alpha}\|_2 = 0.$$

By Lemma 4, either $f|_S$ or $g|_S$ is in $H^\infty|_S$.

To establish the theorem, we need to show that $\mathcal{M}(H^\infty[f] \cap H^\infty[g])$ contains $\mathcal{M}(B)$, since the maximal ideal space of a Douglas algebra completely determines the algebra. Because on the support set for each point m in $\mathcal{M}(B)$ either f or g is in H^∞ , the representing measure of m is multiplicative either on $H^\infty[f]$ or on $H^\infty[g]$ and hence on $H^\infty[f] \cap H^\infty[g]$. Thus $\mathcal{M}(H^\infty[f] \cap H^\infty[g])$ contains $\mathcal{M}(B)$. This completes the proof of the theorem.

One may expect that the converse of Theorem 5 holds. But when B is not the minimal Douglas algebra $H^\infty + C(\partial D)$ or L^∞ , $\mathcal{I}(C_B)$ is not an ideal of the Toeplitz algebra $\mathcal{F}(L^\infty)$, though it is the commutator ideal of $\mathcal{F}(C_B)$.

5. THE DISTRIBUTION FUNCTION INEQUALITY

In this section we will get a distribution function inequality involving the Lusin area integral and a certain maximal function. Some notations are needed.

For w a point of ∂D , we let Γ_w denote the angle with vertex w and opening $\pi/2$ which is bisected by the radius to w . The set of points z in Γ_w satisfying $|z - w| < \varepsilon$ will be denoted by $\Gamma_{w, \varepsilon}$. For h in $L^1(\partial D)$, we define the truncated Lusin area integral of h to be

$$A_\varepsilon(h)(w) = \left[\int_{\Gamma_{w, \varepsilon}} |\text{grad}h(z)|^2 dA(z) \right]^{1/2}$$

where $h(z)$ means the harmonic extension of h to D via the Poisson integral:

$$h(z) = \int_{\partial D} h(w) \frac{(1 - |z|^2)}{|1 - w\bar{z}|^2} d\sigma(w).$$

Here $dA(z)$ denotes the normalized Lebesgue measure on the unit disk D and $d\sigma(w)$ denotes the normalized Lebesgue measure in the unit circle ∂D . The Hardy–Littlewood maximal function of the function h will be denoted by h^* , and for $r > 1$, we let $A_r h = [(|h|^r)^*]^{1/r}$. For $z \in D$, we let I_z denote the closed subarc of ∂D with center $z/|z|$ and measure $\delta(z) = 1 - |z|$. The Lebesgue measure of the subset E of ∂D will be denoted by $|E|$.

Let f and g be in L^2 and $l > 2$. Define

$$\Xi_l(z) = [|f_- - f_-(z)|^l(z)]^{1/l} [|g_- - g_-(z)|^l(z)]^{1/l}$$

for z in D .

We have the following distribution function inequality.

THEOREM 6. *Let f and g be in L^2 , and ϕ and ψ in the Hardy space H^2 . Fix $l > 2$. Then there are numbers $p, r \in (1, 2)$ with $1/l + 1/r = 1/p$, such that for $|z| > 1/2$ and $a > 0$ sufficiently large,*

$$\begin{aligned} & | \{ w \in I_z : A_{2\delta(z)}(H_f \phi)(w) A_{2\delta(z)}(H_g \psi)(w) \\ & < a^2 \Xi_l(z) \inf_{w \in I_z} A_r(\phi)(w) \inf_{w \in I_z} A_r(\psi)(w) \} | \geq C_a |I_z|. \end{aligned}$$

Moreover, the constant C_a can be chosen to satisfy $\lim_{a \rightarrow \infty} C_a = 1$.

Proof. For a fixed z in D and $a > 0$ let $E(a)$ be the set of points in I_z where

$$A_{2\delta(z)}(H_f \phi)(w) \leq a [|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r(\phi)(w)$$

and $F(a)$ the set of points in I_z where

$$A_{2\delta(z)}(H_g \psi)(w) \leq a [|g_- - g_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r(\psi)(w).$$

Then we will get the following distribution function inequalities for $a > 0$ sufficiently large:

$$|E(a)| \geq K_a |I_z|, \tag{*}$$

and $|F(a)| \geq K_a |I_z|$, with $\lim_{a \rightarrow \infty} K_a = 1$. For simplicity we will present only the details of the proof of (*). The same method will prove the second distribution function inequality.

First we show how Theorem 6 follows from these two distribution inequalities. It is easy to see that

$$\begin{aligned}
E(a) \cap F(a) \subset & \{w \in I_z : A_{2\delta(z)}(H_f \phi)(w) A_{2\delta(z)}(H_g \psi)(w) \\
& < a^2 [|f_- - f_-(z)| l(z)]^{1/l} [|g_- - g_-(z)|^l(z)]^{1/l} \\
& \times \inf_{w \in I_z} A_r(\phi)(w) \inf_{w \in I_z} A_r(\psi)(w)\}.
\end{aligned}$$

Since $\lim_{a \rightarrow \infty} K_a = 1$, Theorem 6 follows from

$$\begin{aligned}
& |\{w \in I_z : A_{2\delta(z)}(H_f \phi)(w) A_{2\delta(z)}(H_g \psi)(w) \\
& < a^2 [|f_- - f_-(z)|^l(z)]^{1/l} [|g_- - g_-(z)|^l(z)]^{1/l} \\
& \times \inf_{w \in I_z} A_r(\phi)(w) \inf_{w \in I_z} A_r(\psi)(w)\}| \\
& \geq |E(a)| + |F(a)| - |I_z| \geq (2K_a - 1) |I_z|
\end{aligned}$$

if $C_a = 2K_a - 1$.

Now we turn to the proof of (*). The proof consists of three steps. Let χ_E denote the characteristic function of the subset E of ∂D . In order to prove (*) we write $H_f \phi$ as $H_f \phi = (1 - P)\phi_1 + (1 - P)\phi_2$ where $\phi_1 = [f_- - f_-(z)](\chi_{2I_z} \phi)$, and $\phi_2 = [f_- - f_-(z)](\chi_{\partial D/2I_z} \phi)$.

Step 1. For $l > 2$, there is a positive constant C and $r \in (1, 2)$ such that

$$\left[\int_{I_z} A_\varepsilon((1 - P)\phi_1)^p d\sigma(w) \right]^{1/p} \leq C |I_z|^{1/p} [|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r \phi(w),$$

where $1/l + 1/r = 1/p$, and $p > 1$.

For $l > 2$, we can always find $l' > 2$ and $p > 1$ so that $l = l'p$ and $r = pl'/l' - 2 < 2$. By the theorem of Marcinkiewicz and Zygmund, the truncated Lusin area integral $A_\varepsilon f(w)$ is L^p -bounded for $1 < p < \infty$. So for $l > 2$, we have

$$\begin{aligned}
& \int_{I_z} [A_\varepsilon(1 - P)(\phi_1)(w)]^p d\sigma(w) \\
& \leq C \int_{\partial D} |\phi_1|^p d\sigma(w) \\
& = C \int_{2I_z} |f_-(w) - f_-(z)|^p |\phi(w)|^p d\sigma(w) \\
& \leq |2I_z| \left[\frac{1}{|2I_z|} \int_{2I_z} |f_-(w) - f_-(z)|^{pl'} d\sigma(w) \right]^{1/l'} \\
& \quad \times \left[\frac{1}{|2I_z|} \int_{2I_z} |\phi|^r d\sigma(w) \right]^{p/r}.
\end{aligned}$$

Let $P(z, w)$ denote the Poisson kernel for the point z . Since

$$\left[\frac{1}{|2I_z|} \int_{2I_z} |\phi|^r d\sigma(w) \right]^{1/r} \leq A_r \phi(w)$$

for each $w \in 2I_z$, and an elementary estimate shows that for $w \in 2I_z$, $P(z, w) > C/|2I_z|$, it follows that

$$\begin{aligned} & \left[\int_{I_z} A_\varepsilon((1 - P)(\phi_1)(w))^p d\sigma(w) \right]^{1/p} \\ & \leq C |I_z|^{1/p} [|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r \phi(w). \end{aligned}$$

Step 2. For $l > 2$, on I_z ,

$$A_\varepsilon((1 - P)(\phi_2))(u) \leq C [|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_{l'}(\phi)(w),$$

for some $C > 0$ and $1/l + 1/l' = 1$. For ϕ_2 , we shall use a pointwise estimate of the norm of the gradient of $(1 - P)\phi_2$. It is easy to see that

$$(1 - P)(\phi_2)(w) = \frac{1}{2\pi} \int \frac{\bar{w}\xi\phi_2(\xi)}{1 - \bar{w}\xi} d\sigma(\xi).$$

So the function $(1 - P)(\phi_2)(w)$ is anti-holomorphic in D . Thus

$$\begin{aligned} |\text{grad}(1 - P)\phi_2(w)| & \leq C \int \frac{|\phi_2(\xi)|}{|1 - \bar{w}\xi|^2} d\sigma(\xi) \\ & \leq C \int_{\partial D/2I_z} \frac{|[f_-(\xi) - f_-(z)]\phi(\xi)|}{|1 - \bar{w}\xi|^2} d\sigma(\xi) \end{aligned}$$

On the other hand, there is a constant $C > 0$ so that

$$\left| \frac{1 - \xi\bar{z}}{1 - \xi\bar{w}} \right| \geq C$$

for all ξ in $\partial D/2I_z$ and w in $\Gamma_{u, 2\varepsilon}$. Thus we obtain

$$|\text{grad}(1 - P)\phi_2(w)| \leq C \int_{\partial D/2I_z} \frac{|[f_-(\xi) - f_-(z)]\phi(\xi)|}{|1 - \bar{z}\xi|^2} d\sigma(\xi).$$

Applying the Hölder inequality yields

$$|\text{grad}(1 - P)\phi_2(u)| \leq \frac{C}{1 - |z|^2} [|f_- - f_-(z)|^l(z)]^{1/l} [(|\phi|^{l'})(z)]^{1/l'}.$$

Because the nontangential maximal function is bounded by a constant times the Hardy–Littlewood maximal function, and because z belongs $\Gamma_{u, 2\varepsilon}$, the last factor on the right is no larger than $CA_{l'}\phi(u)$, and the desired inequality is established.

Step 3. This step will complete the proof of the distribution function inequality (*) by combining the last two steps. Since $H_f\phi = (1 - P)\phi_1 + (1 - P)\phi_2$, we have $A_{2\delta(z)}(H_f\phi)(w) \leq A_{2\delta(z)}((1 - P)\phi_1)(w) + A_{2\delta(z)}((1 - P)(\phi_2))(w)$. So for any $\lambda > 0$,

$$\bigcap_{i=1}^2 \left\{ w \in I_z : A_{2\delta(z)}((1 - P)\phi_i) \leq \frac{\lambda}{2} \right\} \subset \{ w \in I_z : A_{2\delta(z)}(H_f\phi) \leq \lambda \}.$$

Let $E_i(a)$ be the subset of I_z such that

$$A_{2\delta(z)}((1 - P)\phi_i) \leq a[|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r(\phi)(w)$$

for $i \leq 2$.

Then we have

$$\bigcap_{i=1}^2 E_i(a/2) \subset E(a).$$

Since

$$\begin{aligned} &|I_z/E_1(a/2)|^{1/p} a[|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_r(\phi)(w) \\ &\leq \left[\int_{I_z} A_{2\delta(z)}[(1 - P)\phi_1]^p d\sigma(w) \right]^{1/p}, \end{aligned}$$

it follows from Step 1 that

$$|I_z/E_1(a/2)| \leq |I_z| a^{-p}K$$

for some positive constant K which is independent of a . Hence $|E_1(a/2)| \geq (1 - a^{-p}K)|I_z|$ for a sufficiently large a .

By Step 2, for $a > 0$ sufficiently large we have

$$A_{2\delta(z)}[(1 - P)\phi_2](u) < a[|f_- - f_-(z)|^l(z)]^{1/l} \inf_{w \in I_z} A_l\phi(w)$$

everywhere on I_z , which implies $E_2(a/2) = I_z$. So

$$|E(a)| \geq (1 - a^{-p}K)|I_z|.$$

This completes the proof of (*) if we choose $K_a = 1 - a^{-p}K$.

6. A SUFFICIENT CONDITION FOR BOUNDEDNESS

In this section we apply the distribution inequality in Section 5 and the following well-known identity, the so-called Littlewood–Paley formula, to get a sufficient condition for the boundedness of $H_f^* H_g$. The idea to use the distribution inequality in the theory of Toeplitz operators and Hankel operators was first appeared in [1].

THE LITTLEWOOD–PALEY FORMULA. *If h_1 and h_2 are in L^2 and $h_1(0) = h_2(0) = 0$, then*

$$\langle h_1, h_2 \rangle = \frac{1}{\pi} \iint_D \langle \text{grad} h_1(z), \text{grad} h_2(z) \rangle \log \frac{1}{|z|^2} dA(z).$$

The Littlewood–Paley formula is a bridge from the unit circle to the unit disk which plays an important role in analysis on the unit disk [9].

THEOREM 7. *Let f and g be in L^2 . Let $l > 2$. If $\sup_{z \in D} \Xi_l(z) < \infty$, then $H_f^* H_g$ is bounded.*

Proof. Let ϕ and ψ be in H^2 . Then

$$\langle H_f^* H_g \psi, \phi \rangle = \langle H_g \psi, H_f \phi \rangle.$$

Using the Littlewood–Paley formula, we have

$$\langle H_f^* H_g \psi, \phi \rangle = \iint \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

Define

$$I = \iint_{|z| > 1/2} \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

and

$$II = \iint_{|z| < 1/2} \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

It is easy to verify that there is a compact operator T on H^2 such that

$$II = \langle T\psi, \phi \rangle.$$

We claim that there is a constant $C > 0$ such that

$$|I| \leq C \sup_{z \in D} \Xi_l(z) \|\psi\|_2 \|\phi\|_2.$$

So $\|H_f^* H_g\| \leq \|T\| + C \sup_{z \in D} \Xi_l(z)$. Now we turn to the proof of the claim. Fix an $a > 0$ for which the distribution function inequality holds. For $w \in \partial D$, let $\rho(w)$ denote the maximum of those numbers ε for which

$$A_\varepsilon(H_f \phi)(w) A_\varepsilon(H_g \psi)(w) \leq a^2 \sup_{z \in D} \Xi_l(z) A_r \phi(w) A_r \psi(w).$$

Thus

$$\begin{aligned} & \int_{\partial D} A_{\rho(w)}(H_f \phi)(w) A_{\rho(w)}(H_g \psi)(w) d\sigma(w) \\ & \leq a^2 \sup_{z \in D} \Xi_l(z) \int_{\partial D} A_r(\phi)(w) A_r(\psi) d\sigma(w) \\ & \leq a^2 \sup_{z \in D} \Xi_l(z) \|A_r(\phi)\|_2 \|A_r(\psi)\|_2 \\ & \leq a^2 \sup_{z \in D} \Xi_l(z) \|\phi\|_2 \|\psi\|_2. \end{aligned}$$

The last inequality holds because the Hardy–Littlewood maximal function is bounded on $L^{2/r}$, since $2/r > 1$.

On the other hand, letting $\chi_w(z)$ denote the characteristic function of $\Gamma_{w, \rho(w)}$, we have

$$\begin{aligned} & \int_{\partial D} A_{\rho(w)}(H_f \phi)(w) A_{\rho(w)}(H_g \psi)(w) d\sigma(w) \\ & = \int_{\partial D} \left(\int_{\Gamma_{w, \rho(w)}} |\text{grad}(H_f \phi)(z)|^2 dA(z) \right)^{1/2} \\ & \quad \times \left(\int_{\Gamma_{w, \rho(w)}} |\text{grad}(H_g \psi)(z)|^2 dA(z) \right)^{1/2} \\ & \geq \int_{|z| > 1/2} \int_{\partial D} \chi_w(z) |\text{grad}(H_f \phi)(z)| |\text{grad}(H_g \psi)(z)| d\sigma(w) dA(z). \end{aligned}$$

Now the distribution function inequality tells us that $\rho(w) \geq 2(1 - |z|)$ on a subset of I_z whose measure is at least $C_a(1 - |z|)$. If $w \in I_z$ and $\rho(w) \geq 2(1 - |z|)$, then z in $\Gamma_{w, \rho(w)}$. Thus $\chi_w(z) = 1$ on a subset of I_z of measure at least $C_a(1 - |z|)$. Combining this observation with the previous inequality, we obtain

$$\begin{aligned} & \int_{\partial D} A_{\rho(w)}(H_f \phi)(w) A_{\rho(w)}(H_g \psi)(w) d\sigma(w) \\ & \geq C_a \int_{|z| > 1/2} |\text{grad}(H_f \phi)(z)| |\text{grad}(H_g \psi)(z)| (1 - |z|) dA(z) \\ & \geq C_a |I|. \end{aligned}$$

So

$$|I| \leq C \sup_{z \in D} \Xi_l(z) \|\phi\|_2 \|\psi\|_2.$$

This completes the proof of the theorem.

To end the section we make a remark on $\Xi_l(z)$ for special functions f and g . For a function f anti-holomorphic on D , we can get easily

$$|f_- - f_-(z)|^l(z) = [|f - f(z)|^l(z)]^{1/l}.$$

So in the case that both f and g are anti-holomorphic on D , $\Xi_l(z)$ can be written as

$$\Xi_l(z) = [|f - f(z)|^l(z)]^{1/l} [|g - g(z)|^l(z)]^{1/l}.$$

This indicates that $\Xi_l(z)$ depends only the behavior of f and g on the disk. In the next section we use this representation of $\Xi_l(z)$ to study the product of two unbounded Toeplitz operators.

7. THE PRODUCT OF TWO TOEPLITZ OPERATORS

In the section we deal with the product of two unbounded Toeplitz operators. The problem posed by Sarason in [15] is the main motivation of this paper. It arose in [14], which contains a class of examples for which the product $T_f T_{\bar{g}}$ is bounded even though at least one of the factors is not.

Because of the identity $H_{\bar{f}}^* H_{\bar{g}} = T_{f\bar{g}} - T_f T_{\bar{g}}$, we see that the result in the above section can be applied to the product of two Toeplitz operators. To compare with Sarason's conjecture we state the following theorem in terms of a small number ε .

THEOREM 8. *Let f and g be two outer functions in H^2 . If, for some $\varepsilon > 0$,*

$$\sup_{z \in D} [|f|^{2+\varepsilon}(z)][|g|^{2+\varepsilon}(z)] < \infty,$$

then $T_f T_{\bar{g}}$ is bounded.

Proof. First we are going to show that $H_f^* H_{\bar{g}}$ is bounded. From Theorem 7 it is sufficient to show that $\sup_{z \in D} \Xi_{2+\varepsilon}(z) < \infty$. By the remark at the end of the last section, we have

$$\Xi_{2+\varepsilon}(z)^{2+\varepsilon} = [|f - f(z)|^{2+\varepsilon}(z)][|g - g(z)|^{2+\varepsilon}(z)].$$

But

$$|f - f(z)|^{2+\varepsilon}(z) \leq C[|f|^{2+\varepsilon}(z) + |f(z)|^{2+\varepsilon}] \leq 2C|f|^{2+\varepsilon}(z).$$

Thus

$$\Xi_{2+\varepsilon}(z)^{2+\varepsilon} \leq C[|f|^{2+\varepsilon}(z)][|g|^{2+\varepsilon}(z)].$$

So $\sup_{z \in D} \Xi_{2+\varepsilon}(z)^{2+\varepsilon} < \infty$. It follows from Theorem 7 that $H_f^* H_{\bar{g}}$ is bounded. Because of the identity $H_f^* H_{\bar{g}} = T_{f\bar{g}} - T_f T_{\bar{g}}$, we need to show $f\bar{g}$ is in L^∞ to complete the proof of the Theorem. The boundedness of the function $f\bar{g}$ follows easily from

$$\begin{aligned} |f(z) \bar{g}(z)| &= \left| \int f(w) g(w) \frac{1 - |z|^2}{|1 - z\bar{w}|^2} d\sigma(w) \right| \\ &\leq [|f|^{2+\varepsilon}(z)]^{1/(2+\varepsilon)} [|g|^{2+\varepsilon}(z)]^{1/(2+\varepsilon)}. \end{aligned}$$

The following corollary is a consequence of either the Helson–Szegő theorem or the Hunt–Muckenhoupt–Wheeden theorem ([9], [12]), as pointed out in [15]. But it is a direct consequence of Theorem 8.

COROLLARY 1. *Let f be an outer function in H^2 . If the function $|f|^2$ satisfies Muckenhoupt’s condition (A_2) , i.e.,*

$$\sup_{z \in D} [|f|^2(z)][|f|^{-2}(z)] < \infty,$$

then $T_f T_{\bar{f}^{-1}}$ is bounded.

Proof. Since Coifman and Fefferman [6] showed that if a weight w satisfies (A_p) , then w also satisfies $(A_{p-\varepsilon})$ for some $\varepsilon > 0$, it is easy to see that f also satisfies

$$\sup_{z \in D} [|f|^{2+\varepsilon}(z)][|f|^{-(2+\varepsilon)}(z)] < \infty$$

for some $\varepsilon > 0$. So the corollary follows from Theorem 8.

From the result of Coifman–Fefferman [6], one may expect that there is a so-called reverse Hölder inequality that for two outer functions f and g : if

$$\sup_{z \in D} [|f|^2(z)][|g|^2(z)] < \infty$$

then

$$\sup_{z \in D} [|f|^{2+\varepsilon}(z)][|g|^{2+\varepsilon}(z)] < \infty$$

for some $\varepsilon > 0$. But this is not true. Wolff showed us the following counterexample.

EXAMPLE. Let f be the an outer function such that $|f(w)| = 1/|\theta|^{1/2} |\log(|\theta|/2)|$ for $w = e^{i\pi\theta}$ and $\theta \in (0, 1) \cup (-1, 0)$, and $g(w) = 1 - w$. So $f(z)g(z)$ is in L^∞ . But f is not in $H^{2+\varepsilon}$ for any $\varepsilon > 0$. Of course,

$$\sup_{z \in D} [|f|^{2+\varepsilon}(z)][|g|^{2+\varepsilon}(z)] < \infty$$

doesn't hold.

On the other hand, for p in H^2 with $p(0) = 0$, we have $T_{\bar{g}}p = \bar{g}p$. So $T_f T_{\bar{g}}p = f\bar{g}p$. Thus

$$\|T_f T_{\bar{g}}p\|_2 \leq \|fg\|_\infty \|p\|_2.$$

Therefore $T_f T_{\bar{g}}$ is bounded. From [15] it follows that $\sup_{z \in D} [|f|^2(z)][|g|^2(z)] < \infty$.

When $T_f T_{\bar{g}}$ is invertible, Sarason's conjecture was proved in [7]. A characterization is found for $T_f T_{\bar{g}}$ to be bounded and invertible in [7]. Here we give another characterization as follows.

COROLLARY 2. *Let f and g be two outer functions. Then $T_f T_{\bar{g}}$ is bounded and invertible if and only if f and g satisfy the following conditions*

$$\sup_{z \in D} (|f|^2(z))(|g|^2(z)) < \infty, \tag{1}$$

and

$$\inf_{z \in D} (|f|^2(z))(|g|^2(z)) > 0. \tag{2}$$

Proof. If $T_f T_{\bar{g}}$ is bounded, Treil's result [15] tells us that (1) holds. If $T_f T_{\bar{g}}$ is invertible, then there is a constant $C > 0$ such that

$$\|T_f T_{\bar{g}}k_z\| > C.$$

But $T_f T_{\bar{g}} k_z = \overline{g(z)} f k_z$. Thus $|g(z)|^2 (|f|^2(z)) > C^2$. Also we have $|g(z)|^2 \leq (|g|^2(z))$. So $(|g|^2(z))(|f|^2(z)) > C^2$ for all $z \in D$. This implies that (2) holds.

Conversely if both (1) and (2) hold, we claim that there are two constants $C_1 > 0$ and $C_2 > 0$ such that $C_1 \leq |f(z) g(z)| \leq C_2$. Then (1) gives us that

$$\infty > \sup_{z \in D} (|f|^2(z))(|f^{-1}fg|^2(z)) \geq \sup_{z \in D} (|f|^2(z))(|f|^{-2}(z)) C_1^2.$$

By Corollary 1, we see that $T_f T_{\bar{f}^{-1}}$ is bounded. So is $T_f T_{\bar{g}}$ since

$$T_f T_{\bar{g}} = T_f T_{\bar{f}^{-1}} T_{\bar{f}g}$$

and $T_{\bar{f}g}$ is bounded. One easily verifies that $T_{(\bar{f}g)^{-1}} T_f T_{\bar{g}} = T_f T_{\bar{g}} T_{(\bar{f}g)^{-1}} = 1$. Thus $T_f T_{\bar{g}}$ is invertible.

Now we are going to prove the claim. Because $|h(z)|^2 \leq |h|^2(z)$ for all z in D , (1) implies that

$$|f^2(z) g^2(z)| \leq \sup_{z \in D} (|f|^2(z))(|g|^2(z)) = C_2.$$

To get $|f(z) g(z)| \geq C_1 > 0$, first we get $1/(fg) \in L^\infty$ by taking radial limits of $|f|^2(z)$ and $|g|^2(z)$ and (2). Then we use the fact that fg is outer to conclude that $1/(fg)$ is in H^∞ . Hence $|f(z) g(z)| > C_1 > 0$. The proof is completed.

8. A SUFFICIENT CONDITION FOR COMPACTNESS

In this section we present an elementary sufficient condition for the product of two Toeplitz operator to be a compact perturbation of a Toeplitz operator. From the above section we see that the compact perturbation problem is equivalent to the problem of the compactness of the product of two Hankel operators.

The following theorem is the main result in the section.

THEOREM 9. *Let f and g be functions in L^2 satisfying the condition in Theorem 7 for some $l > 2$. If*

$$\lim_{z \rightarrow \partial D} \bar{E}_l(z) = 0,$$

then $H_f^ H_g$ is compact.*

Proof. Let ϕ and ψ be in H^2 . Then

$$\langle H_f^* H_g \psi, \phi \rangle = \langle H_g \psi, H_f \phi \rangle.$$

Using the Littlewood–Paley formula, we have

$$\langle H_f^* H_g \psi, \phi \rangle = \iint \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

For any $s \in (0, 1)$ we define

$$I_s = \iint_{|z| > s} \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

and

$$II_s = \iint_{|z| < s} \langle \text{grad}(H_g \psi)(z), \text{grad}(H_f \phi)(z) \rangle \log \frac{1}{|z|} dA(z).$$

It is easy to verify that there is a compact operator T_s on H^2 such that

$$II_s = \langle T_s \psi, \phi \rangle.$$

As in the proof of Theorem 7, we can show that

$$|I_s| \leq C \sup_{|z| > s} \Xi_f(z) \|\phi\|_2 \|\psi\|_2.$$

Thus

$$\|H_f^* H_g - T_s\| \leq C \sup_{|z| > s} \Xi_f(z).$$

So $\lim_{s \rightarrow 1} \|H_f^* H_g - T_s\| = 0$. Because the set of compact operators is closed, $H_f^* H_g$ is compact.

Theorem 9 immediately leads to the following result.

THEOREM 10. *Let f and g be in L^∞ . If $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C(\partial D)$, then $H_f^* H_g$ is compact.*

Proof. Without loss of generality we may assume that f and g are unimodular and invertible in $H^\infty[f]$ and $H^\infty[g]$, respectively. If $H^\infty[f] \cap H^\infty[g] \subset H^\infty + C(\partial D)$ then, by Lemma 2 in [1], for each support set S , either $f|_S$ or $g|_S$ is in $H^\infty|_S$, and hence, by the invertibility, either $f|_S$ or $g|_S$ is a unimodular constant. Thus

$$\lim_{z \rightarrow \partial D} \min\{1 - |f(z)|^2, 1 - |g(z)|^2\} = 0.$$

By Lemma 3 in Section 3, we have

$$\lim_{z \rightarrow \partial D} \|H_f k_z\|_2 \|H_g k_z\|_2 = 0.$$

Since f and g are in L^∞ , it follows that

$$\lim_{z \rightarrow \partial D} |f_- - f_-(z)|^l(z) |g_- - g_-(z)|^l(z) = 0$$

for all $l > 2$. By Theorem 9 we complete the proof of the theorem.

To conclude this section we mention that the above techniques can be applied in the unit sphere in higher dimensions to get a sufficient condition for the product of two Hankel operators to be compact on the Hardy space of the unit sphere [19].

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