Math 256 - Assignment 1  
Simple models

Enter your name here

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Instructions

Save a copy of this notebook, complete the exercises, save and submit on OAK your final version by Mon. Jan. 13.

Edit the line above that reads Enter your name here, inserting instead your name. You may also want to save to a filename that includes your name. By adding your name into the file and filename, you reduce the chance I will mix-up your submission with someone else’s.

Exercises

1. What is a 15% tip on a $40 meal? How many such tips an hour would a server need to make to earn as much as a $300 an hour lawyer?

This is a really simple (almost silly) mathematical model, but it points up some of the common elements of mathematical models. We compare a real situation, how much people leave in tips in relation to their bills, translated to a numerical relationship, that of a proportion between the amount of the tip and the amount of the bill, specified in this case as 15%. We can make a mathematical (numerical) prediction based on assumptions, namely the assumption that the proportion is a constant, the constant a parameter of the model whose value is assumed, with the predicted tip determined as a function of the bill, taken to be a particular value in this example. But the prediction is only an approximation to the real experience. Some tips are larger, some smaller. This model is so well known and understood that most people use it as a guide to tipping. But is it really an accurate prediction? If a server has to deal with too many tables, the average tip rate may decline due to a decline in the quality of service. Are relatively small individual bills likely to get a larger or smaller rate of tipping? Is there any reason for tips to be in proportion to bill, rather than say number of visits to a table? Similarly, is there any reason lawyers should be paid at an hourly rate rather than by the job? Is it reasonable that lawyers will sometimes work for a share of any award they win? What about doctors or economic consultants or ...? Compensation schemes is a fair topic for economists. The mathematical model is an analogy between
some aspect of a real world situation and mathematical objects and relationships. It translates a real world question into a mathematical question where mathematical techniques can be applied. The model will likely only be an approximation to the truth, but it can illuminate the real world situation because it reveals the underlying logic relating the assumptions to the conclusions.

2. A small gourmet restaurant might serve 30 people at a time, for three seatings a night, five nights a week, 52 weeks a year, averaging say $40 a meal. Variable costs, that is all the costs that would decrease if the number of customers were reduced, presumably in proportion to a change in the number of customers, might be $15 per meal served, for food and cooking supplies, dish washing and the like. Fixed costs would include personnel, cooks, servers, busboys etc. (if an adjustment to personnel is not a likely response to a change in the number of customers), rent for the space for a 30 person dining area plus kitchen and perhaps office space, utilities, marketing and miscellaneous administrative expenses. Imagine 10 employees averaging $35000 a year salary (more for the head chef, less for a busboy). Suppose a retail space of 2500 sq. ft. renting for $5 per sq. ft. per month. For this analysis ignore other costs and taxes. What is the total annual revenue for the restaurant? What is the total annual expenses? And what profit does the restaurant make for the owner each year?

Give an income and expenses statement for the restaurant. Of course, I’ve simplified things considerably, and made up a lot of numbers. These numbers may be way off. The question is one of basic accounting, adding up total income, and subtracting total costs, to see what the net profit of a business should be. If the difference doesn’t show up in the owner’s bank account at the end of the year, the accountant should be fired or arrested for embezzlement. (Wait, we forgot to pay for an accountant, oh well. What if we could move to a larger location, or open seven days a week, but then we would need more employees, but would we have more customers, or could we lower prices, or...)

3. This small business probably doesn’t hold many assets, unless it owns a location instead of renting perhaps. But a reputation and supply of regular customers that yields an income is of value. A way of accounting for this value is to imagine how much money you would have to invest, at some rate of return, to get the same income. Alternatively, the value of a certain income stream is the principal of a loan whose interest payments would be just met by that income stream, the maximum amount you could pay if you were to borrow the money to buy the business and pay back the loan out of the income. This present value
of future income depends critically on the discount rate applied to the future income, the interest rate in the calculation. Suppose you could count on the income in the above calculation year in and year out, and you use a 10% per year discount rate. Imagining simple interest on the value of the business each year is equal to the profit the business makes, what is the value of the business?

If a business made a certain $100,000 a year and you could borrow money for 10% per year, wouldn’t you borrow $500,000 to buy the business pay $50,000 interest and pocket the $50,000 difference between the income and loan payment? Wouldn’t someone else be willing to outbid you, borrow and pay $700,000 for the business, pay $70,000 in interest, and pocket instead just $30,000? And then the bidding would continue up to the limiting value of $1,000,000 when the interest payed matches the income. This would work except that in all practical cases, the income is uncertain, potential difficulty in reselling the business at a future time or else a relatively short time horizon over which one would want to have payed back also the principal of a loan. All of these adjustments can be folded into an appropriate discount rate to apply; compare this to the P/E ratio, or its reciprocal, earnings yield, for stocks.

4. What happens to the income statement if during a recession the number of customers decreases by 10%? What is the value of the business then, assuming the 10% decrease is permanent?

5. (Challenge) A better model of present value of an income stream would account for the uncertainty in future income and potentially include variable and uncertain rates of discounting over time. We will consider models with random elements soon enough. For now, consider a continuous periodically varying income stream, say from a predictable business cycle or other periodic event (Olympics, presidential campaigns, etc.). Say that the rate of income at a time \( t \) years in the future is given by \( 1 + 0.5 \sin(2 \pi t/7) \) million dollars per year, for a seven year business cycle say, allowing \( t \) varying continuously over time. Over a seven year period, income averages 1 million dollars per year, for a total of 7 million dollars every seven years. But in the first year, income is greater as income slowly climbs. Discount future income to the present continuously at a compound rate corresponding to 10% per year effective interest rate, weighting future income \( t \) years in the future by a factor of \( 1.1^{-t} \). Assuming this same income stream indefinitely into the future, what is the present value of this income stream? How does this value change over the business cycle?