BEURLING TYPE THEOREM ON THE BERGMAN SPACE VIA THE HARDY SPACE OF THE BIDISK

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ABSTRACT. In this paper, by lifting the Bergman shift as the compression of an isometry on a subspace of the Hardy space of the bidisk, we give a proof of the Beurling type theorem on the Bergman space of Aleman, Richter and Sundberg [1] via the Hardy space of the bidisk.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The *Bergman space* L^2_a is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on $\mathbb{D}$. The famous Beurling theorem [3] classifies the invariant subspaces of the multiplication operator by the coordinate function z on the Hardy space of the unit disk, the unilateral shift. For an operator T on a Hilbert space H, the subspace $H \ominus TH$ is called a *wandering subspace* of T on H. The Beurling theorem says that all invariant subspaces of the unilateral shift are generated by their wandering subspaces with dimension 1. This result has played an important role in function theory and operator theory. Recently Aleman, Richter and Sundberg have established a remarkable Beurling type theorem [1] for the multiplication operator by the coordinate function z on the Bergman space, the Bergman shift even if the dimension of the wandering subspace ranges from 1 to ∞ [2]. Their result states that all invariant subspaces of the Bergman shift are also generated by their wandering subspaces. This result is a breakthrough in understanding of the invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on the Bergman space [6], [10]. Different proofs of the Beurling type theorem were given in [11], [12], [15] later. The goal of this paper is to give a new proof of the Beurling type theorem of Aleman, Richter and Sundberg [1] via the Hardy space of the bidisk. Our main objective is the study of the Bergman space of the unit disk and its operators via the Hardy space of the bidisk. Our project starts in [9], [17], [18] to study multiplication operators on L^2_a by bounded analytic functions on the unit disk $\mathbb{D}$ via the Hardy space of the bidisk. The theme is to use the theory of multivariable operators and functions to study a single operator and the functions of one variable. Our main idea is to lift the Bergman shift up as the compression of a commuting pair of isometries on a subspace of the Hardy space of the bidisk. This idea was used in studying the Hilbert modules by R. Douglas and V. Paulsen [4],

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operator theory in the Hardy space over the bidisk by R. Douglas and R. Yang [5], [19], [20] and [21]; the higher-order Hankel forms by S. Ferguson and R. Rochberg [7] and [8] and the lattice of the invariant subspaces of the Bergman shift by S. Richter [14].

Let $\mathbb{T}$ denote the unit circle. The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$. The *Hardy space* $H^2(\mathbb{T}^2)$ over the bidisk is $H^2(\mathbb{T}) \otimes H^2(\mathbb{T})$. For each integer $n \geq 0$, let

$$p_n(z, w) = \sum_{i=0}^{n} z^i w^{n-i}.$$

Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$. Thus every function in $\mathcal{H}$ is symmetric with respect to z and w. Let $[z-w]$ denote the closure of $(z-w)H^2(\mathbb{T}^2)$ in $H^2(\mathbb{T}^2)$. As every function in $[z-w]$ is orthogonal to each p_n, we easily see

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus [z-w].$$

Let $\mathcal{A}$ be a subspace of $L^2(\mathbb{T}^2)$ and let $P_{\mathcal{A}}$ denote the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $\mathcal{A}$. The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2)$ is defined by

$$T_f(h) = P_{H^2(\mathbb{T}^2)}(fh),$$

for h in $H^2(\mathbb{T}^2)$. It is not difficult to see that T_z and T_w are a pair of doubly commuting pure isometries on $H^2(\mathbb{T}^2)$. A little computation gives

$$P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}.$$

We use $\mathcal{B}$ to denote the operator above. It was shown explicitly in [16] and implicitly in [4] that $\mathcal{B}$ is unitarily equivalent to the *Bergman shift*, the multiplication operator by the coordinate function z on the Bergman space L_a^2 via the following unitary operator $U : L_a^2(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z, w)}{n+1}.$$

So the Bergman shift is lifted up as the compression of an isometry on a nice subspace $\mathcal{H}$ of $H^2(\mathbb{T}^2)$. In the rest of the paper we identify the Bergman shift with the operator $\mathcal{B}$.

This paper is organized as follows. In Section 2, as in [14] we lift each invariant subspace of $\mathcal{B}$ as an invariant subspace of the isometry T_z, do the Wold decomposition and identify the the wandering subspace. In Section 3, using the structure of the wandering space and establishing an identity (in Step 8 in the proof of Theorem 3.1) we give a proof of Aleman, Richter and Sundberg theorem [1].

2. WOLD DECOMPOSITION AND WANDERING SUBSPACES

First we introduce notation to lift each invariant subspace of $\mathcal{B}$ as an invariant subspace of T_z. For an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, define the lifting $\widetilde{\mathcal{M}}$ to be the direct sum

$$\mathcal{M} \oplus [z-w].$$

In Theorem 6.8 [14] Richter showed that the mapping

$$\eta : \mathcal{M} \to \widetilde{\mathcal{M}}$$

is a one-to-one correspondence between invariant subspaces of $\mathcal{B}$ and invariant subspaces of T_z containing $[z-w]$. In this section we will get the Wold decomposition of the isometry T_z on $\widetilde{\mathcal{M}}$ and identify the wandering subspace of T_z on $\widetilde{\mathcal{M}}$.

Although for an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, $\mathcal{M}$ may not be invariant for T_z^*, let the operator $\mathcal{B}_{\mathcal{M}}^*$ on $\mathcal{M}$ denote the compression of T_z^* on $\mathcal{M}$, i.e.,

$$\mathcal{B}_{\mathcal{M}}^* q = P_{\mathcal{M}} T_z^* q = P_{\mathcal{M}} \mathcal{B}^* q$$

for q in $\mathcal{M}$. Since the Bergman shift is bounded below, we have the following lemma.

Lemma 2.1. *Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$. Then $\mathcal{B}_{\mathcal{M}}^*\mathcal{B}$ is invertible on $\mathcal{M}$.*

Proof. Since $\mathcal{B}$ is unitarily equivalent to the Bergman shift, an easy computation gives

$$\|\mathcal{B}f\| \geq \frac{1}{\sqrt{2}}\|f\|$$

for each f in $\mathcal{M}$. The Cauchy-Schwarz inequality gives

$$\begin{aligned} \|\mathcal{B}_{\mathcal{M}}^*\mathcal{B}f\|\|f\| &\geq |\langle \mathcal{B}_{\mathcal{M}}^*\mathcal{B}f, f\rangle \\ &= \|\mathcal{B}f\|^2 \\ &\geq \frac{1}{2}\|f\|^2. \end{aligned}$$

Thus we have

$$\|\mathcal{B}_{\mathcal{M}}^*\mathcal{B}f\| \geq \frac{1}{2}\|f\|$$

for each f in $\mathcal{M}$. So $\mathcal{B}_{\mathcal{M}}^*\mathcal{B}$ is bounded below on $\mathcal{M}$. To show that $\mathcal{B}_{\mathcal{M}}^*\mathcal{B}$ is invertible on $\mathcal{M}$, we need only show that $\mathcal{B}_{\mathcal{M}}^*\mathcal{B}$ is onto. If it is not so, then there is a nonzero function f in $\mathcal{M}$ such that

$$\langle \mathcal{B}_{\mathcal{M}}^*\mathcal{B}x, f\rangle = 0$$

for all x in $\mathcal{M}$. In particular, letting x be equal to f in the above equality, we have

$$\|\mathcal{B}f\|^2 = 0,$$

to get that f is zero since $\mathcal{B}$ is injective on $\mathcal{M}$. This shows that $\mathcal{B}_{\mathcal{M}}^*\mathcal{B}$ is onto on $\mathcal{M}$, to complete the proof.

The following lemma will be used in the proof of Theorem 2.3. First we introduce a notation. For two functions $x,\ y$ in $H^2(\mathbb{T}^2)$, the symbol $x \otimes y$ is the operator on $H^2(\mathbb{T}^2)$ defined by

$$(x \otimes y)g = [\langle g, y\rangle_{H^2(\mathbb{T}^2)}]x$$

for $g \in H^2(\mathbb{T}^2)$.

Lemma 2.2. *On the Hardy space $H^2(\mathbb{T}^2)$, the identity operator equals*

$$I \;=\; T_z T_z^* + \sum_{l\geq 0} w^l \otimes w^l = T_w T_w^* + \sum_{l \geq 0} z^l \otimes z^l.$$

Proof. We will just verify the first equality in the lemma since the same argument leads to the proof of the second equality.

To do this, let h be in $H^2(\mathbb{T}^2)$. Writing $h(z,w) = \sum_{j=0}^{\infty} h_j(w)z^j$ for some functions $h_j(w)$ in $H^2(\mathbb{T})$, we have

$$T_zT_z^*h = \sum_{j=0}^{\infty} h_j(w)T_zT_z^*z^j = \sum_{j=1}^{\infty} h_j(w)z^j.$$

Using the identity

$$\begin{aligned}(w^l \otimes w^l)h &= \langle h, w^l\rangle w^l \\ &= \sum_{j=0}^{\infty}\langle h_j(w)z^j, w^l\rangle w^l = \langle h_0(w), w^l\rangle w^l,\end{aligned}$$

we have

$$(\sum_{l\geq 0} w^l \otimes w^l)h = \sum_{l\geq 0}\langle h_0(w), w^l\rangle w^l = h_0(w),$$

to conclude that for each h in $H^2(\mathbb{T}^2)$,

$$\begin{aligned}[T_zT_z^* + \sum_{l\geq 0} w^l \otimes w^l]h &= \sum_{j=1}^{\infty} h_j(w)z^j + h_0(w) \\ &= \sum_{j=0}^{\infty} h_j(w)z^j = h.\end{aligned}$$

This completes the proof.

Let $\mathcal{M}_0$ be the wandering space of $\mathcal{B}$ on $\mathcal{M}$. Clearly, the wandering subspace $\mathcal{L}_{\widetilde{\mathcal{M}}}$ of T_z on $\widetilde{\mathcal{M}}$ contains $\mathcal{M}_0$. To get understanding $\mathcal{L}_{\widetilde{\mathcal{M}}}$, we need to find out the orthogonal complement of $\mathcal{M}_0$ in $\mathcal{L}_{\widetilde{\mathcal{M}}}$. Let

$$\mathcal{M}_{00} = \{-h_g + zP_{\mathcal{H}}g - wg(w) : \ (h_g, g) \in \mathcal{BM} \times H^2(\mathbb{T}) \text{ and } h_g = \mathcal{B}[\mathcal{B}^*_{\mathcal{M}}\mathcal{B}]^{-1}P_{\mathcal{M}}g\}.$$

The following theorem gives the Wold decomposition of T_z on $\widetilde{\mathcal{M}}$ and shows that the orthogonal complement of $\mathcal{M}_0$ in $\mathcal{L}_{\widetilde{\mathcal{M}}}$ equals $\mathcal{M}_{00}$.

Theorem 2.3. *Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$. Let $\widetilde{\mathcal{M}}$ be the lifting of $\mathcal{M}$. Then $\widetilde{\mathcal{M}}$ is an invariant subspace of the isometry T_z and has the following decomposition:*

$$\widetilde{\mathcal{M}} = \oplus_{n=0}^{\infty} z^n \mathcal{L}_{\widetilde{\mathcal{M}}}$$

where $\mathcal{L}_{\widetilde{\mathcal{M}}}$ is the wandering space of T_z on $\widetilde{\mathcal{M}}$ given by

$$\mathcal{L}_{\widetilde{\mathcal{M}}} = \mathcal{M}_0 \oplus \mathcal{M}_{00}.$$

Proof. Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$ and $\widetilde{\mathcal{M}}$ denote the lifting of $\mathcal{M}$.

It was shown in [14] that $\widetilde{\mathcal{M}}$ is an invariant subspace of T_z. For completeness we include a proof. To do this, for each f in $\widetilde{\mathcal{M}}$, we write

$$f = f_1 + f_2$$

for f_1 in $\mathcal{M}$ and f_2 in $[z-w]$. Since $[z-w]$ is an invariant subspace of T_z, $T_z f_2 = z f_2$ is in $[z-w]$. A little computation gives

$$\begin{aligned} T_z f_1 &= P_{\mathcal{H}}(z f_1) + P_{[z-w]}(z f_1) \\ &= \mathcal{B} f_1 + P_{[z-w]}(z f_1) \\ &\in \mathcal{M} \oplus [z-w] = \widetilde{\mathcal{M}}. \end{aligned}$$

The last equality follows from that $\mathcal{M}$ is an invariant subspace of $\mathcal{B}$. Thus we have

$$\begin{aligned} T_z f &= T_z f_1 + T_z f_2 \\ &= \mathcal{B} f_1 + T_z f_2 \\ &\in \mathcal{M} \oplus [z-w] = \widetilde{\mathcal{M}}, \end{aligned}$$

to get that $\widetilde{\mathcal{M}}$ is an invariant subspace of the isometry T_z.

Since for every function f in $\cap_{n=0}^{\infty} T_z^n \widetilde{\mathcal{M}}$, f and its derivatives vanish at $z=0$, we have

$$\cap_{n=0}^{\infty} T_z^n \widetilde{\mathcal{M}} = \{0\}.$$

By the Wold decomposition theorem [13], we have

$$\widetilde{\mathcal{M}} = \oplus_{n=0}^{\infty} z^n \mathcal{L}_{\widetilde{\mathcal{M}}}$$

where $\mathcal{L}_{\widetilde{\mathcal{M}}}$ is the wandering subspace of T_z on $\widetilde{\mathcal{M}}$.

To finish the proof, we need identify the wandering subspace $\mathcal{L}_{\widetilde{\mathcal{M}}}$ by showing

$$\mathcal{L}_{\widetilde{\mathcal{M}}} = \mathcal{M}_0 \oplus \mathcal{M}_{00}.$$

First we show that $\mathcal{M}_0$ is orthogonal to $\mathcal{M}_{00}$. To do this, let m be in $\mathcal{M}_0$ and $-h_g + zP_{\mathcal{H}} g - wg(w)$ in $\mathcal{M}_{00}$. An easy calculation gives

$$\begin{aligned} \langle m, -h_g + zP_{\mathcal{H}} g - wg(w) \rangle &= -\langle m, h_g \rangle \\ &= 0. \end{aligned}$$

The first equality follows from that $zP_{\mathcal{H}} g - wg(w)$ is in $[z-w]$ and the last equality follows from that h_g is in $\mathcal{B}\mathcal{M}$. Thus $\mathcal{M}_0$ is orthogonal to $\mathcal{M}_{00}$.

Second we show

$$\mathcal{M}_0 \oplus \mathcal{M}_{00} \subset \mathcal{L}_{\widetilde{\mathcal{M}}}.$$

To do this, let m be in $\mathcal{M}_0$ and $-h_g + zP_{\mathcal{H}} g - wg(w)$ in $\mathcal{M}_{00}$. For each f_1 in $\mathcal{M}$ and f_2 in $[z-w]$, we have that $P_{\mathcal{M}} T_z f_1 = \mathcal{B} f_1$ and $z f_2$ is contained in $[z-w]$, to get

$$\langle m, T_z f_1 \rangle = \langle m, \mathcal{B} f_1 \rangle = 0,$$

and

$$\langle m, T_z f_2 \rangle = \langle m, z f_2 \rangle = 0.$$

These give

$$\langle m, T_z (f_1 + f_2) \rangle = 0.$$

An easy computation gives

$$\begin{aligned}\langle -h_g + zP_{\mathcal{H}}g - wg(w), T_z f_1\rangle &= \langle -T_z^* h_g + P_{\mathcal{H}}g, f_1\rangle - \langle T_z^*(wg(w)), f_1\rangle \\ &= \langle -T_z^* h_g + P_{\mathcal{H}}g, P_{\mathcal{M}} f_1\rangle \\ &= \langle -\mathcal{B}_{\mathcal{M}}^* h_g + P_{\mathcal{M}}g, f_1\rangle \\ &= 0.\end{aligned}$$

The second equality follows from that $T_z^*(wg(w)) = 0$ and f_1 is in $\mathcal{M}$. The third equality follows from that $\mathcal{B}_{\mathcal{M}}^*$ is the compression of T_z^* on $\mathcal{M}$. Since $[z-w]$ is an invariant subspace of T_z and $T_z^*(wg(w)) = 0$, we also have

$$\begin{aligned}\langle -h_g + zP_{\mathcal{H}}g - wg(w), T_z f_2\rangle &= -\langle h_g, T_z f_2\rangle + \langle zP_{\mathcal{H}}g, T_z f_2\rangle - \langle wg(w), T_z f_2\rangle \\ &= \langle P_{\mathcal{H}}g, f_2\rangle - \langle T_z^*(wg(w)), f_2\rangle \\ &= \langle P_{\mathcal{H}}g, f_2\rangle = 0.\end{aligned}$$

Thus these give

$$\langle -h_g + zP_{\mathcal{H}}g - wg(w), T_z(f_1 + f_2)\rangle = 0.$$

So we obtain

$$\mathcal{M}_0 \oplus \mathcal{M}_{00} \subset \mathcal{L}_{\widetilde{\mathcal{M}}}.$$

Third we show

$$\mathcal{L}_{\widetilde{\mathcal{M}}} \subset \mathcal{M}_0 \oplus \mathcal{M}_{00}.$$

For q in $\mathcal{L}_{\widetilde{\mathcal{M}}}$, write

$$q = q_1 + q_2$$

for q_1 in $\mathcal{M}$ and q_2 in $[z-w]$. Noting that q_1 is orthogonal to $T_z[z-w]$ and

$$T_z\widetilde{\mathcal{M}} = T_z\mathcal{M} \oplus T_z[z-w],$$

we have that q_2 is orthogonal to $T_z[z-w]$ and $q_1 + q_2$ is orthogonal to $T_z\mathcal{M}$.

Using Lemma 2.2, we will derive a special representation of q_2. Since q_2 is orthogonal to $T_z[z-w]$, we have that $T_z^* q_2$ is orthogonal to $[z-w]$. Letting $G = T_z^* q_2$, then G is in $\mathcal{H}$ and

$$\begin{aligned}zG &= T_z T_z^* q_2 \\ &= q_2 - q_2(0, w).\end{aligned}$$

The last equality follows from Lemma 2.2:

$$I = T_z T_z^* + \sum_{n=0}^{\infty} w^n \otimes w^n.$$

Since q_2 is in $[z-w]$, by Lemma 8 in [9], we have that $q_2(z, z) = 0$ to get

$$zG(z, z) = q_2(z, z) - q_2(0, z) = -q_2(0, z).$$

Letting $g(z) = G(z, z)$, we have that $g(z)$ is in $H^2(\mathbb{T})$ since $q_2(z, w)$ is in $H^2(\mathbb{T}^2)$. Rewrite the above equality as

$$q_2(z, w) = zG(z, w) - wg(w). \tag{2.1}$$

Applying the projection $P_{\mathcal{H}}$ to both sides of the above equality gives

$$\begin{aligned} 0 &= P_{\mathcal{H}} q_2 \\ &= P_{\mathcal{H}}[zG - wg(w)] \\ &= P_{\mathcal{H}} T_z G - P_{\mathcal{H}}[w(P_{\mathcal{H}} g(w) + P_{\mathcal{H}^\perp} g(w))] \\ &= \mathcal{B}G - P_{\mathcal{H}}(w(P_{\mathcal{H}} g(w)) \\ &= \mathcal{B}G - \mathcal{B}(P_{\mathcal{H}} g(w)) \\ &= \mathcal{B}(G - (P_{\mathcal{H}} g(w))). \end{aligned}$$

The first equality follows from that q_2 is in $[z-w]$ and the fourth equality follows from that $[z-w]$ is an invariant subspace of T_w. Using the fact that $\mathcal{B}$ is injective on $\mathcal{H}$, we have

$$G = P_{\mathcal{H}} g.$$

By (2.1), we obtain

$$q_2 = zP_{\mathcal{H}} g - wg(w). \tag{2.2}$$

Since $q_1 + q_2$ is orthogonal to $T_z\mathcal{M}$, a simple computation gives

$$\begin{aligned} 0 &= \langle q_1 + q_2, T_z f\rangle \\ &= \langle q_1 + zP_{\mathcal{H}} g - wg(w), T_z f\rangle \\ &= \langle T_z^* q_1 + P_{\mathcal{H}} g, f\rangle \\ &= \langle P_{\mathcal{M}} T_z^* q_1 + P_{\mathcal{M}} g, f\rangle \end{aligned}$$

for each f in $\mathcal{H}$. This gives

$$\mathcal{B}_{\mathcal{M}}^* q_1 + P_{\mathcal{M}} g = P_{\mathcal{M}} T_z^* q_1 + P_{\mathcal{M}} g = 0. \tag{2.3}$$

Noting

$$\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{B}\mathcal{M},$$

we write

$$q_1 = m_0 - \mathcal{B}h_1 \tag{2.4}$$

for some h_1 in $\mathcal{M}$ and m_0 in $\mathcal{M}_0$. Applying $\mathcal{B}_{\mathcal{M}}^*$ to both sides of the above equality and using (2.3) give

$$\mathcal{B}_{\mathcal{M}}^* \mathcal{B} h_1 = P_{\mathcal{M}} g.$$

Thus

$$h_1 = (\mathcal{B}_{\mathcal{M}}^* \mathcal{B})^{-1} P_{\mathcal{M}} g.$$

Letting $h_g = \mathcal{B}h_1$ and using (2.4) and (2.2), we have

$$q = q_1 + q_2 = m_0 - h_g + zP_{\mathcal{H}} g - wg(w) \in \mathcal{M}_0 \oplus \mathcal{M}_{00}$$

to complete the proof.

3. A PROOF OF ALEMAN, RICHTER AND SUNDBERG THEOREM

In this section we will give a proof of the following Beurling type theorem of Aleman, Richter and Sundberg in [1]. Let S be a subspace of $\mathcal{H}$. We use $[S]$ to denote the smallest invariant subspace of $\mathcal{B}$ containing S, i.e.,

$$[S] = \vee_{n\geq 0}\mathcal{B}^n S,$$

where $\vee_{n\geq 0}\mathcal{B}^n S$ denotes the closed subspace of $\mathcal{H}$ spanned by the set $\{\mathcal{B}^n S : n \geq 0\}$.

Theorem 3.1. *Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$. Then $\mathcal{M} = [\mathcal{M} \ominus \mathcal{B}\mathcal{M}]$.*

Proof. Let $\mathcal{M}_0$ denote the wandering subspace $\mathcal{M} \ominus \mathcal{B}\mathcal{M}$ of $\mathcal{B}$ on $\mathcal{M}$. Let $\mathcal{N}$ be the orthogonal complement of $[\mathcal{M}_0]$ in $\mathcal{M}$.

We will show that $\mathcal{N} = \{0\}$. The proof is long and will be divided into several steps.

Step 1. First we show

$$\mathcal{N} \subset \{\sum_{n=0}^{\infty} z^n u_n : u_n = -h_n + zP_{\mathcal{H}}g_n - wg_n(w) \in \mathcal{M}_{00}\}.$$

To do this, let q be a function in $\mathcal{N}$. By Theorem 2.3, we have

$$q = \sum_{k=0}^{\infty} z^k \tilde{m}_k + \sum_{n=0}^{\infty} z^n u_n,$$

where $\tilde{m}_k$ is in $\mathcal{M}_0$ and $u_n = (-h_n + zP_{\mathcal{H}}g_n - wg_n(w)) \in \mathcal{M}_{00}$. Since q is in $\mathcal{N}$ and orthogonal to $\vee_{n\geq 0}\mathcal{B}^n\mathcal{M}_0$, taking inner product of q with $\mathcal{B}^k\tilde{m}_k$ gives

$$\begin{aligned} 0 &= \langle q, \mathcal{B}^k \tilde{m}_k\rangle \\ &= \langle q, P_{\mathcal{H}} z^k \tilde{m}_k\rangle \\ &= \langle q, z^n \tilde{m}_k\rangle \\ &= \langle z^k \tilde{m}_k, z^k \tilde{m}_k\rangle + \langle z^k u_k, z^k \tilde{m}_k\rangle \\ &= \|\tilde{m}_k\|^2 + \langle u_k, \tilde{m}_k\rangle \\ &= \|\tilde{m}_k\|^2. \end{aligned}$$

The second equality follows from that $\mathcal{B}^k$ equals the compression of T_{z^k} on $\mathcal{H}$ and the fourth equality follows from that u_k is orthogonal to $\tilde{m}_k$. This gives that $\tilde{m}_k = 0$ for all $k \geq 0$. Thus each function q in $\mathcal{N}$ has the following form

$$q = \sum_{n=0}^{\infty} z^n u_n. \tag{3.1}$$

For a function q in $\mathcal{N}$ with the above representation, let

$$q_1 = \sum_{n=1}^{\infty} z^{n-1} u_n. \tag{3.2}$$

Step 2. Next we show that for each q in $\mathcal{N}$,

$$q_1 = \mathcal{B}_{\mathcal{M}}^* q$$

and q_1 is still in $\mathcal{N}$.

An easy calculation gives

$$\begin{aligned} T_z^* q &= P_{H^2(\mathbb{T}^2)}[\bar{z}u_0 + \bar{z}\sum_{n=1}^{\infty} z^n u_n] \\ &= P_{H^2(\mathbb{T}^2)}[\bar{z}(-h_0 + zP_{\mathcal{M}}g_0 - wg_0)] + \sum_{n=1}^{\infty} z^{n-1}u_n \end{aligned}$$

Since q and h_0 are in $\mathcal{M} \subset \mathcal{H}$ and $\mathcal{H}$ is invariant under T_z^*, we have that both $T_z^* q$ and $T_z^* h_0$ are also in $\mathcal{H}$. Noting that $\widetilde{M} = \mathcal{M} \oplus [z-w]$ and both $\mathcal{M}$ and $\mathcal{H}$ are orthogonal to $[z-w]$, we have that $P_{\widetilde{\mathcal{M}}}|_{\mathcal{H}} = P_{\mathcal{M}}|_{\mathcal{H}}$, to get

$$P_{\mathcal{M}} T_z^* q = P_{\widetilde{\mathcal{M}}} T_z^* q \tag{3.3}$$

and

$$P_{\mathcal{M}} T_z^* h_0 = P_{\widetilde{\mathcal{M}}} T_z^* h_0. \tag{3.4}$$

Applying the operator $\mathcal{B}_{\mathcal{M}}^*$ to q and using (3.3) give

$$\begin{aligned} \mathcal{B}_{\mathcal{M}}^* q &= P_{\mathcal{M}} T_z^* q = P_{\widetilde{\mathcal{M}}} T_z^* q \\ &= P_{\widetilde{\mathcal{M}}}\{P_{H^2(\mathbb{T}^2)}[\bar{z}(-h_0 + zP_{\mathcal{M}}g_0 - wg_0)] + \sum_{n=1}^{\infty} z^{n-1}u_n\} \\ &= P_{\widetilde{\mathcal{M}}}[-T_z^* h_0 + P_{\mathcal{M}}g_0] + \sum_{n=1}^{\infty} z^{n-1}u_n \\ &= -P_{\widetilde{\mathcal{M}}} T_z^* h_0 + P_{\mathcal{M}}g_0 + \sum_{n=1}^{\infty} z^{n-1}u_n \\ &= -P_{\mathcal{M}} T_z^* h_0 + P_{\mathcal{M}}g_0 + \sum_{n=1}^{\infty} z^{n-1}u_n \\ &= -\mathcal{B}_{\mathcal{M}}^* h_0 + P_{\mathcal{M}}g_0 + \sum_{n=1}^{\infty} z^{n-1}u_n \\ &= \sum_{n=1}^{\infty} z^{n-1}u_n. \end{aligned}$$

The sixth equality follows from (3.4) and the last equality follows from that $h_0 = \mathcal{B}(\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1}P_{\mathcal{M}}g_0$. This gives

$$\mathcal{B}_{\mathcal{M}}^* q = q_1. \tag{3.5}$$

To show that q_1 is still in $\mathcal{N}$, let m be in $\mathcal{M}_0$ and k nonnegative integers. Easy calculations give

$$\begin{aligned}\langle q_1, \mathcal{B}^k m\rangle &= \langle \mathcal{B}^*_{\mathcal{M}} q, \mathcal{B}^k m\rangle \\ &= \langle P_{\mathcal{M}} T_z^* q, \mathcal{B}^k m\rangle \\ &= \langle T_z^* q, \mathcal{B}^k m\rangle \\ &= \langle q, T_z \mathcal{B}^k m\rangle \\ &= \langle q, \mathcal{B}^{k+1} m\rangle = 0.\end{aligned}$$

This says that q_1 is in $\mathcal{N}$.

Step 3. We show

$$q = \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} q_1.$$

To do so, using (3.2) we have

$$\begin{aligned} q &= zq_1 + [-h_0 + zP_{\mathcal{H}} g_0 - wg_0(w)] \\ &= \mathcal{B}q_1 - h_0 + [(z-\mathcal{B})q_1 + zP_{\mathcal{H}} g_0 - wg_0(w)].\end{aligned}$$

Note that q and $\mathcal{B}q_1 - h_0$ are in $\mathcal{M}$ and $(z-\mathcal{B})q_1 + zP_{\mathcal{H}}g_0 - wg_0(w)$ is in $[z-w]$ and hence orthogonal to $\mathcal{M}$. Taking the projection at both sides of the above equality onto $\mathcal{M}$ and $[z-w]$ respectively gives

$$q = \mathcal{B}q_1 - h_0 \tag{3.6}$$

and

$$(z-\mathcal{B})q_1 = -zP_{\mathcal{H}} g_0 + wg_0(w).$$

By the fact that $T_z^*(wg_0(w)) = 0$, applying T_z^* to both sides of the above equality, we have

$$(1 - T_z^*\mathcal{B})q_1 = -P_{\mathcal{H}} g_0,$$

to get

$$P_{\mathcal{M}} g_0 = -(1 - \mathcal{B}^*_{\mathcal{M}}\mathcal{B})q_1.$$

Since $h_0 = \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} P_{\mathcal{M}} g_0$, we have

$$h_0 = -\mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}(1 - \mathcal{B}^*_{\mathcal{M}}\mathcal{B})q_1. \tag{3.7}$$

So combining (3.6) with (3.7) gives

$$\begin{aligned} q &= \mathcal{B}q_1 - h_0 \\ &= \mathcal{B}q_1 + \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}(1 - \mathcal{B}^*_{\mathcal{M}}\mathcal{B})q_1 \\ &= \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} q_1.\end{aligned}$$

Step 4. We show that $\mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} q$ is in $\mathcal{N}$ for each q in $\mathcal{N}$. By **Step 3**, hence $\mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}|_{\mathcal{N}}$ is the inverse of $\mathcal{B}^*_{\mathcal{M}}|_{\mathcal{N}}$.

Let

$$\tilde{q} = \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} q.$$

Clearly, $\tilde{q}$ is in $\mathcal{M}$. On the other hand, for each m in $\mathcal{M}_0$ and $n > 0$, an easy computation gives

$$\begin{aligned}\langle \tilde{q}, \mathcal{B}^n m\rangle &= \langle \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q, \mathcal{B}^n m\rangle \\ &= \langle q, \mathcal{B}^{n-1}m\rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\langle \tilde{q}, m\rangle &= \langle \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q, m\rangle \\ &= \langle q, (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}\mathcal{B}^* m\rangle = 0,\end{aligned}$$

to get that $\tilde{q}$ is in $\mathcal{N}$.

Step 5. For each $q = \sum_{n=0}^{\infty} z^n u_n$ in $\mathcal{N}$ as in **Step 1**, let $q_k = (\mathcal{B}^*_{\mathcal{M}})^k q$. Then

$$\|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2 = \|u_{k-1}\|^2 - \|u_k\|^2.$$

To prove the above equality, by **Step 2** we have that $q_k = \sum_{n=k}^{\infty} z^{n-k} u_n$, and hence

$$\|q_k\|^2 = \sum_{n=k}^{\infty} \|u_n\|^2,$$

to obtain

$$\begin{aligned}&\|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2 \\ =& \sum_{n=k-1}^{\infty} \|u_n\|^2 + \sum_{n=k+1}^{\infty} \|u_n\|^2 - 2\sum_{n=k}^{\infty} \|u_n\|^2 \\ =& \|u_{k-1}\|^2 - \|u_k\|^2.\end{aligned}$$

Step 6. For each q in $\mathcal{N}$, let $q_k = (\mathcal{B}^*_{\mathcal{M}})^k q$. Then

$$\|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2 = \langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q_k, q_k\rangle + \langle (\mathcal{B}\mathcal{B}^*_{\mathcal{M}})q_k, q_k\rangle - 2\langle q_k, q_k\rangle. \quad (3.8)$$

By **Step 4**, we have

$$q_{k-1} = \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q_k$$

to obtain

$$\begin{aligned}\|q_{k-1}\|^2 &= \langle \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q_k, \mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q_k\rangle \\ &= \langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q_k, q_k\rangle.\end{aligned}$$

This gives (3.8).

The Dirichlet space $\mathcal{D}$ consists of analytic functions on the unit disk whose derivatives are in the Bergman space L^2_a. We will get a representation of functions in $\mathcal{M}$.

Step 7. For each f in $\mathcal{M}$, there is a function $g(z)$ in $H^2(\mathbb{T}) \cap \mathcal{D}$ such that

$$f(z, w) = -P_{\mathcal{H}} g - \frac{zg(z) - wg(w)}{z - w}. \quad (3.9)$$

For f in $\mathcal{M}$ and each h in $\mathcal{H}$,

$$\begin{aligned}\langle (\mathcal{B} - T_z)f, h\rangle &= \langle (P_{\mathcal{H}}T_z - T_z)f, h\rangle \\ &= \langle T_z f - T_z f, h\rangle = 0.\end{aligned}$$

This gives that $(\mathcal{B}-T_z)f$ is in $[z-w]$. On the other hand, for each F in $H^2(\mathbb{T}^2)$,

$$\begin{aligned}\langle(\mathcal{B}-T_z)f, z(z-w)F\rangle &= \langle -T_z f, z(z-w)F\rangle \\ &= \langle f, (z-w)F\rangle = 0.\end{aligned}$$

Thus $(\mathcal{B}-T_z)f$ is in the wandering subspace $\mathcal{L}_{\widetilde{\{0\}}} = \mathcal{L}_{[z-w]}$. By Theorem 2.3, there is a function g in $H^2(\mathbb{T})$ such that

$$(\mathcal{B}-T_z)f = zP_{\mathcal{H}}g - wg(w).$$

So

$$\mathcal{B}f = zP_{\mathcal{H}}g - wg(w) + zf.$$

Noting that $\mathcal{B}f$, f, and $P_{\mathcal{H}}g$ are in $\mathcal{H}$ and hence they are symmetric functions of z and w, we also have

$$\mathcal{B}f = wP_{\mathcal{H}}g - zg(z) + wf.$$

Taking the difference of the above equalities gives

$$0 = (z-w)P_{\mathcal{H}}g + zg(z) - wg(w) + (z-w)f.$$

Hence we have

$$f = -P_{\mathcal{H}}g - \frac{zg(z)-wg(w)}{z-w}.$$

Since f is in $\mathcal{H}$, Theorem 9 in [9] gives that g is also in the Dirichlet space $\mathcal{D}$. This completes the proof.

Clearly, for each function $\tilde{f}(z)$ in $\mathcal{D}$, $\frac{\tilde{f}(z)-\tilde{f}(w)}{z-w}$ is in $\mathcal{H}$. For a function q in the invariant subspace $\mathcal{M}$ of $\mathcal{B}$, let $f = (\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1}q$. By (3.9) in **Step 7,** there is a function g in $H^2(\mathbb{T})\cap\mathcal{D}$ such that

$$f(z,w) = -P_{\mathcal{H}}g - \frac{zg(z)-wg(w)}{z-w}.$$

Step 8. Let $\mathcal{M}^\perp$ denote the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. Then

$$\begin{aligned}&\langle(\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1}q,q\rangle + \langle(\mathcal{B}\mathcal{B}_{\mathcal{M}}^*)q,q\rangle - 2\langle q,q\rangle \\ = \ &\frac{1}{2}\|P_{\mathcal{M}^\perp}\mathcal{B}P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2 - \|P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2.\end{aligned}$$

The above identity may be interesting in its own and immediately gives

$$\langle(\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1}q,q\rangle + \langle(\mathcal{B}\mathcal{B}_{\mathcal{M}}^*)q,q\rangle - 2\langle q,q\rangle \leq 0 \tag{3.10}$$

since

$$\|P_{\mathcal{M}^\perp}\mathcal{B}P_{\mathcal{M}^\perp}\| \leq 1.$$

(3.10) is the key ingredient in the new proof of the Beurling type theorem of Aleman, Richter and Sundberg in [15].

To prove the above identity, we need to identify $(\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1}q$, q and $\mathcal{B}_{\mathcal{M}}^*q$. Easy calculations give

$$\begin{aligned}
\mathcal{B}f &= P_{\mathcal{H}}[T_z(-P_{\mathcal{H}}g - \frac{zg(z)-wg(w)}{z-w})] \\
&= -\mathcal{B}P_{\mathcal{H}}g - P_{\mathcal{H}}[\frac{z^2g(z)-zwg(w)}{z-w}] \\
&= -\mathcal{B}P_{\mathcal{H}}g - P_{\mathcal{H}}[\frac{z^2g(z)-w^2g(w)+(w^2-zw)g(w)}{z-w}] \\
&= -\mathcal{B}P_{\mathcal{H}}g - \frac{z^2g(z)-w^2g(w)}{z-w} + P_{\mathcal{H}}(wg(w)) \quad (\text{by } \frac{z^2g(z)-w^2g(w)}{z-w} \in \mathcal{H}) \\
&= -\mathcal{B}P_{\mathcal{H}}g - \frac{z^2g(z)-w^2g(w)}{z-w} + \mathcal{B}P_{\mathcal{H}}g \quad (\text{by } wP_{\mathcal{H}}^{\perp}g \in [z-w]) \\
&= -\frac{z^2g(z)-w^2g(w)}{z-w},
\end{aligned}$$

and hence we have

$$\begin{aligned}
q &= \mathcal{B}_{\mathcal{M}}^*\mathcal{B}f \\
&= P_{\mathcal{M}}T_z^*[-\frac{z^2g(z)-w^2g(w)}{z-w}] \\
&= -P_{\mathcal{M}}P_{H^2(\mathbb{T}^2)}[\frac{zg(z)-w^2\bar{z}g(w)}{z-w}] \\
&= -P_{\mathcal{M}}P_{H^2(\mathbb{T}^2)}[\frac{zg(z)-wg(w)+w(1-w\bar{z})g(w)}{z-w}] \\
&= -P_{\mathcal{M}}[\frac{zg(z)-wg(w)}{z-w}] - P_{\mathcal{M}}P_{H^2(\mathbb{T}^2)}(\bar{z}wg(w)) \\
&= -P_{\mathcal{M}}[\frac{zg(z)-wg(w)}{z-w}] \\
&= P_{\mathcal{M}}[-P_{\mathcal{H}}g - \frac{zg(z)-wg(w)}{z-w} + P_{\mathcal{H}}g] \\
&= P_{\mathcal{M}}[f + P_{\mathcal{H}}g] \qquad (\text{by } (3.9)) \\
&= f + P_{\mathcal{M}}g \\
&= -P_{\mathcal{M}^{\perp}}g - \frac{zg(z)-wg(w)}{z-w}.
\end{aligned}$$

The ninth equality follows from that f is in $\mathcal{M}$. Since $\mathcal{M}$ is an invariant subspace of $\mathcal{B}$, $\mathcal{M}^{\perp}$ is an invariant subspace of $\mathcal{B}^*$. Thus

$$P_{\mathcal{M}}\mathcal{B}^*(P_{\mathcal{M}^{\perp}}g) = 0.$$

So we have

$$\begin{aligned}\mathcal{B}_{\mathcal{M}}^* q &= P_{\mathcal{M}}\mathcal{B}^* q \\ &= P_{\mathcal{M}}[-\mathcal{B}^* P_{\mathcal{M}^\perp} g - \mathcal{B}^* \frac{zg(z) - wg(w)}{z-w}] \\ &= -P_{\mathcal{M}}[\frac{g(z)-g(w)}{z-w}],\end{aligned}$$

to get

$$\begin{aligned}\langle \mathcal{B}\mathcal{B}_{\mathcal{M}}^* q, q\rangle &= \langle \mathcal{B}_{\mathcal{M}}^* q, \mathcal{B}_{\mathcal{M}}^* q\rangle \\ &= \|P_{\mathcal{M}}[\frac{g(z)-g(w)}{z-w}]\|^2,\end{aligned}$$

$$\begin{aligned}&\langle (\mathcal{B}_{\mathcal{M}}^*\mathcal{B})^{-1} q, q\rangle = \langle f, q\rangle \\ = \ & \langle f, -P_{\mathcal{M}}[\frac{zg(z)-wg(w)}{z-w}]\rangle \\ = \ & \langle f, -[\frac{zg(z)-wg(w)}{z-w}]\rangle \qquad \text{(by } f \in \mathcal{M}) \\ = \ & \langle -P_{\mathcal{H}} g - \frac{zg(z)-wg(w)}{z-w}, -[\frac{zg(z)-wg(w)}{z-w}]\rangle \\ = \ & \langle P_{\mathcal{H}} g, [\frac{zg(z)-wg(w)}{z-w}]\rangle + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \\ = \ & \langle g, [\frac{zg(z)-wg(w)}{z-w}]\rangle + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \quad \text{(by } \frac{zg(z)-wg(w)}{z-w} \in \mathcal{H}) \\ = \ & \|g\|^2 + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2,\end{aligned}$$

and

$$\begin{aligned}\langle q, q\rangle &= \langle q, -P_{\mathcal{M}}[\frac{zg(z)-wg(w)}{z-w}]\rangle \\ &= \langle q, -[\frac{zg(z)-wg(w)}{z-w}]\rangle \\ &= \langle -P_{\mathcal{M}^\perp} g - \frac{zg(z)-wg(w)}{z-w}, -[\frac{zg(z)-wg(w)}{z-w}]\rangle \\ &= \langle P_{\mathcal{M}^\perp} g, [\frac{zg(z)-wg(w)}{z-w}]\rangle + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \\ &= -\langle P_{\mathcal{M}^\perp}[\frac{zg(z)-wg(w)}{z-w}], [\frac{zg(z)-wg(w)}{z-w}]\rangle + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \\ &= -\|P_{\mathcal{M}^\perp}[\frac{zg(z)-wg(w)}{z-w}]\|^2 + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \\ &= -\|P_{\mathcal{M}^\perp} g\|^2 + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2.\end{aligned}$$

The fifth equality and the last equality follow from

$$\begin{aligned} 0 &= P_{\mathcal{M}^\perp} f \\ &= P_{\mathcal{M}^\perp}[-P_{\mathcal{H}} g - \frac{zg(z)-wg(w)}{z-w}] \\ &= -P_{\mathcal{M}^\perp} g - P_{\mathcal{M}^\perp}[\frac{zg(z)-wg(w)}{z-w}]. \end{aligned}$$

Therefore

$$\begin{aligned} &\langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q, q\rangle + \langle (\mathcal{B}\mathcal{B}^*_{\mathcal{M}})q, q\rangle - 2\langle q, q\rangle \\ = \;& \|g\|^2 + \|[\frac{zg(z)-wg(w)}{z-w}]\|^2 + \|P_{\mathcal{M}}[\frac{g(z)-g(w)}{z-w}]\|^2 \\ &\qquad + 2\|P_{\mathcal{M}^\perp} g\|^2 - 2\|[\frac{zg(z)-wg(w)}{z-w}]\|^2 \\ = \;& -\|[\frac{g(z)-g(w)}{z-w}]\|^2 + \|P_{\mathcal{M}}[\frac{g(z)-g(w)}{z-w}]\|^2 + 2\|P_{\mathcal{M}^\perp} g\|^2 \\ = \;& 2\|P_{\mathcal{M}^\perp} g\|^2 - \|P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2. \end{aligned}$$

The second equality follows from

$$\|[\frac{zg(z)-wg(w)}{z-w}]\|^2 = \|g\|^2 + \|[\frac{g(z)-g(w)}{z-w}]\|^2.$$

The above identity comes directly from the computation by using the Fourier series expansion of the function $g(z)$ and the fact that $\{\frac{p_n(z,w)}{\sqrt{n+1}}\}_{n=0}^{\infty}$ form an orthonormal basis of $\mathcal{H}$. Since

$$\begin{aligned} \mathcal{B}[\frac{g(z)-g(w)}{z-w}] &= [\frac{zg(z)-wg(w)}{z-w}] - P_{\mathcal{H}} g \\ &= -f - 2P_{\mathcal{H}} g, \end{aligned}$$

and f is in $\mathcal{M}$, we have

$$P_{\mathcal{M}^\perp}\mathcal{B}[\frac{g(z)-g(w)}{z-w}] = -2P_{\mathcal{M}^\perp} g.$$

Thus

$$\begin{aligned} \|P_{\mathcal{M}^\perp} g\|^2 &= \frac{1}{4}\|P_{\mathcal{M}^\perp}\mathcal{B}[\frac{g(z)-g(w)}{z-w}]\|^2 \\ &= \frac{1}{4}\|P_{\mathcal{M}^\perp}\mathcal{B}P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2. \end{aligned}$$

The last equality follows from that $\mathcal{B}P_{\mathcal{M}}[\frac{g(z)-g(w)}{z-w}]$ is in $\mathcal{M}$. So we obtain

$$\begin{aligned} &\langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q, q\rangle + \langle (\mathcal{B}\mathcal{B}^*_{\mathcal{M}})q, q\rangle - 2\langle q, q\rangle \\ = \;& \frac{1}{2}\|P_{\mathcal{M}^\perp}\mathcal{B}P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2 - \|P_{\mathcal{M}^\perp}[\frac{g(z)-g(w)}{z-w}]\|^2. \end{aligned}$$

Step 9. Finally we show that $\mathcal{N} = \{0\}$.

For each q in $\mathcal{N}$, by **Step 1**, we write $q = \sum_{n=0} z^n u_n$ with $\|q\|^2 = \sum_{n=0}^{\infty} \|u_n\|^2$, for u_n in $\mathcal{M}_{00}$. Let $q_k = (\mathcal{B}_{\mathcal{M}}^*)^k q$. **Step 8** gives that

$$\langle (\mathcal{B}_{\mathcal{M}}^* \mathcal{B})^{-1} q_k, q_k \rangle + \langle (\mathcal{B}\mathcal{B}_{\mathcal{M}}^*) q_k, q_k \rangle - 2\langle q_k, q_k \rangle \leq 0.$$

By **Steps 5 and 6**, we have

$$\|u_{k-1}\|^2 - \|u_k\|^2 \leq 0,$$

to get that the sequence $\{\|u_k\|^2\}$ of nonnegative numbers increases, but is summable. Hence $\|u_k\|^2 = 0$ for $k \geq 0$. This implies that $q = 0$, to complete the proof.

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