MATH 250 — Introduction to Mathematical Logic.

Most intermediate and advanced mathematics courses involve proving theorems. Mathematical logic is a formal analysis of proofs. It starts by formalizing language. For instance, in everyday English, the statement

> if it is sunny in the morning when I leave my house, then I do not take along my umbrella

is equivalent to

> if I have my umbrella with me during the day, then it was not sunny in the morning when I left my house.

This equivalence can be represented symbolically in a formal language as $(A \to \neg B) \leftrightarrow (B \to \neg A)$ where the associations of symbols to meanings are:

$$\neg \text{ (not)}, \qquad \to \text{ (implies)}, \qquad \leftrightarrow \text{ (is equivalent to)}.$$

Formal statements like this (along with rules used to deduce other statements) are referred to as *syntax*. The meanings behind the symbols, such as A means "I have my umbrella with me during the day" and can either be true or false, are referred to as *semantics*. Mathematical logic studies the syntactic approach, the semantic approach, and the connection between the two.

- Syntactic logic is concerned with *deductions*, or formal proofs, transforming one or more formulas into other formulas, with the aid of *axioms* (assumed formulas). The *theorems* are the deduced formulas, i.e., those that can be derived starting from just the axioms.
- Semantic logic is concerned with the evaluation of formulas. For instance, in the classical two-valued semantics, the only possible values are false and true, which can be represented numerically by 0 and 1. In a fuzzy logic, typically, the values are all the numbers in the interval $[0, 1]$, with 1 being the only value that is "completely true." In a relevant logic or a comparative logic, one might use for values all the integers, with higher values being "more true"; there is no "truest" value. For the values in constructive logic one might use sets of numbers — more specifically, the open subsets of the real line; larger sets are "more true," and the entire line is the only set that is "completely true." A *tautology* in a semantic logic is a formula that is always true in that logic.
- Mathematical modeling is concerned with pairing up syntactic logic with semantic logic. Typical questions about these pairings are:
  - Is every theorem a tautology? i.e., is every provable thing also true? (*soundness*)
  - Is every tautology a theorem? i.e., is every true thing also provable? (*adequacy*)

The concepts described above — syntax, semantics, deduction, axioms, theorems, soundness, adequacy — can be conveyed using different examples. The choice of examples will depend on the tastes and pedagogical philosophy of your instructor. In recent years, our classes have been evenly split between these two substantially different approaches:

- "Traditional" (depth first): classical logic only, at both propositional and predicate levels
- "Pluralist" (breadth first): classical and nonclassical, but only at the propositional level

The latter approach is described further at `www.math.vanderbilt.edu/~schectex/logics/`.