Math 256 - Assignment 2
Profit maximizing firms

Enter your name here

1/13/14

Instructions

Save a copy of this notebook, complete the exercises, save and submit on OAK your final version by Wed. Jan. 22.

Edit the line above that reads Enter your name here, inserting instead your name. You may also want to save to a filename that includes your name.

Problem

A firm has a unique product it will sell to consumers at a uniform price. How should it set the price so as to maximize its profit?

Model

In running a firm, managers must decide many things. What to sell, how to produce what it sells, how much to sell, and where and how to sell what it does. But one of the most important things it must decide is the price to charge for its products. A firm selling something that another firm could also supply is limited in the price it can charge by what the competition charges; since consumers are likely to switch to the cheaper of equivalent products. A firm with a unique product to sell is limited by how many consumers will by at any given price and how much it costs to produce a given quantity of product, without concern for (or with much less concern about) how the competition might respond. We start by considering the simple case of a monopoly supplier of a single product.

Suppose the firm sets the price at \( p \). We describe the consumer demand for the product as a function of this variable, \( q(p) \), the amount that can be sold at price \( p \), say in each month. The firm should know how much raw material and labor it will take to produce a given quantity \( q \), and so will have some idea of the cost \( C(q) \) of producing this much product. The profit of the firm is then \( \Pi(p) = pq(p) - C(q(p)) \) each month. We assume that the firm will seek to maximize profit, concluding that the price charged
will be determined by the demand and cost functions, \( q(p) \) and \( C(q) \).

We have a simple model of how firms set prices, expressed as a simple profit maximization problem. Mathematically, we solve for the profit maximizing price using simple calculus. At a critical point, 
\[
0 = \frac{\partial \Pi(p)}{\partial p} = q(p) + p \frac{\partial q(p)}{\partial p} - C'(q(p)) \frac{\partial q(p)}{\partial p}
\]

We can identify important factors in determining the best price. For example, \( C'(q) \) is the change in cost per unit change in quantity, what economists call the marginal cost. We might rearrange the terms in this condition to conclude
\[
\frac{p-C(q,p)}{p} = \frac{-1}{\varepsilon}
\]

where the quantity on the left is the profit margin and
\[
\varepsilon = \frac{\partial q(p)}{\partial p} \frac{p}{q(p)}
\]

is the own price elasticity of demand, the fractional change in demand as a multiple of the fractional change in price.

To solve for the profit maximizing price we would need the functions \( q(p) \) and \( C(q) \). To get a practical answer for a real world problem we would need to estimate at least the elasticity of demand and marginal cost. To understand how demand and cost determine the optimum price, we can assume simple forms for the demand and cost functions, with parameters that can be adjusted to fit various scenarios. The accuracy of the conclusions drawn will depend on whether the assumed forms are flexible enough and whether the parameters can be estimated with any precision. By stating the assumptions of the model, making explicit the functional forms and parameter values, and solving various scenarios, we can better evaluate the implications of the model in the real world.

---

**Computation**

**Linear demand model**

The simplest model for demand would be to take a linear function of price. Imagine that 20% of potential consumers would buy a product at a price of $50 but 25% would buy at a price of $40. Suppose there are a total of 10000 potential consumers (per month say).

Using the method of undetermined coefficients, we write the unknown demand of the desired form with variable coefficients.

\[
\text{In[1]}= \text{demandform[p_] = demandslope \cdot p + demandintercept}
\]

\[
\text{Out[1]}= \text{demandintercept + demandslope p}
\]

Next we right the given conditions for the demand is assumed to satisfy.

\[
\text{In[2]}= \text{population} = 10000;
\]

\[
\text{In[3]}= \text{demandconditions} = \\
\{\text{demandform[50.]} = 0.20 \cdot \text{population}, \text{demandform[40.]} = 0.25 \cdot \text{population}\}
\]

\[
\text{Out[3]}= \{\text{demandintercept + 50. \cdot demandslope = 2000.}, \text{demandintercept + 40. \cdot demandslope = 2500.}\}
\]

Now use Solve to find coefficients satisfying the stated conditions. If there are as many free variables as equations, then there is a good chance the conditions the coefficients can be found satisfying the conditions. Solve tries to find all solutions symbolically. In other cases, we may only be able to approxi-
mate a solution numerically close to some starting estimate using the function FindRoot.

In[4] :=\[demandsolns = Solve[demandconditions, \{demandslope, demandintercept\}]

Out[4] =\{
demandslope \rightarrow -50., demandintercept \rightarrow 4500.\}\}

Solve returns a list of solutions, each solution a list of rules for the variables. A Rule is typed as var \rightarrow value with \rightarrow becoming \rightarrow. In this case, there is just one solution which we can extract using double brackets to index in the list demandsolns[[1]]. If we have an expression involving some variables and a list of rules giving values for the variables we can use ReplaceAll to substitute the values for the variables in the expression. The shorthand notation for ReplaceAll is expr /. rules thus,


In[6] = Plot[demand[p], \{p, 30, 60\}]

We must also assume a cost function for the producer. For simplicity, we simply assume a linear function. The intercept is the cost at zero production, the fixed cost. The slope is the incremental cost of each additional unit of production, the marginal cost. Again, we may calibrate the cost function to particular sample points. For this example, we will simply assume a cost function.

In[7] = fixedcost = 20 000.;

In[8] = marginalcost = 8.;

In[9] = cost[q_] = fixedcost + marginalcost * q

Out[9] = 20 000. + 8. q

Next define the profit function to be maximized. We can plot the profit function and see that it has a maximum in the range used to calibrate the demand function.


The profit function is maximum at a critical point. From the optimum price we compute the resulting demand, cost, revenue and profit.

\[ \text{profitmaxcondition} = \frac{\text{D}[\text{profit}[p], p]}{p} = 0 \]

\[ \text{Out}[12]= 4900. - 100. \cdot p = 0 \]

\[ \text{profitmaxsolns} = \text{Solve}[\text{profitmaxcondition}, p] \]

\[ \text{Out}[13]= \{ \{ p \to 49. \} \} \]

\[ \text{profitmaxprice} = p \bigg/ \text{profitmaxsolns}[1] \]

\[ \text{Out}[14]= 49. \]

\[ \text{profitmaxquantity} = \text{demand}[\text{profitmaxprice}] \]

\[ \text{Out}[15]= 2050. \]

\[ \text{profitmaxcost} = \text{cost}[\text{profitmaxquantity}] \]

\[ \text{Out}[16]= 36400. \]

\[ \text{profitmaxrevenue} = \text{profitmaxprice} \cdot \text{profitmaxquantity} \]

\[ \text{Out}[17]= 100450. \]

\[ \text{profitmaxrevenue} - \text{profitmaxcost} \]

\[ \text{Out}[18]= 64050. \]

This checks against the profit function. Increasing or decreasing the price by a small amount decreases the total profit.

\[ \text{profitmaxprofit} = \text{profit}[\text{profitmaxprice}] \]

\[ \text{Out}[19]= 64050. \]

\[ \text{profit}[\text{profitmaxprice} + 0.1] \]

\[ \text{Out}[20]= 64049.5 \]
In[21] = profit[profitmaxprice - 0.1]
Out[21] = 64049.5

The profit margin is the excess of the price over the marginal cost as a fraction of the price.

In[22] = profitmaxprofitmargin = (profitmaxprice - marginalcost) / profitmaxprice
Out[22] = 0.836735

The elasticity of demand measures the (fractional) sensitivity of demand to (fractional) price changes.

In[23] = elasticityformula[p_] = D[demand[p], p] * p / demand[p]
Out[23] = -\frac{50.\ p}{4500.-50.\ p}

In[24] = profitmaxelasticity = elasticityformula[profitmaxprice]
Out[24] = -1.19512

The profit margin is inversely related to the elasticity when the firm is profit maximizing.

In[25] = -1 / profitmaxelasticity
Out[25] = 0.836735

Alternative demand model

There is no requirement in this calculation that we take a linear demand model. The maximum amount a given consumer might be willing to pay, the consumer's personal value for the product, might be normally distributed. *Mathematica* represents statistical distributions by a symbol, in this case the name NormalDistribution, applied to parameters that define the distribution. The function NormalDistribution does not evaluate, but represents the distribution as an argument in statistical functions. For a normal distribution the parameters are the mean and standard deviation. The fraction of values in the population less than a given \( x \) is the cumulative distribution function, the CDF. Assuming 20% buy at $50 means 80% have values less than $50, and for 25% to buy at $40 means 75% have values less than $40. We translate these calibration conditions. The demand is the population times 1 minus the CDF.

In[26] = valuedistribution = NormalDistribution[valuemean, valuesd]
Out[26] = NormalDistribution[valuemean, valuesd]

In[27] = valueconditions = {CDF[valuedistribution, 50.] == 1 - 0.20, CDF[valuedistribution, 40.] == 1 - 0.25}
Out[27] = \{\frac{1}{2}\text{Erfc}\left[\frac{-50.+\valuemean}{\sqrt{2}\ \text{valuesd}}\right] = 0.8, \frac{1}{2}\text{Erfc}\left[\frac{-40.+\valuemean}{\sqrt{2}\ \text{valuesd}}\right] = 0.75\}\n
In[28] = valuesolns = Solve[valueconditions, {valuemean, valuesd}]
Out[28] = \{\{\text{valuemean} \to -0.356834, \text{valuesd} \to 59.8331\}\}

*Mathematica* warns that inverse functions are being used, but gets a solution anyways. It may be that we will have a system of equations that cannot be solved symbolically. In such a case it is possible to
solve the system of equations numerically using a variant of Newton's method to refine a approximate guess to get a precise solution. The *Mathematica* function for numerical root finding is FindRoot. Suppose we thought the mean value was around 20 with an sd of 30. Using this as an initial guess and using FindRoot finds an answer.

In[29]:= valuesoln1 = FindRoot[valueconditions, {{valuemean, 20}, {valuesd, 30}}]

Out[29]= {valuemean -> -0.356834, valuesd -> 59.8331}

Now this answer may seem a little odd. The mean value is near zero, meaning half the population would have a negative value for the product. But we only imagine the approximation to a normal distribution of values will work near the calibration data, extrapolating all the way to zero values probably would not work.

In[30]= population = 10000;

In[31]= normaldemand[p_] = population * (1 - CDF[valueldistribution, p]) /. valuesolns[[1]]

Out[31]= 10000 \left(1 - \frac{1}{2} \text{Erfc}[0.011818 (-0.356834 - p)]\right)

In[32]= Plot[normaldemand[p], {p, 0, 100}]

Out[32]=

In the range we have been considering the new demand is approximates the linear demand, after all the two functions both go through the same two values at $40$ and $50$.

In[33]= Plot[{normaldemand[p], demand[p]}, {p, 30, 60}]

Out[33]=
Assuming the same cost function as before we then maximize profit.

```
In[34]= normalprofit[p_] = p*normaldemand[p] - cost[normaldemand[p]]
```

```
Out[34]= -20000. - 80000. \left[1 - \frac{1}{2} \text{Erfc}[0.011818 (-0.356834 - p)] \right] + \\
10000 p \left[1 - \frac{1}{2} \text{Erfc}[0.011818 (-0.356834 - p)] \right]
```

```
In[35]= Plot[normalprofit[p], \{p, 30, 60\}]
```

```
Out[35]=
```

```
In[36]= normalprofitmaxcondition = D[normalprofit[p], p] = 0
```

```
Out[36]= 533.407 e^{-0.000139665 (-0.356834-p)^2} - 66.6758 e^{-0.000139665 (-0.356834-p)^2} p + \\
10000 \left[1 - \frac{1}{2} \text{Erfc}[0.011818 (-0.356834 - p)] \right] = 0
```

And it is very unlikely Solve will work here so we use FindRoot instead, with a starting value for p of say 50.

```
In[37]= normalprofitmaxsoln1 = FindRoot[normalprofitmaxcondition, \{p, 50.\}]
```

```
Out[37]= \{p \to 50.5324\}
```

```
In[38]= normalprofitmaxprice = p /. normalprofitmaxsoln1
```

```
Out[38]= 50.5324
```

```
In[39]= normalprofitmaxquantity = normaldemand[normalprofitmaxprice]
```

```
Out[39]= 1975.18
```

```
In[40]= normalprofitmaxcost = cost[normalprofitmaxquantity]
```

```
Out[40]= 35801.5
```

```
In[41]= normalprofitmaxrevenue = normalprofitmaxprice * normalprofitmaxquantity
```

```
Out[41]= 99810.7
```

```
In[42]= normalprofitmaxrevenue - normalprofitmaxcost
```

```
Out[42]= 64009.2
```
Constant elasticity demand model

In the linear demand model, a constant slope of demand is assumed, that is, the additional number of consumers that will buy for a given dollar amount decrease in price is the same at any price level. In the second example, the slope of demand curve is not constant, but the linear approximation is not too different, at least within a certain range, if we calibrate the demand curves to the same conditions. Of course, there's no particular reason to think the alternative demand model is better given the information available. Another possibility is to imagine that, instead of the slope of demand being constant, the (price) elasticity of demand is a constant. That is, we might consider instead a demand function \( q(p) \) such that elasticity

\[
\varepsilon = \frac{\partial q(p)}{\partial p} \frac{p}{q(p)}
\]

is a constant independent of price. What is the form of such a function? This a condition on the derivative of \( q(p) \), defining a differential equation. Mathematica can solve many differential equations symbolically, given an equation (or initial value problem) in terms of a symbolic function in some independent variable. Look up DSolve to see how.

\[
\text{Input} \quad \text{constantelasticitydiffeqn = \{D[q[p], p] \star p / q[p] == elast\}}
\]

\[
\text{Output} \quad \{\frac{p \, q'[p]}{q[p]} = \text{elast}\}}
\]

\[
\text{Input} \quad \text{constantelasticitysoln = DSolve[constantelasticitydiffeqn, q[p], p]}\]

\[
\text{Output} \quad \{\{q[p] \rightarrow p^\text{elast} \, C[1]\}}\}
\]

\[
\text{Input} \quad \text{cedemandform[p_] = p^elast \, scale}\]

\[
\text{Output} \quad p^\text{elast} \, \text{scale}\]

Check.

\[
\text{Input} \quad \text{D[cedemandform[p], p] \star p / cedemandform[p]}\]

\[
\text{Output} \quad \text{elast}\]

So the form of the demand function is a power of price times an arbitrary constant defining the scale of demand. This amounts to the log of demand being a linear function in the log of price.

\[
\text{Input} \quad \text{PowerExpand[Log[cedemandform[p]]]}\]

\[
\text{Output} \quad \text{elast} \, \text{Log}[p] + \text{Log}[\text{scale}]\]

---

Exercises

1. Find the coefficients in a linear demand function assuming demand is 10000 at
a price of $50 and 8000 at a price of $55.

2. Solve for the profit maximizing price for this demand, assuming a constant marginal cost of $25.

3. Suppose the marginal cost increases from $25 to $26. Then the optimal price to charge should also go up, though perhaps not by a full dollar. The pass-through rate is the amount the (optimal) price increases for each dollar increase in the marginal cost (or better, it is the derivative of optimal price as a function of the marginal cost parameter). Repeat the profit maximizing calculations with the higher marginal cost and determine (approximately) the pass-through rate.

4. Consider the effect of imposing a corporate profits tax of 10%. I.e., how do the optimal price and quantity change if the firm maximizes the 90% of its profits that it gets to keep?

5. Consider the effect of imposing a sales tax of 10%. Suppose that whatever price the firm decides on, the final price the consumer sees is 10% higher and that price is what determines the demand for the product, i.e., assume the same consumer demand function as in 1 is applied to 1.1 times the price the firm sets (and receives) for the product. What happens to optimal price and quantity if the firm selects a price that maximizes its profit given this change in consumer demand due to the tax?

6. This example has assumed that demand is linear. An alternative is to assume that demand is constant elasticity. The demand function in this case would be given by the formula given earlier. Repeat 1-5 with this demand form, calibrating the demand function to the same conditions, a demand of 10000 at $50 and 8000 at $55.