ESSENTIALLY COMMUTING HANKEL AND TOEPLITZ OPERATORS

KUNYU GUO AND DECHAO ZHENG

Abstract. We characterize when a Hankel operator and a Toeplitz operator have a compact commutator.

Let $d\sigma(w)$ be the normalized Lebesgue measure on the unit circle ∂D. The Hardy space H^2 is the subspace of $L^2(\partial D, d\sigma)$, denoted by L^2, which is spanned by the space of analytic polynomials. So there is an orthogonal projection P from L^2 onto the Hardy space H^2, the so-called Hardy projection. Let f be in L^∞. The Toeplitz operator T_f and the Hankel operator H_f with symbol f are defined by $T_f h = P(fh)$, and $H_f h = P(Ufh)$, for h in H^2. Here U is the unitary operator on L^2 defined by

$$Uh(w) = \bar{w}\tilde{h}(w).$$

Clearly,

$$H_f^* = H_{f^*},$$

where $f^*(w) = \overline{f(\bar{w})}$. U is a unitary operator which maps H^2 onto $[H^2]^\perp$ and has the following useful property:

$$UP = (1-P)U.$$

These two classes of operators, Hankel operators and Toeplitz operators have played an especially prominent role in function theory on the unit circle and there are many fascinating problems about those two classes of operators [7], [16], [17], [18] and [19]. It is natural to ask about the relationships between these two classes of operators. In this paper we shall concentrate mainly on the following problem:

When is the commutator $[H_g, T_f] = H_g T_f - T_f H_g$ of the Hankel operator H_g and Toeplitz operator T_f compact?

This problem is motivated by Martinez-Avendaño's recent paper [15] solving the problem of when a Hankel operator commutes with a Toeplitz operator. Martinez-Avendaño showed that H_g commutes with T_f if and only if either $g \in H^\infty$ or there exists a constant λ such that $f + \lambda g$ is in H^∞, and both $f + \tilde{f}$ and $f\tilde{f}$ are constants. Here $\tilde{f}(z)$ denotes the function $f(\bar{z})$. An equvialent statement is : H_g and T_f commute if and only if one of the following three conditions is satisfied:

$(M1)$. g is in H^∞.

$(M2)$. f and $\tilde{f}$ are in H^∞.

$(M3)$. There exists a nonzero constant λ such that $f + \lambda g$ $f + \tilde{f}$ and $f\tilde{f}$ are in H^∞.

[0]The first author is partially supported by NNSFC(10171019), Shuguang project in Shanghai, and Young teacher Fund of higher school of National Educational Ministry of China. The second author was partially supported by the National Science Foundation.

Note that $\tilde{f}$ is in $\overline{H^\infty}$ whenever f is in H^∞. Clearly, $(M2)$ means that f is constant; $(M3)$ implies that $f+\tilde{f}$ and $f\tilde{f}$ are constant since

$$\widetilde{f+\tilde{f}} = f+\tilde{f}, \qquad \widetilde{f\tilde{f}} = f\tilde{f}.$$

One may conjecture that H_g and T_f have a compact commutator if and only if Martinez-Avendaño's conditions hold on the boundary of the unit disk in some sense. In Theorem 2 we confirm this conjecture with $(M2)$ replaced by the following condition:

$(M2')$. f and $\tilde{f}$ are in H^∞, and $(f-\tilde{f})g$ is in H^∞.

To state our main results we will also need results about Douglas algebras. Let H^∞ be the subalgebra of L^∞ consisting of bounded analytic functions on the unit disk D. A Douglas algebra is, by definition, a closed subalgebra of L^∞ that contains H^∞. Let $H^\infty[f]$ denote the Douglas algebra generated by the function f in L^∞, and $H^\infty[f,g,h]$ the Douglas algebra generated by the functions f, g and h in L^∞.

Theorem 1. *The commutator $[H_g, T_f] = H_gT_f - T_fH_g$ of the Hankel operator H_g and Toeplitz operator T_f is compact if and only if*

$$H^\infty[g] \bigcap H^\infty[f,\tilde{f},(f-\tilde{f})g] \bigcap \cap_{|\lambda|>0} H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}] \subseteq H^\infty + C. \tag{0.1}$$

Here $H^\infty + C$ denotes the minimal Douglas algebra, i.e., the sum of H^∞ and the algebra $C(\partial D)$ of continuous functions on the unit circle.

This theorem completely solves the problem we mentioned before. In Section 3, we show that (0.1) can be restated as a local condition involving support sets (see Section 3 for the definition).

Theorem 2. *The commutator $[H_g, T_f] = H_gT_f - T_fH_g$ of the Hankel operator H_g and Toeplitz operator T_f is compact if and only if for each support set S, one of the following holds:*

(1). $g|_S$ is in $H^\infty|_S$.

(2). $f|_S$, $\tilde{f}|_S$ and $[(f-\tilde{f})g]|_S$ are in $H^\infty|_S$.

(3). There exists a nonzero constant λ_S such that $[f+\lambda_S g]|_S$, $[f+\tilde{f}]|_S$ and $[f\tilde{f}]|_S$ are in $H^\infty|_S$.

Theorems 1 and 2 are applications of the main result in [12], which characterizes when those compact perturbations of Toeplitz operators on the Hardy space that can be written as a finite sum of finite products of Toeplitz operators. Examples exist [2] of some f and g such that $K = H_gT_f - T_fH_g$ is not in the Toeplitz algebra, the C^*-algebra generated by the bounded Toeplitz operators; see Section 2. Clearly, such a K is not a finite sum of finite products of Toeplitz operators. But we will show that K^*K is a finite sum of finite products of Toeplitz operators.

Our work is inspired by the following beautiful theorem of Axler, Chang and Sarason [1] and Volberg [21], stated below, on the compactness of the semicommutator $T_{fg} - T_fT_g$ of two Toeplitz operators.

Axler-Chang-Sarason-Volberg Theorem. *$T_{fg} - T_fT_g$ is compact if and only if*

$$H^\infty[\bar{f}] \cap H^\infty[g] \subseteq H^\infty + C.$$

One of our motivations is the solution of the compactness of the commutator $T_fT_g - T_gT_f$ of two Toeplitz operators T_f and T_g in [9]:

Theorem 3. *The commutator* $[T_f, T_g]$ *of two Toeplitz operators is compact if and only if*

$$H^\infty[f,g] \bigcap H^\infty[\overline{f},\overline{g}] \bigcap \cap_{|a|+|b|>0} H^\infty[af+bg, \overline{af+bg}] \subseteq H^\infty + C.$$

Another motivation is the characterization of the compactness of a finite sum of products of two Hankel operators in [11].

1. Elementary identities

In this section we will obtain some identities needed in the proof of Theorem 2. Hankel operators and Toeplitz operators are closely related. First we introduce some notation.

For each z in the unit disk D, let k_z denote the normalized reproducing kernel at z:

$$k_z(w) = \frac{(1-|z|^2)^{1/2}}{1-\bar{z}w},$$

and ϕ_z be the Möbius transform:

$$\phi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

Define a unitary operator U_z on L^2 by

$$U_z f(w) = f(\phi_z(w))k_z(w),$$

for $f \in L^2$. Since $\phi_z \circ \phi_z(w) = w$ and $k_z \circ \phi_z k_z = 1$, we have

$$U_z^* = U_z = U_z^{-1}.$$

For each $f \in L^2$, we use f_+ to denote $P(f)$ and f_- to denote $(1-P)(f)$. The operator U_z has the following useful properties:

Lemma 4. *For each* $z \in D$,
(1) U_z *commutes with* P, *and*
(2) $U_z U = -U U_{\bar{z}}$.

Proof. First we show that U_z commutes with P. Let f be in L^2. Thus

$$U_z P(f) = f_+(\phi_z)k_z,$$

and

$$PU_z(f) = P(f_+(\phi_z)k_z + f_-(\phi_z)k_z) = f_+(\phi_z)k_z.$$

The last equality follows because $f_+(\phi_z)k_z$ is in H^2 and

$$\overline{\phi_z(w)}k_z(w) = -\bar{w}k_{\bar{z}}(\bar{w})$$

is perpendicular to H^2. So we obtain

$$U_z P(f) = PU_z(f)$$

for each $f \in L^2$. Hence U_z commutes with P.

Next we turn to the proof of the statement (2). For each f in L^2, an easy calculation gives

$$\begin{aligned} U_z U f &= U_z(\bar{w}\tilde{f}) = \overline{\phi_z} k_z \tilde{f}(\phi_z) \\ &= -\bar{w}k_{\bar{z}}(\bar{w}) f(\overline{\phi_z}) = -\bar{w}k_{\bar{z}}(\bar{w}) f(\phi_{\bar{z}}(\bar{w})), \end{aligned}$$

and

$$UU_{\bar{z}}f = U(f(\phi_{\bar{z}})k_{\bar{z}}) = \bar{w}f(\phi_{\bar{z}}(\bar{w}))k_{\bar{z}}(\bar{w}).$$

This implies

$$U_z U f = -U U_{\bar{z}} f.$$

So we conclude that $U_z U = -U U_{\bar{z}}$, to complete the proof of the lemma.

Let x and y be two vectors in L^2. Define $x \otimes y$ to be the following operator of rank one: for $f \in L^2$,

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

Lemma 5. *For fixed $z \in D$,*

$$H_{\overline{\phi_z}} = -k_{\bar{z}} \otimes k_z.$$

Proof. Let $\{w^n\}_0^\infty$ be the basis for H^2. For $n \geq 0$,

$$PU(\bar{w} w^n) = P(\bar{w}^n).$$

This gives

$$H_{\bar{w}} w^n = PU(\bar{w} w^n) = 0$$

for $n > 1$, and

$$H_{\bar{w}} 1 = PU(\bar{w}) = 1.$$

Hence we have

$$H_{\bar{w}} = 1 \otimes 1.$$

By Lemma 4, we have that U_z commutes with P and

$$U_{\bar{z}} U = -U U_z$$

giving

$$U_{\bar{z}} H_{\bar{w}} U_z = -H_{\overline{\phi_z}},$$

since $U_z^2 = I$. On the other hand, an easy calculation leads to

$$U_{\bar{z}}[1 \otimes 1] U_z = [U_{\bar{z}} 1] \otimes [U_z 1] = k_{\bar{z}} \otimes k_z.$$

This completes the proof.

To get the relationship between these two classes of operators, we consider the multiplication operator M_ϕ on L^2 for $\phi \in L^\infty$, defined by

$$M_\phi h = \phi h$$

for $h \in L^2$. If M_ϕ is expressed as an operator matrix with respect to the decomposition $L^2 = H^2 \oplus [H^2]^\perp$, the result is of the form

$$M_\phi = \begin{pmatrix} T_\phi & H_{\tilde{\phi}} U \\ U H_\phi & U T_{\tilde{\phi}} U \end{pmatrix}. \tag{1.1}$$

If f and g are in L^∞, then $M_{fg} = M_f M_g$, and therefore (multiply matrices and compare upper or lower left corners)

$$T_{fg} = T_f T_g + H_{\tilde{f}} H_g \tag{1.2}$$

and

$$H_{\tilde{f}g} = T_f H_g + H_{\tilde{f}} T_g. \tag{1.3}$$

The second equality implies that if $\tilde{f}$ is in H^∞, then

$$T_f H_g = H_g T_{\tilde{f}}, \tag{1.4}$$

for $g \in L^\infty$. These identities can be found in [3] and [17]. They will play an important role and be used often in this paper

Lemma 6. *Suppose that f and g are in L^∞. For each $z \in D$,*

$$T_{\tilde{\phi}_z} H_g T_f T_{\overline{\phi_z}} =$$

$$H_g T_f - [H_g T_f k_z] \otimes k_z + [H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}].$$

Proof. Since $\tilde{\phi}_z$ is in $\overline{H^\infty}$ for each $z \in D$, (1.4) gives

$$T_{\tilde{\phi}_z} H_g = H_g T_{\phi_z}.$$

So we obtain

$$T_{\tilde{\phi}_z} H_g T_f T_{\overline{\phi_z}} = H_g T_{\phi_z} T_f T_{\overline{\phi_z}} = H_g T_f T_{\phi_z} T_{\overline{\phi_z}} - H_g H_{\tilde{\phi}_z} H_f T_{\overline{\phi_z}}.$$

The last equality follows from the consequence of (1.2):

$$T_{\phi_z} T_f = T_{f\phi_z} - H_{\tilde{\phi}_z} H_f = T_f T_{\phi_z} - H_{\tilde{\phi}_z} H_f,$$

since ϕ_z is in H^∞. By (1.2) again, we obtain

$$T_{\phi_z} T_{\overline{\phi_z}} = 1 - H_{\tilde{\phi}_z} H_{\overline{\phi_z}}.$$

Lemma 5 implies

$$H_{\overline{\phi_z}} = -k_{\bar{z}} \otimes k_z,$$

and

$$H_{\tilde{\phi}_z} = H_{\overline{\phi_{\bar{z}}}} = -k_z \otimes k_{\bar{z}}.$$

Therefore we conclude

$$T_{\tilde{\phi}_z} H_g T_f T_{\overline{\phi_z}} = H_g T_f - [H_g T_f k_z] \otimes k_z + [H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}].$$

Lemma 7. *Suppose that f and g are in L^∞. For each $z \in D$,*

$$T_{\tilde{\phi}_z} T_f H_g T_{\overline{\phi_z}} = T_f H_g - [T_f H_g k_z] \otimes k_z - [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}].$$

Proof. Let z be in D. (1.2) gives

$$T_{\tilde{\phi}_z} T_f H_g T_{\overline{\phi_z}} = T_f T_{\tilde{\phi}_z} H_g T_{\overline{\phi_z}} + H_{\tilde{f}} H_{\tilde{\phi}_z} H_g T_{\overline{\phi_z}}$$

$$= T_f H_g T_{\phi_z} T_{\overline{\phi_z}} + H_{\tilde{f}} H_{\tilde{\phi}_z} H_g T_{\overline{\phi_z}}.$$

The last equality comes from (1.4). As in the proof of Lemma 6, by Lemma 5 we obtain

$$T_{\tilde{\phi}_z} T_f H_g T_{\overline{\phi_z}} = T_f H_g - [T_f H_g k_z] \otimes k_z - [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}].$$

This gives the desired result.

The next lemma will be used in the proof of Theorem 2.

Lemma 8. *Suppose that f and g are in L^∞. Let $K = H_g T_f - T_f H_g$. Then*

(1) For each $z \in D$,

$$K T_{\phi_z} = T_{\tilde{\phi}_z} K - [H_g k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}].$$

(2) Let $\lambda \neq 0$ be a constant. For each $z \in D$,

$$\lambda K T_{\phi_z} = T_{\tilde{\phi}_z} \lambda K + [H_{\tilde{f} - \lambda g} k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H_{f+\lambda g}^* k_{\bar{z}}].$$

Proof. Since ϕ_z is in H^∞ and $\tilde{\phi}_z$ is in $\overline{H^\infty}$ for each $z \in D$, by (1.2) and (1.4),

$$T_f T_{\phi_z} = T_{\phi_z} T_f + H_{\tilde{\phi}_z} H_f, \tag{1.5}$$

$$T_f T_{\tilde{\phi}_z} = T_{\tilde{\phi}_z} T_f - H_{\tilde{f}} H_{\tilde{\phi}_z}, \tag{1.6}$$

and

$$T_{\tilde{\phi}_z} H_f = H_f T_{\phi_z}. \tag{1.7}$$

Thus we have

$$\begin{aligned} KT_{\phi_z} &= H_g T_f T_{\phi_z} - T_f H_g T_{\phi_z} \\ &= H_g T_{\phi_z} T_f + H_g H_{\tilde{\phi}_z} H_f - T_f T_{\tilde{\phi}_z} H_g \\ &= T_{\tilde{\phi}_z} H_g T_f + H_g H_{\tilde{\phi}_z} H_f - T_{\tilde{\phi}_z} T_f H_g + H_{\tilde{f}} H_{\tilde{\phi}_z} H_g \\ &= T_{\tilde{\phi}_z} K + H_g H_{\tilde{\phi}_z} H_f + H_{\tilde{f}} H_{\tilde{\phi}_z} H_g \\ &= T_{\tilde{\phi}_z} K - [H_g k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}]. \end{aligned}$$

The second equality comes from (1.5) and (1.7). The third equality follows from (1.6) and (1.7). The last equality follows from Lemma 5. This completes the proof of (1).

To prove (2), for the given constant $\lambda \neq 0$, by (1.3), write

$$\begin{aligned} H_{\tilde{f}f} &= T_f H_f + H_{\tilde{f}} T_f \\ &= \lambda[H_g T_f - T_f H_g] + T_f H_{f+\lambda g} + H_{\tilde{f}-\lambda g} T_f, \end{aligned}$$

to obtain

$$\lambda K = H_{\tilde{f}f} - T_f H_{f+\lambda g} - H_{\tilde{f}-\lambda g} T_f.$$

Similarly, use of (1.5), (1.6) and (1.7) gives

$$\lambda K T_{\phi_z} = T_{\tilde{\phi}_z} \lambda K + [H_{\tilde{f}-\lambda g} k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H_{f+\lambda g}^* k_{\bar{z}}].$$

This completes the proof.

2. Compact operators

We begin with a necessary condition for a bounded operator to be compact on H^2. The proof of the following lemma is analogous to the proof of Lemma 6.1 in [20].

Lemma 9. *If $K : H^2 \to H^2$ is a compact operator, then*

$$\lim_{|z| \to 1^-} \| K - T_{\tilde{\phi}_z} K T_{\bar{\phi}_z} \| = 0.$$

Proof. Since operators of finite rank are dense in the set of compact operators, given $\epsilon > 0$ there exist $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ in H^2 so that

$$\|K - \sum_{i=1}^n f_i \otimes g_i\| < \epsilon.$$

Thus this lemma follows once we prove it for operators of rank one.

If $f \in L^2$ and $|z| \to 1^-$, then for every w on ∂D we have $z - \phi_z(w) = (1 - |z|^2)w/(1 - \bar{z}w) \to 0$ and $z - \tilde{\phi}_z(w) = (1 - |z|^2)\bar{w}/(1 - \bar{z}\bar{w})$, so by the Lebesgue Dominated Convergence Theorem, $\|zf - \phi_z f\|_2 \to 0$ and $\|zf - \tilde{\phi}_z f\|_2 \to 0$, as $|z| \to 1^-$. It follows that $\|\zeta f - \phi_z f\|_2 \to 0$ and $\|\zeta f - \tilde{\phi}_z f\|_2 \to 0$, if $z \in D$ tends to ζ.

If $f \in H^2$, we apply the Hardy projection P to obtain

$$\|\zeta f - T_{\phi_z} f\|_2 = \|\zeta f - P(\phi_z f)\|_2 \to 0,$$

and

$$\|\zeta f - T_{\tilde{\phi}_z} f\|_2 = \|\zeta f - P(\tilde{\phi}_z f)\|_2 \to 0,$$

as z in D tends to ζ. If $f, g \in H^2$, then writing

$$\begin{aligned}\|f \otimes g - T_{\tilde{\phi}_z}(f \otimes g) T_{\bar{\phi}_z}\| &= \|(\zeta f) \otimes (\zeta g) - (T_{\tilde{\phi}_z} f) \otimes (T_{\phi_z} g)\| \\ &\le \|(\zeta f - T_{\tilde{\phi}_z} f) \otimes (\zeta g)\| + \|(T_{\tilde{\phi}_z} f) \otimes (\zeta g - T_{\phi_z} g)\| \\ &\le \|\zeta f - T_{\tilde{\phi}_z} f\|_2 \|g\|_2 + \|f\|_2 \|\zeta g - T_{\phi_z} g\|_2,\end{aligned}$$

we see that

$$\|f \otimes g - T_{\tilde{\phi}_z}(f \otimes g) T_{\bar{\phi}_z}\| \to 0$$

as $|z| \to 1^-$. This completes the proof of the lemma.

By making use of the above lemma, we obtain a necessary condition for $H_g T_f - T_f H_g$ to be compact.

Lemma 10. *Suppose that f and g are in L^∞. If $H_g T_f - T_f H_g$ is compact, then*

$$\lim_{|z| \to 1^-} \|[H_g k_z] \otimes [H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}]\| = 0.$$

Proof. Suppose that $H_g T_f - T_f H_g$ is compact. Letting $K = H_g T_f - T_f H_g$, by Lemma 9, we obtain

$$\lim_{|z| \to 1^-} \| K - T_{\tilde{\phi}_z} K T_{\bar{\phi}_z}\| = 0.$$

On the other hand, Lemmas 6 and 7 give

$$\begin{aligned}T_{\tilde{\phi}_z} K T_{\bar{\phi}_z} &= [H_g T_f - [H_g T_f k_z] \otimes k_z + \\ [H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}]] &- [T_f H_g - [T_f H_g k_z] \otimes k_z - [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}]] \\ &= K - [K k_z] \otimes k_z + [H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}].\end{aligned}$$

Noting that k_z converges weakly to zero as $|z| \to 1^-$, we have

$$K k_z \to 0$$

giving

$$\lim_{|z| \to 1^-} \|[H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}]\| = 0.$$

Since

$$\begin{aligned}&\|[H_g k_z] \otimes [H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}]\| = \\ &\|[[H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}]] T_{\phi_z}\| \\ &\le \|[[H_g k_z] \otimes [T_{\phi_z} H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [T_{\phi_z} H_g^* k_{\bar{z}}]]\| \|T_{\phi_z}\|,\end{aligned}$$

we conclude that

$$\lim_{|z| \to 1^-} \|[H_g k_z] \otimes [H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}]\| = 0.$$

The next lemma gives a close relationship between H_f and H_f^*.

Lemma 11. *Suppose that f is in L^∞. For each $z \in D$,*

$$\|H_f^* k_{\bar{z}}\|_2 = \|H_f k_z\|_2.$$

The above lemma is the special case of the following lemma with $g = k_z$. We thank the referee for suggesting the following lemma.

Lemma 12. *Let H_f be a bounded Hankel operator and let $g \in H^2$. Then $H_f^* g^* = (H_f g)^*$ and thus $\|H_f^* g^*\| = \|H_f g\|$.*

Proof. Notice that for all $g \in L^2$, $(Ug)^* = Ug^*$ and $Pg^* = (Pg)^*$. Therefore,

$$H_f^* g^* = H_{f^*} g^* = PU(f^* g^*) = P(Ufg)^* = (PUfg)^* = (H_f g)^*.$$

Since $\|g^*\| = \|g\|$ for all $g \in L^2$, we have

$$\|H_f^* g^*\| = \|(H_f g)^*\| = \|H_f g\|,$$

to complete the proof.

To prove the sufficiency part of Theorem 2 we need the following result [12] which characterizes when an operator is the compact perturbation of a Toeplitz operator if the operator is a finite sum of finite products of Toeplitz operators.

Theorem 13. *A finite sum T of finite products of Toeplitz operators is a compact perturbation of a Toeplitz operator if and only if*

$$\lim_{|z| \to 1} \|T - T_{\phi_z}^* T T_{\phi_z}\| = 0. \tag{2.1}$$

Theorem 13 is a variant of Theorem 4 in the paper [10] of C. Gu. However, some crucial details are omitted from the proof in [10], especially details in the proof of a key distribution function inequality. An alternative proof of Theorem 13 can be found in the authors paper [12]. Theorem 13 can not be applied to $H_g T_f - T_f H_g$ directly since $H_g T_f - T_f H_g$ may not be a finite sum of finite products of Toeplitz operators. The following example shows that there are f and g in L^∞ such that $H_g T_f - T_f H_g$ is not a finite sum of finite products of Toeplitz operators.

Example: Let $\{a_n\}$ be a Blaschke sequence in the unit disk such that

$$\lim_{n \to \infty} a_n = 1,$$

and

$$\frac{|1 - a_n|}{1 - |a_n|} \geq 2^n.$$

Let b be the Blaschke product associated with the sequence. Let f be the function constructed in [2] with the following properties

(A) f is in $QC(= [H^\infty + C] \cap [\overline{H^\infty + C}])$.

(B) $\tilde{f} = -f$.

(C) $f(a_n) \to 1$.

Let $g = \bar{b}$, and $K = H_g T_f - T_f H_g$. It was shown that K is not compact in [2]. By making use of Theorem 2, we will show that K is not in the Toeplitz algebra. Suppose that K is in the Toeplitz algebra. We will derive a contradiction. By (1.3), we see

$$K = H_{(f - \tilde{f})g} + H_{\tilde{f}} T_g - T_{\tilde{g}} H_f.$$

It is well known that both H_f and $H_{\tilde{f}}$ are compact. Letting

$$O = H_{\tilde{f}} T_g - T_{\tilde{g}} H_f,$$

we have that O is compact and

$$K = H_{(f - \tilde{f})g} + O.$$

By a lemma in [2], which states that if a bounded operator K on H^2 is in the Toeplitz algebra, then $KT_f - T_f K$ is compact for every function $f \in QC$, we have that $H_{(f - \tilde{f})g} T_f - T_f H_{(f - \tilde{f})g}$ is compact. Let m be in the closure of $\{a_n\}$ in the maximal ideal space $M(H^\infty)$ of H^∞. Let S be the support set of m. Noting that $g|_S = \bar{b}|_S$ is not in $H^\infty|_S$, and $[(f - \tilde{f})g]|_S = 2\bar{b}|_S$, we have that for any nonzero constant λ

$$[f + \lambda(f - \tilde{f})g]|_S = (1 + 2\lambda\bar{b})|_S$$

is not in $H^\infty|_S$. By Theorem 2, we see that only Condition (2) in Theorem 2 may hold. That is,

$$(f - \tilde{f})^2 g|_S \in H^\infty|_S.$$

But (B) and (C) imply that $f - \tilde{f} = 2f$ and $f|_S = 1$. This leads to

$$4g|_S \in H^\infty|_S,$$

which is a contradiction.

The following lemma shows that K^*K is a finite sum of finite products of Toeplitz operators.

Lemma 14. *Suppose that f and g are in L^∞. Let $K = H_gT_f - T_fH_g$. Then K^*K is a finite sum of finite products of Toeplitz operators.*

Proof. Letting $K = H_gT_f - T_fH_g$, by (1.3) we write K as

$$K = -H_{\tilde{f}g} + H_gT_f + H_{\tilde{f}}T_g.$$

Taking adjoint both sides of the above equality gives

$$K^* = -H_{(\tilde{f}g)^*} + T_g^* H_{\tilde{f}^*} + T_f^* H_{g^*}$$

The last equality follows from

$$H_f^* = H_{f^*},$$

where $f^*(w) = \overline{f(\overline{w})}$. This leads to

$$K^*K = H_{(\tilde{f}g)^*}H_{\tilde{f}g} - H_{(\tilde{f}g)^*}[H_gT_f + H_{\tilde{f}}T_g]$$

$$-[T_g^*H_{\tilde{f}^*} + T_f^*H_{g^*}]H_{\tilde{f}g} + [T_g^*H_{\tilde{f}^*} + T_f^*H_{g^*}][H_gT_f + H_{\tilde{f}}T_g]. \tag{2.2}$$

The first term in the right hand side of (2.2) is a semicommutator of two Toeplitz operators since for two functions ϕ and ψ in L^∞, by (1.3)

$$H_{\phi^*}H_\psi = T_{\overline{\phi}\psi} - T_{\overline{\phi}}T_\psi;$$

both the second and the third terms are products of a Toeplitz operator and a semicommutator of two Toeplitz operators; the fourth term is the product of two Toeplitz operators and a semicommutator of two Toeplitz operators. Therefore (2.2) gives that K^*K is a finite sum of finite products of Toeplitz operators. This completes the proof of the lemma.

We thank the referee for pointing out that any product of Hankel and Toeplitz operators that has an even number of Hankel operators is a finite sum of finite products of Toeplitz operators.

A symbol mapping was defined on the Toeplitz algebra in [7]. It was extended to a $*$-homomorphism on the Hankel algebra in [3]. One of the important properties of the symbol mapping is that the symbols of both compact operators and Hankel operators are zero ([7], [3]). Note K is in the Hankel algebra and equals $H_gT_f - T_fH_g$. Clearly, the symbol of K is zero. By Theorem 13 we see that K is compact if and only if

$$\lim_{|z|\to 1} \|K^*K - T_{\phi_z}^* K^*KT_{\phi_z}\| = 0.$$

3. Proof of main results

To prove Theorems 1 and 2 we need some notation. The Gelfand space (space of nonzero multiplicative linear functionals) of the Douglas algebra B will be denoted by $M(B)$. If B is a Douglas algebra, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on B, and we identify the function f with its Gelfand transform on $M(B)$. In particular, $M(H^\infty+C)=M(H^\infty)-D$, and a function $f\in H^\infty$ may be thought of as a continuous function on $M(H^\infty+C)$. A subset of $M(L^\infty)$ is called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty+C)$. For a function F on the unit disk D and $m\in M(H^\infty+C)$, we say

$$\lim_{z\to m} F(z)=0$$

if for every net $\{z_\alpha\}\subset D$ converging to m,

$$\lim_{z_\alpha\to m} F(z_\alpha)=0.$$

The following lemma in [9] (Lemma 2.5) will be used several times later.

Lemma 15. *Let f be in L^∞ and $m\in M(H^\infty+C)$, and let S be the support set for m. Then $f|_S\in H^\infty|_S$ if and only if*

$$\underline{\lim}_{z\to m}\|H_f k_z\|_2=0.$$

Clearly, Theorem 1 follows from Theorem 2 and the following lemma.

Lemma 16. *Let $f,g\in L^\infty$. Then*

$$H^\infty[g]\bigcap H^\infty[f,\tilde f,(f-\tilde f)g]\bigcap \cap_{|\lambda|>0}H^\infty[f+\lambda g, f+\tilde f, f\tilde f]\subseteq H^\infty+C \tag{3.1}$$

if and only if for each support set S one of the following holds:

(1). $g|_S$ is in $H^\infty|_S$.

(2). $f|_S$, $\tilde f|_S$ and $[(f-\tilde f)g]|_S$ are in $H^\infty|_S$.

(3). There exists nonzero constant λ_S, such that $[f+\lambda_S g]|_S$ is in H^∞ and both $[f+\tilde f]|_S$ and $[f\tilde f]|_S$ are in $H^\infty|_S$.

Proof. Without loss of generality we may assume that $\|f\|_\infty<1/4$ and $\|g\|_\infty<1/4$. Let A denote the Douglas algebra

$$H^\infty[g]\bigcap H^\infty[f,\tilde f,(f-\tilde f)g]\bigcap \cap_{|\lambda|>0}H^\infty[f+\lambda g, f+\tilde f, f\tilde f].$$

By the Sarason Theorem (Lemma 1.3 in [9]), we get that $M(A)$ equals

$$M(H^\infty[g])\bigcup M(H^\infty[f,\tilde f,(f-\tilde f)g])\bigcup \overline{\cup_{|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde f, f\tilde f])}.$$

Suppose that (3.1) holds. Then $A\subset H^\infty+C$, and so $M(H^\infty+C)\subset M(A)$. Let $m\in M(H^\infty+C)$. Then m is an element of

$$M(H^\infty[g])\bigcup M(H^\infty[f,\tilde f,(f-\tilde f)g])\bigcup \overline{\cup_{|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde f, f\tilde f])}.$$

If m is in either of the first two sets, Lemma 1.5 in [9] gives that either Condition (1) or Condition (2) holds. Thus, we may assume that

$$m\in\overline{\cup_{|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde f, f\tilde f])}.$$

Note

$$\cap_{|\lambda|>0}H^\infty[f+\lambda g, f+\tilde f, f\tilde f]=$$

$$\cap_{1\geq|\lambda|>0}H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}]\bigcap\cap_{1\geq|\lambda|>0}H^\infty[\lambda f+g, f+\tilde{f}, f\tilde{f}],$$

since $(f+\lambda g)=\lambda(\frac{f}{\lambda}+g)$. Thus

$$\overline{\cup_{|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])}=\overline{\cup_{1\geq|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])}$$

$$\bigcup\overline{\cup_{1\geq|\lambda|>0}M(H^\infty[\lambda f+g, f+\tilde{f}, f\tilde{f}])}.$$

So m must be either in

$$\overline{\cup_{1\geq|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])}$$

or in

$$\overline{\cup_{1\geq|\lambda|>0}M(H^\infty[\lambda f+g, f+\tilde{f}, f\tilde{f}])}.$$

Now we only consider the case that m is in

$$\overline{\cup_{1\geq|\lambda|>0}M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])}.$$

If m is in the second set, use the same argument that we will use below.

We shall show that $m\in M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])$ for some λ with $|\lambda|\leq 1$. By Lemma 15 and Lemma 1.5 in [9], it suffices to show that for some λ with $|\lambda|\leq 1$,

$$\lim_{z\to m}\|H_{f+\lambda g}k_z\|_2=0,$$

$$\lim_{z\to m}\|H_{f+\tilde{f}}k_z\|_2=0,$$

and

$$\lim_{z\to m}\|H_{f\tilde{f}}k_z\|_2=0.$$

We only prove the first limit; the second and third limits follow by the same argument.

Hence there exist constants λ_α and points $m_\alpha\in M(H^\infty[f+\lambda_\alpha g, f+\tilde{f}, f\tilde{f}])$ such that $m_\alpha\to m$. We may assume that $\lambda_\alpha\to\lambda$, for some complex number λ. Clearly, $|\lambda|\leq 1$.

Note that since $m_\alpha\in M(H^\infty[f+\lambda_\alpha g, f+\tilde{f}, f\tilde{f}])$,

$$\lim_{z\to m_\alpha}\|H_{f+\lambda_\alpha g}k_z\|_2=0.$$

Since

$$dist_{L^\infty}(f+\lambda g, H^\infty)\leq\|f+\lambda g\|_\infty<1/2,$$

as a consequence of the Adamian-Arov-Krein Theorem [8], [16], there exists a unimodular function u_λ in $f+\lambda g+H^\infty$. Lemma 2 [22] gives

$$\|H_{f+\lambda g}k_z\|_2\leq(1-|u_\lambda(z)|^2)^{1/2}\leq 3\|H_{f+\lambda g}k_z\|_2, \tag{3.2}$$

where $u_\lambda(z)$ denotes the value of the harmonic extension of u_λ at z.

Note

$$\|H_{f+\lambda g}k_z\|_2\leq\|H_{f+\lambda_\alpha g}k_z\|_2+|\lambda-\lambda_\alpha|.$$

Thus we have

$$\limsup_{z\to m_\alpha}\|H_{f+\lambda g}k_z\|_2\leq\limsup_{z\to m_\alpha}\|H_{f+\lambda_\alpha g}k_z\|_2+|\lambda-\lambda_\alpha|=|\lambda-\lambda_\alpha|.$$

(3.2) gives

$$\limsup_{z\to m_\alpha}(1-|u_\lambda(z)|^2)^{1/2}\leq 3\limsup_{z\to m_\alpha}\|H_{f+\lambda g}k_z\|_2\leq 3|\lambda-\lambda_\alpha|.$$

Since $u_\lambda(m)$ is continuous on $M(H^\infty)$ [13], we have

$$(1-|u_\lambda(m_\alpha)|^2)^{1/2} \leq 3|\lambda-\lambda_\alpha|.$$

Taking the limit on both sides of the above inequality gives

$$(1-|u_\lambda(m)|)^{1/2} = \limsup_{m_\alpha\to m}(1-|u_\lambda(m_\alpha)|^2)^{1/2} \leq \limsup_{m_\alpha\to m} 3|\lambda-\lambda_\alpha| = 0.$$

We obtain

$$(1-|u_\lambda(m)|)^{1/2} = 0.$$

On the other hand, (3.2) gives

$$\begin{aligned}\limsup_{z\to m}\|H_{f+\lambda g}k_z\|_2 &\leq \limsup_{z\to m}(1-|u_\lambda(z)|)^{1/2}\\ &= (1-|u_\lambda(m)|)^{1/2} = 0.\end{aligned}$$

This gives the desired result.

Conversely, let S be the support set for an element $m \in M(H^\infty + C)$ and suppose that one of Conditions (1), (2) and (3) holds for m. Then by Lemma 1.5 in [9], either $m \in M(H^\infty[g])$ or $m \in M(H^\infty[f, \tilde{f}, (f-\tilde{f})g])$, or there exists a nonzero constant λ, such that $m \in M(H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}])$. Thus m is in

$$M(H^\infty[g]\bigcap H^\infty[f,\tilde{f},(f-\tilde{f})g]\bigcap \cap_{|\lambda|>0}H^\infty[f+\lambda g, f+\tilde{f}, f\tilde{f}]).$$

Therefore, $M(H^\infty + C) \subseteq M(A)$. By the Chang-Marshall Theorem ([6], [14]) $A \subseteq H^\infty + C$. The proof of Lemma 16 is completed

Let BMO be the space of functions with bounded mean oscillation on the unit circle. If f is in BMO and analytic or co-analytic on D, the norm $\|f\|_{BMO}$ is equivalent to

$$|f(0)| + \sup_{z\in D}\|f\circ\phi_z - f(z)\|_p$$

for $p \geq 1$. It is well known that the Hardy projection P maps L^∞ into BMO ([8] and [18]).

Lemma 17. *Suppose that f and g are in L^∞. If*

$$\lim_{z\to m}\|H_g k_z\|_2 = 0,$$

then

$$\lim_{z\to m}\|H_g T_f k_z\|_2 = 0.$$

Proof. Write

$$g = g_+ + g_-$$

where $g_+ = P(g)$ and $g_- = (1-P)(g)$. Since U_z commutes with the Hardy projection P we get

$$H_g k_z = H_{g_-}k_z = H_{g_-}U_z 1 = -U_{\bar{z}}H_{g_-\circ\phi_z}1.$$

Thus we have

$$\|H_g k_z\|_2 = \|U_{\bar{z}}H_{g_-\circ\phi_z}1\|_2 = \|H_{g_-\circ\phi_z}1\|_2.$$

The last equality follows because $U_{\bar{z}}$ is a unitary operator on L^2. An easy calculation gives

$$H_{g_-\circ\phi_z}1 = U(g_-\circ\phi_z - g_-(z)).$$

Therefore

$$\lim_{|z|\to 1}\|g_-\circ\phi_z - g_-(z)\|_2 = 0.$$

Similarly we can also get

$$\|H_g T_f k_z\|_2 = \|U_{\bar{z}} H_{g_-\circ\phi_z} T_{f\circ\phi_z} 1\|_2$$

$$= \|H_{g_-\circ\phi_z} T_{f\circ\phi_z} 1\|_2 = \|H_{g_-\circ\phi_z}(f_+ \circ \phi_z + f_-(z))\|_2$$

$$= \|(1-P)(g_- \circ \phi_z - g_-(z))(f_+ \circ \phi_z + f_-(z))\|_2.$$

The first equality holds because $U_{\bar{z}}$ commutes with P and the second equality holds because $U_{\bar{z}}$ is a unitary operator on L^2. The third equality follows from the decomposition of f:

$$f = f_+ + f_-.$$

The Hölder inequality gives

$$\|H_g T_f k_z\|_2 \leq \|g_- \circ \phi_z - g_-(z)\|_4 \|f_+ \circ \phi_z + f_-(z)\|_4.$$

To prove that

$$\lim_{z\to m} \|H_g T_f k_z\|_2 = 0$$

we need only to show that

$$\|f_+ \circ \phi_z + f_-(z)\|_4 \leq C\|f\|_\infty,$$

and

$$\lim_{z\to m} \|g_- \circ \phi_z - g_-(z)\|_4 = 0.$$

$$f_+ \circ \phi_z + f_-(z) = f_+ \circ \phi_z - f_+(z) + f(z),$$

we have

$$\|f_+ \circ \phi_z + f_-(z)\|_4 \leq \|f_+ \circ \phi_z - f_+(z)\|_4 + \|f\|_\infty$$

$$\leq C_1 \|P(f)\|_{BMO} + \|f\|_\infty \leq C\|f\|_\infty,$$

for some positive constants C and C_1. The last inequality follows because P is bounded from L^∞ to BMO. The Hölder inequality gives

$$\|g_- \circ \phi_z - g_-(z)\|_4 \leq \|g_- \circ \phi_z - g_-(z)\|_2^{1/4} \|g_- \circ \phi_z - g_-(z)\|_6^{3/4}$$

$$\leq C\|g_- \circ \phi_z - g_-(z)\|_2^{1/4} \|g_-\|_{BMO}^{1/4} \leq C\|g_- \circ \phi_z - g_-(z)\|_2^{1/4} \|g\|_\infty^{1/4}.$$

The last inequality also follows because P is bounded from L^∞ to BMO. This gives

$$\lim_{z\to m} \|g_- \circ \phi_z - g_-(z)\|_4 = 0,$$

to complete the proof of the lemma.

Combining Lemmas 11 and 17 we have the following lemma needed in the proof of Theorem 2.

Lemma 18. *Suppose that f and g are in L^∞. If*

$$\lim_{z\to m} \|H_g^* k_{\bar{z}}\|_2 = 0,$$

then

$$\lim_{z\to m} \|H_g^* T_f k_{\bar{z}}\|_2 = 0.$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2

First we prove the necessity part of Theorem 2. Suppose that $H_gT_f - T_fH_g$ is compact. Without loss of generality we may assume that $\|f\|_\infty < 1/2$ and $\|g\|_\infty < 1/2$. By Lemma 10 we get

$$\lim_{|z|\to 1^-} \|[H_gk_z]\otimes[H_f^*k_{\bar{z}}] + [H_{\tilde{f}}k_z]\otimes[H_g^*k_{\bar{z}}]\| = 0. \tag{3.3}$$

Let m be in $M(H^\infty + C)$, and let S be the support set of m. By Carleson's Corona Theorem [5], there is a net z converging to m.

Suppose that

$$\underline{\lim}_{z\to m}\|H_gk_z\|_2 = 0.$$

By Lemma 15 we have that $g|_S$ is in $H^\infty|_S$. So Condition (1) holds.

Suppose that there is a constant c such that

$$\underline{\lim}_{z\to m}\|H_gk_z\|_2 \geq c > 0.$$

Let $\lambda_z = \langle H_{\tilde{f}}k_z, H_gk_z\rangle/\|H_gk_z\|^2$. Then $|\lambda_z| \leq \frac{1}{c}$, and so we may assume that $\lambda_z \to c_m$ for some constant c_m.

Applying the operator $[[H_gk_z]\otimes[H_f^*k_{\bar{z}}] + [H_{\tilde{f}}k_z]\otimes[H_g^*k_{\bar{z}}]]^*$ to H_gk_z and multiplying by $\frac{1}{\|H_gk_z\|_2^2}$ we get

$$\lim_{z\to m}\|H_f^*k_{\bar{z}} + \overline{\lambda_z}[H_g^*k_{\bar{z}}]\|_2 = 0.$$

Thus

$$\lim_{z\to m}\|H_{f+c_mg}^*k_{\bar{z}}\|_2 = 0.$$

Lemma 11 implies

$$\lim_{z\to m}\|H_{f+c_mg}k_z\|_2 = 0. \tag{3.4}$$

Now we consider two cases.

Case 1. $c_m = 0$.

In this case, we have

$$\lim_{z\to m}\|H_fk_z\|_2 = 0.$$

(3.3) gives

$$\lim_{z\to m}\|H_{\tilde{f}}k_z\|_2 = 0.$$

Thus by Lemma 15, we obtain

$$f|_S \in H^\infty|_S,$$

and

$$\tilde{f}|_S \in H^\infty|_S.$$

On the other hand, by making use of (1.3) twice we have

$$K = H_{(f-\tilde{f})g} - T_{\tilde{g}}H_f + H_{\tilde{f}}T_g.$$

Since K is compact and k_z converges to 0 weakly as $z \to m$, we have

$$\lim_{z\to m}\|[H_{(f-\tilde{f})g} - T_{\tilde{g}}H_f + H_{\tilde{f}}T_g]k_z\|_2 = 0.$$

Lemma 17 gives

$$\lim_{z\to m}\|[-T_{\tilde{g}}H_f + H_{\tilde{f}}T_g]k_z\|_2 = 0.$$

Thus

$$\lim_{z\to m}\|H_{(f-\tilde f)g}k_z\|_2=0.$$

By Lemma 15, we have

$$[(f-\tilde f)g]|_S\in H^\infty|_S.$$

to get Condition (2).

Case 2. $c_m\neq 0$.

In this case, by Lemma 15 and (3.4), we obtain

$$(f+c_mg)|_S\in H^\infty|_S.$$

Now write

$$[H_gk_z]\otimes[H^*_{\tilde f}k_{\bar z}]+[H_{\tilde f}k_z]\otimes[H^*_gk_{\bar z}]=$$
$$[H_gk_z]\otimes[H^*_{\tilde f+c_mg}k_{\bar z}]+[H_{\tilde f-c_mg}k_z]\otimes[H^*_gk_{\bar z}].$$

Because

$$\lim_{z\to m}\|[H_gk_z]\otimes[H^*_{\tilde f}k_{\bar z}]+[H_{\tilde f}k_z]\otimes[H^*_gk_{\bar z}]\|=0,$$

and

$$\lim_{z\to m}\|[H^*_{\tilde f+c_mg}k_{\bar z}]\|_2=0,$$

we get

$$\lim_{z\to m}\|[H_{\tilde f-c_mg}k_z]\otimes[H^*_gk_{\bar z}]\|=0.$$

Note that

$$\|[H^*_gk_{\bar z}]\|_2=\|H_gk_z\|_2,$$

and

$$\|[H_{\tilde f-c_mg}k_z]\otimes[H^*_gk_{\bar z}]\|=\|[H_{\tilde f-c_mg}k_z]\|_2\|[H^*_gk_{\bar z}]\|_2.$$

We have

$$\lim_{z\to m}\|[H_{\tilde f-c_mg}k_z]\|_2=0.$$

Combining (3.4) with the above limit gives

$$\lim_{z\to m}\|H_{\tilde f+f}k_z\|_2=0$$

since

$$H_{\tilde f+f}k_z=H_{\tilde f-c_mg}k_z+H_{f+c_mg}k_z.$$

Therefore by Lemma 15,

$$(f+\tilde f)|_S\in H^\infty|_S.$$

To prove that $(f\tilde f)|_S\in H^\infty|_S$, by (1.3), write

$$H_{\tilde f f}=T_fH_f+H_{\tilde f}T_f$$

$$=c_m[H_gT_f-T_fH_g]+T_fH_{f+c_mg}+H_{\tilde f-c_mg}T_f \tag{3.5}$$

By Lemma 17, we obtain

$$\lim_{z\to m}\|H_{\tilde f-c_mg}T_fk_z\|_2=0.$$

Apply the bounded operator T_f to $H_{\tilde f+c_mg}T_fk_z$, to get

$$\lim_{z\to m}\|T_fH_{f+c_mg}k_z\|_2=0.$$

Since $[H_gT_f-T_fH_g]$ is compact and k_z weakly converges to zero as $z\to m$, we have

$$\lim_{z\to m}\|[H_gT_f-T_fH_g]k_z\|_2=0.$$

Therefore (3.5) implies

$$\lim_{z\to m}\|H_{f\tilde{f}}k_z\|_2=0.$$

Lemma 15 gives

$$(f\tilde{f})|_S\in H^\infty|_S.$$

So Condition (3) holds. This completes the proof of the necessity part.

Next we shall prove the sufficiency part of Theorem 2. Suppose that f and g satisfy one of Conditions (1)-(3) in Theorem 2.

Let $K=H_gT_f-T_fH_g$ and $T=K^*K$. By Lemma 14, T is a finite sum of finite products of Toeplitz operators with zero symbol. By Theorem 13, we need only to show

$$\lim_{|z|\to 1}\|T-T^*_{\phi_z}TT_{\phi_z}\|=0.$$

By the Carleson Corona Theorem, the above condition is equivalent to the condition that for each $m\in M(H^\infty+C)$,

$$\lim_{z\to m}\|T-T^*_{\phi_z}TT_{\phi_z}\|=0. \tag{3.6}$$

Let m be in $M(H^\infty+C)$, and let S be the support set of m. By Carleson's Corona Theorem, there is a net z converging to m.

Suppose that Condition (1) holds, i.e., $g|_S\in H^\infty|_S$. Lemma 15 gives that

$$\lim_{z\to m}\|H_gk_z\|_2=0.$$

By Lemma 11, we have

$$\lim_{z\to m}\|H_g^*k_{\bar{z}}\|_2=0.$$

Let

$$F_z=-[H_gk_z]\otimes[H_f^*k_{\bar{z}}]-[H_{\tilde{f}}k_z]\otimes[H_g^*k_{\bar{z}}],$$

the above limits give

$$\lim_{z\to m}\|F_z\|=0.$$

This gives

$$\lim_{z\to m}\|[K^*T^*_{\tilde{\phi}_z}]F_z+F_z^*[T_{\tilde{\phi}_z}K]+F_z^*F_z\|=0, \tag{3.7}$$

since

$$\|K\|<\infty, \qquad \sup_{z\in D}\|F_z\|<\infty.$$

By Lemma 8 we also have

$$KT_{\phi_z}=T_{\tilde{\phi}_z}K+F_z,$$

to get

$$\begin{aligned}T^*_{\phi_z}TT_{\phi_z}&=[KT_{\phi_z}]^*[KT_{\phi_z}]=K^*T^*_{\tilde{\phi}_z}T_{\tilde{\phi}_z}K+[K^*T^*_{\tilde{\phi}_z}]F_z+F_z^*[T_{\tilde{\phi}_z}K]+F_z^*F_z\\&=K^*K+[K^*k_{\bar{z}}]\otimes[K^*k_{\bar{z}}]+[K^*T^*_{\tilde{\phi}_z}]F_z+F_z^*[T_{\tilde{\phi}_z}K]+F_z^*F_z.\end{aligned}$$

The last equality comes from

$$T^*_{\tilde{\phi}_z}T_{\tilde{\phi}_z}=1-k_{\bar{z}}\otimes k_{\bar{z}}. \tag{3.8}$$

Lemma 18 gives

$$\lim_{z\to m}\|K^*k_{\bar{z}}\|_2=0.$$

Therefore (3.7) implies (3.6).

Suppose that Condition (2) holds. By Lemma 15, we have

$$\lim_{z\to m} \|H_f k_z\|_2 = 0,$$

$$\lim_{z\to m} \|H_{\tilde{f}} k_z\|_2 = 0,$$

and

$$\lim_{z\to m} \|H_{(f-\tilde{f})g} k_z\|_2 = 0. \tag{3.9}$$

Let

$$F_z = -[H_g k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H_g^* k_{\bar{z}}];$$

the above limits give

$$\lim_{z\to m} \|F_z\| = 0.$$

This gives

$$\lim_{z\to m} \|[K^* T^*_{\tilde{\phi}_z}] F_z + F_z^* [T_{\tilde{\phi}_z} K] + F_z^* F_z\| = 0. \tag{3.10}$$

By Lemma 8 we have

$$K T_{\phi_z} = T_{\tilde{\phi}_z} K + F_z,$$

to get

$$\begin{aligned} T^*_{\phi_z} T T_{\phi_z} &= [K T_{\phi_z}]^* [K T_{\phi_z}] \\ &= K^* T^*_{\tilde{\phi}_z} T_{\tilde{\phi}_z} K + [K^* T^*_{\tilde{\phi}_z}] F_z + F_z^* [T_{\tilde{\phi}_z} K] + F_z^* F_z \\ &= K^* K - [K^* k_{\bar{z}}] \otimes [K^* k_{\bar{z}}] + [K^* T^*_{\tilde{\phi}_z}] F_z + F_z^* [T_{\tilde{\phi}_z} K] + F_z^* F_z. \end{aligned}$$

The last equality comes from (3.8).

By making use of (1.3) twice, we have

$$K = H_{(f-\tilde{f})g} - T_{\tilde{g}} H_f + H_{\tilde{f}} T_g.$$

Lemma 18 and (3.9) give

$$\lim_{z\to m} \|K^* k_{\bar{z}}\|_2 = 0.$$

Therefore (3.10) implies (3.6).

Suppose that Condition (3) holds. Then for some constant $c_m \neq 0$,

$$\lim_{z\to m} \|H_{f-c_m g} k_z\|_2 = 0, \tag{3.11}$$

$$\lim_{z\to m} \|H_{f+\tilde{f}} k_z\|_2 = 0, \tag{3.12}$$

$$\lim_{z\to m} \|H_{f\tilde{f}} k_z\|_2 = 0. \tag{3.13}$$

These give

$$\lim_{z\to m} \|H_{\tilde{f}+c_m g} k_z\|_2 = 0, \tag{3.14}$$

since $\tilde{f} + c_m g = f + \tilde{f} - (f - c_m g)$. By (3.5) and Lemma 18, we have

$$\lim_{z\to m} \|K^* k_{\bar{z}}\| = 0. \tag{3.15}$$

Lemma 8 gives

$$c_m K T_{\phi_z} = T_{\tilde{\phi}_z} c_m K - [H_{\tilde{f}-c_m g} k_z] \otimes [H_f^* k_{\bar{z}}] + [H_{\tilde{f}} k_z] \otimes [H^*_{f+c_m g} k_{\bar{z}}]. \tag{3.16}$$

Letting

$$G_z = [H_{\tilde{f}-c_m g} k_z] \otimes [H_f^* k_{\bar{z}}] - [H_{\tilde{f}} k_z] \otimes [H^*_{f+c_m g} k_{\bar{z}}],$$

we have

$$c_m K T_{\phi_z} = T_{\tilde{\phi}_z} c_m K + G_z$$

and

$$\lim_{z\to m} \|G_z\| = 0.$$

The last limit follows from (3.11) and (3.14) and gives

$$\lim_{z\to m} \|[T_{\tilde{\phi}_z} c_m K]^* G_z + G_z^* T_{\tilde{\phi}_z} c_m K + G_z^* G_z\| = 0. \tag{3.17}$$

(3.16) gives

$$[c_m K T_{\phi_z}]^*[c_m K T_{\phi_z}] = [T_{\tilde{\phi}_z} c_m K + G_z]^*[T_{\tilde{\phi}_z} c_m K + G_z]$$

$$= [T_{\tilde{\phi}_z} c_m K]^*[T_{\tilde{\phi}_z} c_m K] + [T_{\tilde{\phi}_z} c_m K]^* G_z + G_z^* T_{\tilde{\phi}_z} c_m K + G_z^* G_z$$

$$= |c_m|^2 K^* T^*_{\tilde{\phi}_z} T_{\tilde{\phi}_z} K + [T_{\tilde{\phi}_z} c_m K]^* G_z + G_z^* T_{\tilde{\phi}_z} c_m K + G_z^* G_z$$

$$= |c_m|^2 K^* K - |c_m|^2 [K^* k_{\bar{z}}] \otimes [K^* k_{\bar{z}}] + [T_{\tilde{\phi}_z} c_m K]^* G_z + G_z^* T_{\tilde{\phi}_z} c_m K + G_z^* G_z.$$

The last equality comes from (3.8). (3.15) implies that the second term on the right hand side of the above equality converges to zero and (3.17) implies that the third, fourth and fifth terms converge to zero. Thus we conclude

$$\lim_{z\to m} \||c_m|^2 T - |c_m|^2 T^*_{\phi_z} T T_{\phi_z}\| = 0.$$

Since $c_m \neq 0$, the above limit gives (3.6). This completes the proof of Theorem 2.

ACKNOWLEDGMENTS

The authors would like to thank D. Sarason and the referee for their useful suggestions.

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Department of Mathematics, Fudan University, Shanghai, 200433, The People's Republic of China.
E-mail address: kyguo@fudan.edu.cn

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240, USA
E-mail address: zheng@math.vanderbilt.edu