On some small cardinals for Boolean algebras

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Abstract. Assume that all algebras are atomless. (1) $\mathrm{Spind}(A \times B) = \mathrm{Spind}(A) \cup \mathrm{Spind}(B)$. (2) $\mathrm{Spind}(\prod^{\mathrm{w}}_{i\in I} A_i) = \{\omega\} \cup \bigcup_{i\in I} \mathrm{Spind}(A_i)$. Now suppose that κ and λ are infinite cardinals, with κ uncountable and regular and with $\kappa < \lambda$. (3) There is an atomless Boolean algebra A such that $\mathfrak{u}(A) = \kappa$ and $\mathfrak{i}(A) = \lambda$. (4) If λ is also regular, then there is an atomless Boolean algebra A such that $\mathfrak{t}(A) = \mathfrak{s}(A) = \kappa$ and $\mathfrak{a}(A) = \lambda$. All results are in ZFC, and answer some problems posed in Monk [01] and Monk [∞].

Introduction. First we define the cardinal functions considered in this paper. In these definitions, assume that A is an atomless Boolean algebra. A subset X of A *splits* A provided that for every nonzero $a \in A$ there is a $b \in X$ such that $a \cdot b \neq 0 \neq a \cdot -b$. A *partition of unity* of A is a collection X of nonzero pairwise disjoint elements of A with supremum 1. A *tower* is a subset X of $A\backslash\{1\}$ well-ordered by the Boolean ordering, with supremum 1.

$$
\begin{aligned}
\mathfrak{i}(A) &= \min\{|X| : X \text{ is a maximal independent subset of } A\};\\
\mathfrak{u}(A) &= \min\{|X| : X \text{ generates an ultrafilter of } A\};\\
\mathfrak{s}(A) &= \min\{|X| : X \text{ splits } A\};\\
\mathfrak{a}(A) &= \min\{|X| : X \text{ is an infinite partition of unity in } A\};\\
\mathfrak{t}(A) &= \min\{|X| : X \text{ is a tower in } A\};\\
\mathrm{Spind}(A) &= \{|X| : X \text{ is a maximal independent subset of } A\}.
\end{aligned}
$$

The main results are then as indicated in the abstract. As a corollary of (1) we have $\mathfrak{i}(A \times B) = \min\{\mathfrak{i}(A), \mathfrak{i}(B)\}$. (1) answers questions raised in Monk [01] and Monk [∞]. With $A = \mathscr{P}(\omega)/\mathrm{fin}$, models M, N of ZFC in which $\mathfrak{u}(A) < \mathfrak{i}(A)$ and $\mathfrak{s}(A) < \mathfrak{a}(A)$ respectively hold have been known for a long time; see Blass, Shelah [87] and Balcar, Simon [89]. Thus a contribution here is a construction of such Boolean algebras in ZFC. The problems about obtaining such examples in ZFC were also raised in Monk [01]. All of these cardinal functions for Boolean algebras generalize known ones for $\mathscr{P}(\omega)/\mathrm{fin}$, and they are extensively discussed in Monk [01] and Monk [∞]. This article is self-contained, however.

Notation. Our set-theoretic notation is mostly standard. The complement of Y relative to X is denoted by $X\backslash Y$. For any function f and subset X of its domain, $f[X] = \{f(a) : a \in X\}$. For a cartesian product $A \times B$, the two projections are denoted by π_0 and π_1. A set is called *denumerable* iff it is countably infinite. The restriction of a function f to a subset D of its domain is denoted by $f \restriction D$.

For Boolean algebras we follow the notation of Koppelberg [89]. In particular, the fundamental operations of a Boolean algebra are denoted by $+, \cdot, -, 0, 1$. The free product of Boolean algebras A, B is denoted by $A \oplus B$. The subalgebra generated by a set X is $\langle X\rangle$. For an element $a \in A$ we let $a^1 = a$ and $a^0 = -a$. Given a subset X of A, a *monomial* over X is a product of the form $a_0^{\varepsilon(0)}\cdot\ldots\cdot a_{m-1}^{\varepsilon(m-1)}$ where $a_0,\ldots,a_{m-1}$ are distinct elements of X and each $\varepsilon(i)$ is 0 or 1. The *weak product* of a system $\langle A_i : i\in I\rangle$ of Boolean algebras is denoted by $\prod^{\mathrm{w}}_{i\in I} A_i$; it consists of all elements x of the full product such that either $\{i\in I : x_i\neq 0\}$ is finite or $\{i\in I : x_i\neq 1\}$ is finite. If B is a subalgebra of A and $a\in A$, then $B\restriction a$ is the Boolean algebra with underlying set $\{b\cdot a : b\in B\}$ and operations $+$, $\cdot$, 0, a, and with the complement of an element c being $-c\cdot a$, with $-c$ the complement in A.

1. Products and independence. The first lemma is needed for the result (1) above.

Lemma 1. *If A is countable and F is an infinite free BA, then $F\cong A\oplus F$.*

Proof. Let X be a set of free generators of F, and write $X=Y_0\cup Y_1$ with Y_0 denumerable and $Y_0\cap Y_1=\emptyset$. Then $F=\langle Y_0\rangle\oplus\langle Y_1\rangle$. Now $A\oplus\langle Y_0\rangle$ is denumerable and atomless, and hence is isomorphic to $\langle Y_0\rangle$. Hence the conclusion of the lemma follows. □

Theorem 2. *If A_0 and A_1 are atomless Boolean algebras, then* $\mathrm{Spind}(A_0\times A_1)=\mathrm{Spind}(A_0)\cup\mathrm{Spind}(A_1)$.

Proof. $\supseteq$ is easy. Suppose that $\kappa\in\mathrm{Spind}(A_0\times A_1)\backslash(\mathrm{Spind}(A_0)\cup\mathrm{Spind}(A_1))$. First we show that $\kappa>\omega$. In fact, suppose that $\kappa=\omega$. Let X be a denumerable maximal independent subset of $A_0\times A_1$. Then $\pi_0[X]$ is contained in a denumerable atomless subalgebra B of A_0, and since $\mathfrak{i}(A_0)>\omega$ we get an element $a_0\in A_0$ such that $\pi_0(b)\cdot a_0\neq 0\neq\pi_0(b)\cdot -a_0$ for every $b\in X$ such that $\pi_0(b)\neq 0$. Similarly we get an element $a_1\in A_1$ such that $\pi_1(b)\cdot a_1\neq 0\neq\pi_1(b)\cdot -a_1$ for every $b\in X$ such that $\pi_1(b)\neq 0$. Then $(a_0,a_1)\notin X$ and $X\cup\{(a_0,a_1)\}$ is still independent, contradiction. Thus $\kappa>\omega$.

Let $X\subseteq A_0\times A_1$ be maximal independent with $|X|=\kappa$. *Temporarily fix $j\in 2$.* We define

$$D_j=\{w : w \text{ is a monomial over } X \text{ and } \pi_j(v)\neq 0 \text{ for every monomial } v\leq w\}.$$

Note that if $w\in D_j$ and v is a monomial over X such that $v\leq w$, then also $v\in D_j$.

(1) There is a monomial w over X such that $\pi_j(w)=0$.

In fact, suppose not. Then $\langle\pi_j(a) : a\in X\rangle$ is an independent subset of A_j. Since A_j has no maximal independent subset of size κ, it follows that there is a $c\in A_j\backslash\{\pi_j(a) : a\in X\}$ such that $\langle\pi_j(a) : a\in X\rangle^\frown\langle c\rangle$ is independent. Let $c'\in A_0\times A_1$ be such that $c'_j=c$. Then $\langle a : a\in X\rangle^\frown\langle c'\rangle$ is independent, contradicting the maximality of X. So (1) holds.

(2) If w is a monomial over X such that $\pi_j(w)=0$, then $w\in D_{1-j}$.

For, suppose that v is a monomial over X with $v\leq w$ and $\pi_{1-j}(v)=0$. Then $v=0$, contradiction. So (2) holds.

(3) $D_j\neq\emptyset$.

This is true by (1) and (2) (since j is arbitrary).

(4) If w is a monomial over X and $w\notin D_j$, then there is a monomial $v\leq w$ such that $v\in D_{1-j}$.

For, choose a monomial $v \leq w$ such that $\pi_j(v) = 0$. Then $v \in D_{1-j}$ by (2).

Now let M_j be a maximal set of pairwise disjoint members of D_j. So, M_j is countable. Let X_j be a denumerable subset of X such that $M_j \subseteq \langle X_j \rangle$, and let $Y_j = X \backslash X_j$.

(5) There is an element b_j of $\langle X \rangle$ and a subalgebra C_j of $\langle X \rangle \restriction b_j$ such that C_j is free of size κ and the following conditions hold:

(a) If $w \in D_j$, then there is a nonzero $v \in C_j$ such that $v \leq w$.

(b) If c is a nonzero element of C_j, then there is a $w \in D_j$ such that $w \leq c$.

To prove this, we consider two cases.

Case 1. M_j is infinite. Let $b_j = 1$. Let J_j be the ideal of $\langle X_j \rangle$ generated by M_j, and let B_j be the subalgebra of $\langle X_j \rangle$ generated by J_j. Let $C_j = \langle B_j \cup Y_j \rangle$. Now B_j is a denumerable BA, and $b \cdot c \neq 0$ whenever $0 \neq b \in B_j$ and $0 \neq c \in \langle Y_j \rangle$. Thus $C_j = B_j \oplus \langle Y_j \rangle$. So by Lemma 1, C_j is free. Clearly it has size κ.

To check (a), suppose that $w \in D_j$. By the maximality of M_j, there is a member r of M_j such that $w \cdot r \neq 0$. Note that $w \cdot r$ is a monomial over X. Now we can write $w \cdot r = s_0 \cdot s_1$ with s_0 a monomial over X_j and s_1 a monomial over Y_j. So $s_0 \leq r$, and hence s_0 is in J_j and hence is also in B_j. Hence $w \cdot r$ is a member of C_j. Hence (a) holds.

To check (b), suppose that c is a nonzero element of C_j. Choose d, e so that $d \in B_j$, d is a monomial over X_j, e is a monomial over Y_j, and $d \cdot e \leq c$. If $d \in J_j$, then there exist $r_0, \ldots, r_{m-1} \in M_j$ such that $d \leq r_0 + \cdots + r_{m-1}$. Wlog $d \cdot r_0 \neq 0$. So $d \cdot r_0 \cdot e$ is a monomial over X, $d \cdot r_0 \cdot e \leq r_0 \in M_j \subseteq D_j$, so $d \cdot r_0 \cdot e \in D_j$. Since $d \cdot r_0 \cdot e \leq c$, this is as desired.

On the other hand, suppose that $d \notin J_j$. Then $-d \in J_j$, and so we get members $r_0, \ldots, r_{m-1}$ of M_j such that $-d \leq r_0 + \cdots + r_{m-1}$. Since M_j is infinite, there is a member s of M_j different from each r_i, and hence $s \leq d$. Clearly then $s \in D_j$, so $s \cdot e \in D_j$, and $s \cdot e \leq c$, as desired.

Case 2. M_j is finite. Let $b_j = \sum M_j$ and $B_j = \langle X_j \rangle \restriction b_j$. Then B_j is a denumerable atomless BA. Let C_j be the subalgebra of $\langle X \rangle \restriction b_j$ generated by $B_j \cup \{b_j \cdot y : y \in Y_j\}$. Again C_j is free of size κ by Lemma 1. Conditions (a) and (b) can be checked by easy modifications of the arguments in Case 1; they are in fact easier than in Case 1.

This completes the proof of (5).

(6) π_j is injective on C_j.

In fact, suppose that $c \in C_j$ and $c \neq 0$. By (5)(b), choose $w \in D_j$ such that $w \leq c$. Then by the definition of D_j we have $\pi_j(w) \neq 0$, and hence $\pi_j(c) \neq 0$.

Now since $\kappa \notin \mathrm{Spind}(A_j)$, it follows that also $\kappa \notin \mathrm{Spind}(A_j \restriction \pi_j(b_j))$. Since $\pi_j[C_j]$ is free and of size κ, it is not maximal independent. Hence we can choose $w_j \in (A_j \restriction \pi_j(b_j)) \backslash \pi_j[C_j]$ such that w_j is free over $\pi_j[C_j]$.

(7) $w_j \cdot \pi_j(d) \neq 0 \neq -w_j \cdot \pi_j(d)$ for all $d \in D_j$.

For, by (5)(a) choose a nonzero v in C_j such that $v \leq d$. Then $0 \neq w_j \cdot \pi_j(v) \leq w_j \cdot \pi_j(d)$, so $w_j \cdot \pi_j(d) \neq 0$. Similarly, $-w_j \cdot \pi_j(d) \neq 0$.

Now unfix j. Let $w = (w_0, w_1)$. Suppose that v is a monomial over X. If $v \in D_0$, then $w \cdot v \neq 0 \neq -w \cdot v$ by (7). Suppose that $v \notin D_0$. By (4), choose a monomial $s \leq v$ such that $s \in D_1$. Then again $w \cdot v \neq 0 \neq -w \cdot v$ by (7).

This contradicts the maximality of X. □

Corollary 3. *If A_0 and A_1 are atomless BAs, then $\mathfrak{i}(A_0 \times A_1) = \min(\mathfrak{i}(A_0), \mathfrak{i}(A_1))$.* □

Theorem 4. *Suppose that I is an infinite set, and $\langle A_i : i \in I\rangle$ is a system of atomless BAs. Then*

$$\mathrm{Spind}\left(\prod_{i\in I}^{\mathrm{w}} A_i\right) = \{\omega\} \cup \bigcup_{i\in I} \mathrm{Spind}(A_i).$$

Proof. $\supseteq$ holds, using Proposition 8 of Monk [01]. For $\subseteq$, suppose to the contrary that $\kappa \in \left(\prod_{i\in I}^{\mathrm{w}} A_i\right) \backslash (\{\omega\} \cup \bigcup_{i\in I} \mathrm{Spind}(A_i))$. Let X be a maximal independent subset of $\prod_{i\in I}^{\mathrm{w}} A_i$ of size κ. Wlog for all $x \in X$ the set $F_x = \{i \in I : x(i) \neq 0\}$ is finite. Let Y be an uncountable subset of X such that $\langle F_x : x \in Y\rangle$ forms a Δ-system, say with kernel G. That is, $F_x \cap F_y = G$ for any two distinct members $x, y \in Y$. Obviously $G \neq \emptyset$.

(*) $\langle x \restriction G : x \in X\rangle$ is independent in $\prod_{i\in G} A_i$.

In fact, suppose that K is a finite subset of X and $\varepsilon \in {}^K2$. Choose distinct $x, z \in Y\backslash K$. Then $x \cdot z \cdot \prod_{y\in K} y^{\varepsilon(y)} \neq 0$, so $\prod_{y\in K}(y \restriction G)^{\varepsilon(y)} \neq 0$, as desired in (*).

By Theorem 2, there is an element w of $\prod_{i\in G} A_i$ such that $\langle x \restriction G : x \in X\rangle^\frown\langle w\rangle$ is independent. Let v be the member of $\prod_{i\in I}^{\mathrm{w}} A_i$ whose restriction to G is w, and with value 0 outside of G. For any finite subset K of X, any $\varepsilon \in {}^K2$, and any $\delta \in 2$, choose distinct $x, z \in Y\backslash K$; then

$$x \cdot z \cdot \prod_{y\in K} y^{\varepsilon(y)} \cdot v^\delta \neq 0.$$

But this contradicts the maximality of X. □

The following problem remains open.

Problem. *If $\langle A_i : i \in I\rangle$ is a system of atomless Boolean algebras with I infinite, is $\mathfrak{i}\left(\prod_{i\in I} A_i\right) = \min_{i\in I} \mathfrak{i}(A_i)$?*

2. An atomless BA B such that $\mathfrak{u}(B) < \mathfrak{i}(B)$. More precisely, we show:

Theorem 5. *Let κ and λ be cardinals, with $\kappa < \lambda$ and κ regular and uncountable. Then there is a BA B such that $\mathfrak{u}(B) = \kappa$ and $\mathfrak{i}(B) = |B| = \lambda$.*

Proof. The construction goes as follows. Let A be free on the distinct generators

$$Y \stackrel{\text{def}}{=} \{x_\alpha : \alpha < \kappa\} \cup \{y_{\alpha\beta} : \alpha < \kappa,\ \beta < \lambda\}.$$

Then let

$$\begin{aligned} K &= \{x_\beta \cdot -x_\alpha : \alpha < \beta < \kappa\}; \\ L &= \{x_\alpha \cdot -y_{\alpha\beta} : \alpha < \kappa,\ \beta < \lambda\}; \\ I &= \text{ ideal generated by } K \cup L; \\ B &= A/I. \end{aligned}$$

We denote the equivalence class of $a \in A$ under I by $[a]$. Let $u_\alpha = [x_\alpha]$ and $v_{\alpha\beta} = [y_{\alpha\beta}]$ for all $\alpha < \kappa$, $\beta < \lambda$. We make use of the following easy algebraic fact several times:

(1) If $c \in I$, then we can write $c \leq a + b$ with a a finite sum of elements of K and b a finite sum of elements of L.

(2) If $\alpha < \beta < \kappa$, then $u_\beta < u_\alpha$.

In fact, clearly $u_\beta \leq u_\alpha$, and if they are equal, then we can write $x_\alpha \cdot -x_\beta \leq a + b$ as in (1). Let f be the homomorphism of A into 2 such that $f(x_\gamma) = 1$ for all $\gamma \leq \alpha$, $f(x_\gamma) = 0$ for all $\gamma > \alpha$, and $f(y_{\gamma\delta}) = 1$ for all γ, δ. Applying f to the above inequality we get $1 \leq 0$, contradiction.

(3) $\{u_\alpha : \alpha < \kappa\}$ generates an ultrafilter on B.

This is obvious, using (2) to see that the indicated set does not contain 0.

(4) $\prod_{\alpha<\kappa} u_\alpha = 0$.

For, suppose that w is a lower bound for $\{u_\alpha : \alpha < \kappa\}$. By (3) there are two possibilities. First, w is in the indicated ultrafilter. Hence $u_\alpha \leq w$ for some α. Since $w \leq u_{\alpha+1}$, this contradicts (2). So, we must have $-w$ in the ultrafilter, so $u_\alpha \leq -w$ for some α. Hence $w \leq -u_\alpha$. But $w \leq u_\alpha$ too, so $w = 0$, as desired for (4).

(5) If $\alpha < \kappa$, $\beta, \gamma < \lambda$, and $\beta \neq \gamma$, then $v_{\alpha\beta} \neq v_{\alpha\gamma}$.

For, suppose that this is not true. Then we get a, b as above such that $y_{\alpha\beta} \triangle y_{\alpha\gamma} \leq a + b$, where $\triangle$ denotes symmetric difference. Mapping each x_δ to 0 and fixing each $y_{\delta\xi}$, we get an endomorphism f of A, and $0 \neq y_{\alpha\beta} \triangle y_{\alpha\gamma} = f(y_{\alpha\beta} \triangle y_{\alpha\gamma}) \leq 0$, contradiction. Thus (5) holds.

Now each $c \in B$ can be written in the form $\sum_{d \in S_c} [d]$, where S_c is a finite collection of monomials over Y, each $[d] \neq 0$. For each monomial d over Y, let

$$\begin{aligned} T_d = \{x_\alpha : {} & x_\alpha^\varepsilon \text{ is a term of } d \text{ for some } \varepsilon\} \\ & \cup \{y_{\alpha\beta} : y_{\alpha\beta}^\varepsilon \text{ is a term of } d \text{ for some } \varepsilon\}. \end{aligned}$$

(6) $\mathfrak{u}(B) = \kappa$.

To prove this, suppose that X filter-generates an ultrafilter F, and $|X| < \kappa$; we want to get a contradiction. We may assume that X is closed under multiplication. Now $\bigcup_{c \in X} \bigcup_{d \in S_c} T_d$ has size less than κ, and κ is regular, so choose α greater than each index β such that x_β or $y_{\beta\gamma}$ is in this set for some γ. Since F is an ultrafilter and X filter-generates F and is closed under multiplication, there is a $c \in F$ and a $\varepsilon \in 2$ such that $c \cdot v_{\alpha 0}^\varepsilon = 0$. Take any $d \in S_c$. Then there are a, b as above such that $d \cdot y_{\alpha 0}^\varepsilon \leq a + b$. Let f be the homomorphism from A into B such that $f(x_\delta) = u_\delta$ for all $\delta < \alpha$, $f(x_\delta) = 0$ for all $\delta \geq \alpha$, $f(y_{\alpha 0}) = \varepsilon$, and $f(y_{\delta\xi}) = v_{\delta\xi}$ otherwise. Then the inequality gives $[d] = 0$, contradiction. So (6) holds.

(7) $\mathfrak{i}(B) = |B| = \lambda$.

By (5) we clearly have $|B| = \lambda$. Hence it suffices to take an independent subset Z of B with $|Z| < \lambda$, assume that Z is maximal, and get a contradiction.

Let $U = \bigcup_{c \in Z} \bigcup_{d \in S_c} T_d$. Note that $Z \subseteq \langle \{[u] : u \in U\} \rangle$.

(8) For each $\alpha < \kappa$ there is a monomial w over Z such that $w \leq u_\alpha$.

For, by (5) choose β such that

$$v_{\alpha\beta} \notin \langle \{[u] : u \in U\} \rangle.$$

So $Z \cup \{v_{\alpha\beta}\}$ is dependent, and hence there is a monomial w over Z and a $\delta \in 2$ such that $w \cdot v_{\alpha\beta}^\delta = 0$. Choose $t \in \langle U \rangle_A$ such that $w = [t]$. Note that $y_{\alpha\beta} \notin \langle U \rangle_A$. Then there are a, b as above such that $t \cdot y_{\alpha\beta}^\delta \leq a + b$. If $\delta = 1$, let f be the homomorphism from A into B such that $f(y_{\alpha\beta}) = 1$ and $f(s) = [s]$ for any $s \in Y \backslash \{y_{\alpha\beta}\}$. Then $f(t) = w$, and so $w = 0$, contradiction. So $\delta = 0$. Then let $f(y_{\alpha\beta}) = u_\alpha$ and $f(s) = [s]$ for any $s \in Y \backslash \{y_{\alpha\beta}\}$. The inequality gives $w \cdot -u_\alpha = 0$, proving (8).

(9) For every $\alpha < \kappa$ there exist a $\beta \in [\alpha, \kappa)$ and a monomial w over Z such that $w \leq u_\beta$, while for each $\gamma \in (\beta, \kappa)$ we have $w \not\leq u_\gamma$.

For, by (8) choose a monomial w over Z such that $w \leq u_\alpha$. Let $\beta = \sup\{\gamma < \kappa : w \leq u_\gamma\}$. By (2) we have $\alpha \leq \beta < \kappa$. It suffices now to show that $w \leq u_\beta$. Suppose not. Hence β is a limit ordinal and there is a $d \in S_w$ such that $[d] \neq 0$ and for each $\gamma \in [\beta, \kappa)$, the element x_γ is not a factor of d. Then choose $\varepsilon < \beta$ such that no x_γ with $\gamma \in [\varepsilon, \beta)$ is a factor of d. Now $[d] \leq u_\varepsilon$, so we get a, b as above such that $d \cdot -x_\varepsilon \leq a + b$. Let f be the homomorphism of A into B such that $f(x_\gamma) = u_\gamma$ for each $\gamma < \varepsilon$, $f(x_\gamma) = 0$ for all $\gamma \in [\varepsilon, \kappa)$, and $f(y_{\delta\xi}) = v_{\delta\xi}$ for all δ, ξ. Although there may be some $\gamma \in [\varepsilon, \kappa)$ such that $-x_\gamma$ is a factor of d, this still implies that $[d] = 0$, contradiction. Thus (9) holds.

We call a pair (w, β) *special* iff w and β satisfy the conclusion of (9). Since $w = w \cdot u_\beta$ in this case, wlog each $d \in S_w$ has exactly one factor x_γ, and for it, $\gamma \geq \beta$; d has at most one factor $-x_\delta$, and if it has such, then $\delta > \gamma$;and if $y_{\delta\xi}^\varepsilon$ is a factor, then $\delta > \gamma$.

Now an easy recursive definition gives sequences $\langle w_\xi : \xi < \kappa \rangle$ and $\langle \beta_\xi : \xi < \kappa \rangle$ such that the following conditions hold:

(10) $\langle \beta_\xi : \xi < \kappa \rangle$ is strictly increasing;

(11) each (w_ξ, β_ξ) is special;

(12) if $\xi < \eta < \kappa$ and x_γ or $-x_\gamma$ is a factor of some $d \in S_{w_\xi}$, then $\gamma < \beta_\eta$;

(13) if $\xi < \eta < \kappa$ and $y_{\delta\sigma}^\varepsilon$ is a factor of some $d \in S_{w_\xi}$, then $\delta < \beta_\eta$;

Now we claim:

(14) If $\xi < \eta < \kappa$, then $w_\xi \cdot -w_\eta \neq 0$.

For, $w_\xi \not\leq u_{\beta_\eta}$, so $w_\xi \cdot -u_{\beta_\eta} \neq 0$. But $w_\eta \leq u_{\beta_\eta}$, so $-u_{\beta_\eta} \leq -w_\eta$. Hence (15) holds.

Let Γ be the set of all $\xi < \kappa$ such that there is a $d \in S_{w_\xi}$ such that every factor $y_{\rho\sigma}^\varepsilon$ of d has $\varepsilon = 1$, and d does not have a factor $-x_\gamma$.

(15) If $\xi < \eta$ and $\xi \in \Gamma$, then $w_\eta < w_\xi$.

In fact, choose $d \in S_{w_\xi}$ in accordance with the definition of Γ. Then $w_\eta \le [d] \le w_\xi$. Then (15) follows from (14).

Now $\langle Z\rangle$ satisfies ccc, so it follows from (15) that Γ is countable. Let $\nu < \kappa$ be greater than each $\xi \in \Gamma$. Now if $\nu \le \xi < \eta < \kappa$, then each $d \in S_{w_\xi}$ has a factor $-y_{\rho\sigma}$ or a factor $-x_\gamma$, and hence $w_\eta \cdot w_\xi = 0$. This again contradicts ccc.

This finishes the proof. □

3. A Boolean algebra B such that $\mathfrak{s}(B) < \mathfrak{a}(B)$. More precisely, we prove the following:

Theorem 6. *Let κ and λ be regular cardinals, with $\aleph_0 < \kappa < \lambda$. Then there exists a Boolean algebra A such that $\mathfrak{t}(A) = \mathfrak{s}(A) = \kappa$ and $\mathfrak{a}(A) = \lambda$.*

Proof. We begin by defining a base algebra, and then extending it many times to get the desired algebra. The base algebra B_0 is an algebra of subsets of the set ${}^\kappa 2$ of all functions mapping κ into $2 = \{0, 1\}$. We call a set $U \subseteq {}^\kappa 2$ *$(< \kappa)$-defined* iff there is a $D \subseteq \kappa$ of size less than κ such that for all $f, g \in {}^\kappa 2$, if $f \in U$ and $f \restriction D = g \restriction D$, then $g \in U$. Let B_0 be the collection of all $(< \kappa)$-defined subsets of ${}^\kappa 2$. Clearly B_0 is a κ-field of subsets of ${}^\kappa 2$, i.e., it is closed under complements and under unions of fewer than κ sets.

Let

$$S = \{s_\alpha : \alpha < \kappa\} \quad \text{where} \quad s_\alpha = \{f \in {}^\kappa 2 : f(\alpha) = 1\}.$$

Clearly $S \subseteq B_0$. Given $B \ge B_0$, we say that S has the *co-κ splitting property in B* provided that for all nonzero elements b of B, there is a set $T \subseteq \kappa$ of size less than κ such that for all $\beta \in \kappa \backslash T$ we have $b \cdot s_\beta \ne 0 \ne b \cdot -s_\beta$. Clearly this implies that S splits B. It is also clear that S has the co-κ splitting property in B_0.

We are going to construct a tower of Boolean algebras containing B_0, in each of which S has the co-κ splitting property.

(1) *Let $B_0 \le B$, and suppose that X is an infinite partition of unity in B and that S has the co-κ splitting property in B. Then there is an algebra $B' \ge B$, generated by $B \cup \{u\}$ where u is a new element, such that in B', $u \ne 0$ and $u \cdot x = 0$ for all $x \in X$, and such that S has the co-κ splitting property in B'.*

To prove (1), we consider the free extension $B(u)$ (the free product of B and the four-element algebra), and its ideal J generated by all elements of the form $x \cdot u$ with $x \in X$. It turns out that $B' = B/J$ naturally embeds B and has the required properties. All properties are pretty obvious, except that S (or S/J) has the co-κ splitting property in B'. To see that S/J has this property, it suffices to consider separately elements of B' of the form $(b \cdot u)/J$ ($b \in B$) and of the form $(b \cdot -u)$ ($b \in B$).

Consider first $(b \cdot -u)/J \ne 0$ in B', $b \in B$. If $v \in B$ and $v/J \cdot (b \cdot -u)/J = 0$, then $v \cdot b \cdot -u \le w \cdot u$ in $B(u)$, where w is the sum of finitely many elements of X, and hence $v \cdot b \cdot -u = 0$ and so $v \cdot b = 0$. Thus the fact that S has the co-κ splitting property in B implies that with the exception of fewer than κ many $\beta \in \kappa$ we have that $(b \cdot -u)/J$ intersects both s_β/J and its complement.

Now consider $(b \cdot u)/J \ne 0$, $b \in B$. Since this element is nonzero in B' and $\sum X = 1$ in B, it follows that there is a set $\{x_n : n \in \omega\}$ of distinct members of X such that $b \cdot x_n \ne 0$

for all n. Let $T \subseteq \kappa$ be a set of size less than κ such that for all $\beta \in \kappa \backslash T$ and all n we have $b \cdot x_n \cdot s_\beta \neq 0 \neq b \cdot x_n \cdot -s_\beta$. Thus for such β we have $(b \cdot u)/J \cdot s_\beta/J \neq 0 \neq (b \cdot u)/J \cdot -s_\beta/J$. This finishes the proof of (1).

Now we do a similar thing for towers. Note here that in general $\mathfrak{t}(C) \leq \mathfrak{s}(C)$. This step concerning towers can be omitted if $\kappa = \omega_1$, since in general every tower is uncountable if $\mathfrak{a}(C)$ is uncountable.

(2) *Let $B_0 \leq B$, and suppose that X is a tower in B of size less than κ and that S has the co-κ splitting property in B. Then there is an algebra $B' \geq B$, generated by $B \cup \{u\}$ where u is a new element, such that in B', $u \neq 1$ and $x \leq u$ for all $x \in X$, and such that S has the co-κ splitting property in B'.*

The proof is very similar to that of (1). We consider the free extension $B(u)$, and its ideal J generated by all elements of the form $x \cdot -u$ with $x \in X$. Again $B'' = B/J$ naturally embeds B and has the required properties, and we check only that S (or S/J) has the co-κ splitting property in B', considering separately elements of B' of the form $(b \cdot u)/J$ $(b \in B)$ and of the form $(b \cdot -u)$ $(b \in B)$.

Consider first $(b \cdot -u)/J \neq 0$ in B', $b \in B$. Now $(b \cdot -u)/J \neq 0$ implies that $b \cdot -w \neq 0$, for each $w \in X$. So for each $w \in X$ we can choose a subset Y_w of X of size less than κ such that $b \cdot -w \cdot s_\alpha \neq 0 \neq b \cdot -w \cdot -s_\alpha$ for each $\alpha \in \kappa \backslash Y_w$. So if we take any $\alpha \in \kappa \backslash \bigcup_{w \in X} Y_w$ we get $(s_\alpha/J) \cdot (b \cdot -u)/J \neq 0 \neq -(s_\alpha/J) \cdot (b \cdot -u)/J$.

Now consider $(b \cdot u)/J \neq 0$, $b \in B$. If $b \cdot s_\alpha \neq 0 \neq b \cdot -s_\alpha$, then clearly also $(s_\alpha/J) \cdot (b \cdot u)/J \neq 0 \neq -(s_\alpha/J) \cdot (b \cdot u)/J$. Thus (2) holds.

(3) *Suppose that $B_0 \subseteq B$ and S has the co-κ splitting property in B. Then there is an extension B''' of B such that S has the co-κ splitting property in B''', and B''' does not have any partition of unity X such that $X \subseteq B$ and $\aleph_0 \leq |X| < \lambda$, and does not have any tower $X \subseteq B$ of size less than κ.*

To prove this, let $\langle X_\tau : 0 < \tau < \gamma \rangle$ be a list of all the infinite sets X of at most κ pairwise disjoint elements of B, with $|X| < \lambda$, and let $\langle Y_\tau : 0 < \tau < \gamma \rangle$ be a list of all the well-ordered subsets of $B \backslash \{1\}$ of size less than κ. Define C_τ by recursion for $\tau < \gamma$ as follows. Let $C_0 = B$. Suppose that C_τ has been defined for all $\delta < \tau$, where $0 < \tau < \gamma$. Take C'_τ to the the union of $\{C_\delta : \delta < \tau\}$, and take C_τ to be the algebra obtained from C'_τ by applying first (1) to X_τ and then (2) to Y_τ. Finally, take $B''' = \bigcup_{\tau < \gamma} C_\tau$. This proves (3).

Now we can finish the main part of the proof of the theorem as follows. We define a tower $\langle B_\tau : \tau < \lambda \rangle$ and take A to be its union. B_0 was defined at the beginning of the proof. For a successor $\tau = \delta + 1$, take B_τ to be the algebra supplied by (3) with $B = B_\delta$. For limit τ, take $B_\tau = \bigcup_{\delta < \tau} B_\delta$. Clearly this works, as any infinite partition of unity of A of size less than λ is contained in some B_δ, and by construction this is impossible. Similarly for towers of size less than κ. Also, since S splits A, it follows that A is atomless.

The only thing missing is that A may not have a partition of unity of size λ. To assure this property, we extend A further. Take $C = A \oplus D$, where D is the algebra of finite and cofinite subsets of λ. Clearly S still splits C, C has a partition of unity of size λ, and C has a tower of size κ. We need to check that C has no infinite partition of unity of size less than λ, and no tower of size less than κ.

Suppose that X is an infinite partition of unity of C of size less than λ. We may assume that the elements x of X have the form $x = a_x \cdot d_x$ with $a_x \in A$ and $d_x \in D$. Clearly $\bigcup_{x\in X} d_x = \lambda$. It follows that for some element $\beta \in \lambda$, the set $\Gamma = \{x \in X : \beta \in d_x\}$ is infinite. For any two distinct $x, y \in \Gamma$ we have $a_x \cdot a_y = 0$, and so by our partition property for A, there is an element $e \in A$ such that $e \cdot a_x = 0$ for all $x \in \Gamma$. Then $e \cdot \{\beta\}$ is a nonzero element of C disjoint from each $x \in X$, contradiction.

Suppose that X is a tower in C of size less than κ. We can write each element $x \in X$ in the form $\sum_{i<m_x}(a_{i,x} \cdot d_{i,x})$ with $a_{i,x} \in A$, $d_{i,x} \in D$, and the $d_{i,x}$'s disjoint for distinct i's. Moreover, we may assume that the order type of X is regular. It is uncountable since $\mathfrak{t}(C) = \omega$ would imply that $\mathfrak{a}(C) = \omega$. So, we may assume that $m = m_x$ is independent of x. Note that $\sum_{i<m} d_{i,x} \leq \sum_{i<m} d_{i,y}$ if $x, y \in X$ and $x < y$, and $\sum_{x\in X}\sum_{i<m} d_{i,x} = \lambda$. Hence, as $|X| < \kappa < \lambda$, there is an $x \in X$ such that $\sum_{i<m} d_{i,x} = \lambda$. Thus for $y \in X$ and $x \leq y$, there is an $i < m$ such that $d_{i,y}$ is cofinite. We may assume that for each $y \in X$ with $x \leq y$, it is the element $d_{0,y}$ which is cofinite; also, for each $\beta \in \lambda$ let $i(\beta, y)$ be the index less than m such that $\beta \in d_{i(\beta,y),y}$. Now we claim

(*) For every $\beta \in \lambda$ there is a $y \in X$ with $x < y$ such that $a_{i(\beta,z),z} = 1$ for each $z \in X$ with $z \geq y$.

To prove this, for $y, z \in X$ and $x < y < z$ we have

$$a_{i(\beta,y),y} \cdot \{\beta\} = y \cdot \{\beta\} \leq z \cdot \{\beta\} = a_{i(\beta,z),z} \cdot \{\beta\},$$

and hence $a_{i(\beta,y),y} \leq a_{i(\beta,z),z}$. If the conclusion of (*) fails to hold, then because A has no tower of size less than κ, there is a $b \in A$ such that $a_{i(\beta,y),y} < b < 1$ for all $y \in X$ for which $x < y$. Then for any such y we have $y < b \cdot \{\beta\} + (\lambda\backslash\{\beta\}) < 1$, contradiction. So (*) holds.

By (*), for each $\beta \in d_{0,x}$ choose $y_\beta \in X$ with $x < y_\beta$ such that $a_{i(\beta,z),z} = 1$ for each $z \in X$ with $z \geq y$. Since $\kappa < \lambda$, there is an infinite $\Gamma \subseteq d_{0,x}$ and a $y \in X$ with $x < y$ such that $y_\beta = y$ for all $\beta \in \Gamma$. Choose $\beta \in \Gamma\backslash\bigcup_{0<i<m} d_{i,y}$. Then $\beta \in d_{0,y}$, and so $a_{0,y} = 1$. For each $\beta \in \lambda\backslash d_{0,y}$, by (*) choose $z_\beta \in X$ with $z_\beta > y$ such that $a_{i(\beta,w),w} = 1$ for each $w \in X$ with $z_\beta \leq w$. Let $w \in X$ be greater than each z_β with $\beta \in \lambda\backslash d_{0,y}$. Then $w = \lambda$, contradiction. □

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