Operator Theory:
Advances and Applications, Vol. 168, 189–208
© 2006 Birkhäuser Verlag Basel/Switzerland

Operator-valued Symbols for Elliptic and Parabolic Problems on Wedges

Jan Prüss and Gieri Simonett

Dedicated to Philippe Clément

Abstract. We study evolution problems of the type $e^{sx}\partial_t u + h(\partial_x)u = f$ where h is a holomorphic function on a vertical strip around the imaginary axis, and $s > 0$. If P is a second-order polynomial we give a complete characterization of the spectrum of the parameter-dependent operator $\lambda e^{sx} + P(\partial_x)$ in $L_p(\mathbb{R})$. We show the surprising result that the spectrum is independent of λ whenever $|\arg\lambda| < \pi$. Moreover, we also characterize the spectrum of $\partial_t e^{sx} + P(\partial_x)$, and we show that this operator admits a bounded $\mathcal{H}^\infty$-calculus. Finally, we describe applications to free boundary problems with moving contact lines, and we study the diffusion equation in an angle or a wedge domain with dynamic boundary conditions. Our approach relies on the $\mathcal{H}^\infty$-calculus for sectorial operators, the concept of $\mathcal{R}$-boundedness, and recent results for the sum of non-commuting operators.

1. Introduction

In recent years the $\mathcal{H}^\infty$-calculus for sectorial operators in Banach spaces, the concept of $\mathcal{R}$-boundedness, and the method of operator sums have become important tools for proving existence and optimal regularity results for solutions of partial differential and integro-differential equations, as well as for abstract evolutionary problems. We mention here only the references [7, 8, 15, 16, 17] which document some recent work by the authors. In the current paper we apply these techniques to the study of certain operator-valued symbols which arise from elliptic or parabolic equations on angles or wedges with dynamic boundary conditions. Another main objective of this paper is to develop tools for the study of free boundary problems with moving contact lines.

To describe the class of symbols we have in mind, let us first consider the case of dynamic boundary conditions. It is shown in Section 5 that the boundary symbol for the Laplace equation $\Delta u = 0$ on an angle $G = \{(r\cos\phi, r\sin\phi);\ r > 0,\ \phi \in (0,\alpha)\}$ in $\mathbb{R}^2$ with Dirichlet condition $u = 0$ on $\phi = \alpha$ and dynamic boundary condition $\partial_t u + \partial_\nu u = g$ on $\phi = 0$ is given by

$$\partial_t e^x + \psi_0(-(\partial_x+\beta)^2), \quad \psi_0(z) = \sqrt{z}\coth(\alpha\sqrt{z}), \quad z \in \mathbb{C}.$$

Here $\beta \in \mathbb{R}$ is a number whose meaning is explained in Section 5. Similarly, if one considers the one-phase quasi-stationary Stefan problem with surface tension (also sometimes called the Mullins-Sekerka problem) in two dimensions with boundary intersection and prescribed contact angle $\alpha \in (0,\pi]$, one is led to the boundary symbol

$$\partial_t e^{3x} - \psi_1\big(\partial_x+\beta)^2\big)(\partial_x+\beta+1)(\partial_x+\beta+2),$$

where this time $\psi_1(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. The free boundary problem for the stationary Stokes equations with boundary contact and prescribed contact angle in two dimensions leads to

$$\partial_t e^x + \psi(\partial_x+\beta),$$

where

$$\psi(z) = (1+z)\frac{\cos(2\alpha z)-\cos(2\alpha)}{\sin(2\alpha z)+z\sin(2\alpha)},$$

in the slip case and

$$\psi(z) = \frac{(1+z)}{4}\,\frac{\sin(2\alpha z)-z\sin(2\alpha)}{z^2\sin^2(\alpha)-\cos^2(\alpha z)}$$

in the non-slip case. This motivates the study of equations of the type

$$\partial_t e^{sx} + h(\partial_x) \tag{1.1}$$

and its parametric form

$$\lambda e^{sx} + h(\partial_x), \tag{1.2}$$

where $s > 0$, $\lambda \in \mathbb{C}$, and h is a function holomorphic on a vertical strip around the imaginary axis.

In higher dimensions, the boundary symbol for the Laplace equation in a wedge with dynamic boundary condition is given by

$$\partial_t e^x + \psi_0\big(-\Delta_y e^{2x} - (\partial_x+\beta)^2\big),$$

where Δ_y means the Laplacian in the variables y tangential to the edge, whereas that of the quasi-stationary Stefan problem becomes

$$\partial_t e^{3x} + \psi_1\big(-\Delta_y e^{2x} - (\partial_x+\beta)^2\big)[-\Delta_y e^{2x} - (\partial_x+\beta+1)(\partial_x+\beta+2)].$$

Similarly,

$$\partial_t e^x + \psi_0\big((\partial_t - \Delta_y)e^{2x} - (\partial_x+\beta)^2\big)$$

represents the boundary symbol for the diffusion equation in a wedge with dynamic boundary condition. Clearly, these symbols are considerably more complicated than in the two-dimensional case. They also show the importance of the special case $h = P$ where P is a second-order polynomial. In the current paper we concentrate on this case.

It is our ultimate goal to identify function spaces such that the operators defined by the symbols of type (1.1) and (1.2) become topological isomorphisms between these spaces, i.e., to obtain optimal solvability results. We intend to do this in the framework of L_p-spaces. Our main tools are very recent results on sums of sectorial operators, their $\mathcal{H}^\infty$-calculi, and $\mathcal{R}$-boundedness of associated operator families.

Once this goal is achieved, one can go on to study symbols of higher-dimensional (or time-dependent) problems. We will do this here only for those arising from the problems with dynamic boundary conditions. The symbols for the Mullins-Sekerka problem in higher dimensions, for the Stefan problem with surface tension, and for the non-steady Stokes problem with free boundary will be the subject of future work.

Observe that symbols of type (1.1) and (1.2) are highly degenerate, due to the presence of the exponentials. They are not directly accessible by standard methods for pseudo-differential operators. Moreover, the basic ingredients of these symbols, namely e^x and ∂_x, do not commute and so the functional calculus in two variables does not apply. Still, there is a close relation between these operators. In fact, e^{sx} is an eigenfunction of ∂_x with eigenvalue s, or to put it in a different way, the commutator between e^{sx} and ∂_x is se^{sx}. It is this relation we base our approach on. It allows us to apply abstract results on sums of non-commuting operators.

The plan for this paper is as follows. In Section 2 we introduce the necessary notation and state some abstract results needed in the sequel. In Sections 3 and 4 we deal with the important special case where h is a second-order polynomial P. By means of an explicit representation of the resolvent we derive a complete characterization of the spectrum of the operators

$$\lambda e^{sx} + P(\partial_x) \quad \text{in} \quad L_p(\mathbb{R})$$

and

$$\partial_t e^{sx} + P(\partial_x) \quad \text{in} \quad L_p(J \times \mathbb{R}),$$

where $J = (0, T)$. We conclude the paper with applications to the Laplace equation and the diffusion equation in an angle or a wedge with dynamic boundary condition. By means of our results we obtain the complete solvability behavior of this problem in appropriately weighted L_p-spaces.

We mention here that the Laplace and the diffusion equation with dynamic boundary conditions in domains with edges were previously studied in the framework of weighted L_2-spaces by Frolova and Solonnikov, see [10, 18] and the references contained therein.

2. Preliminaries

In the following, $X = (X, |\cdot|)$ always denotes a Banach space with norm $|\cdot|$, and $\mathcal{B}(X)$ stands for the space of all bounded linear operators on X, where we will again use the notation $|\cdot|$ for the norm in $\mathcal{B}(X)$. If A is a linear operator on X, then $\mathsf{D}(A)$, $\mathsf{R}(A)$, $\mathsf{N}(A)$ denote the domain, the range, and the kernel of A, whereas $\rho(A)$, $\sigma(A)$ stand for the resolvent set, and the spectrum of A, respectively. An operator A is called *sectorial* if

- $\mathsf{D}(A)$ and $\mathsf{R}(A)$ are dense in X,
- $(-\infty, 0) \subset \rho(A)$ and $|t(t+A)^{-1}| \leq M$ for $t > 0$.

The class of all sectorial operators is denoted by $\mathcal{S}(X)$. If A is sectorial, then it is closed, and it follows from the ergodic theorem that $\mathsf{N}(A) = 0$. Moreover, by a Neumann series argument one obtains that $\rho(-A)$ contains a sector

$$\Sigma_\phi := \{z \in \mathbb{C} : \ z \neq 0, \ |\arg(z)| < \phi\}.$$

Consequently, it is meaningful to define the *spectral angle* ϕ_A of A by means of

$$\phi_A := \inf\{\phi > 0 : \rho(-A) \supset \Sigma_{\pi-\phi}, \ M_{\pi-\phi} < \infty\},$$

where $M_\theta := \sup\{|\lambda(\lambda+A)^{-1}| : \ \lambda \in \Sigma_\theta\}$. Obviously we have

$$\pi > \phi_A \geq \arg(\sigma(A)) := \sup\{|\arg(\lambda)| : \ \lambda \neq 0, \ \lambda \in \sigma(A)\}.$$

If A is sectorial, the functional calculus of Dunford given by

$$\Phi_A(f) := f(A) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda$$

is a well-defined algebra homomorphism $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$, where $\mathcal{H}_0(\Sigma_\phi)$ denotes the set of all functions $f : \Sigma_\phi \to \mathbb{C}$ that are holomorphic and that satisfy the condition

$$\sup_{\lambda \in \Sigma_\phi} (|\lambda^{-\varepsilon} f(\lambda)| + |\lambda^\varepsilon f(\lambda)|) < \infty \text{ for some } \varepsilon > 0 \text{ and some } \phi > \phi_A.$$

Here Γ denotes a contour $\Gamma = e^{i\theta}(\infty, 0] \cup e^{-i\theta}[0, \infty)$ with $\theta \in (\phi_A, \phi)$. A is said to admit an $\mathcal{H}^\infty$-*calculus* if there are numbers $\phi > \phi_A$ and $M > 0$ such that the estimate

$$|f(A)| \leq M|f|_{\mathcal{H}^\infty(\Sigma_\phi)}, \quad f \in \mathcal{H}_0(\Sigma_\phi), \tag{2.1}$$

is valid. In this case, the Dunford calculus extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$, see for instance [6] for more details. We denote the class of sectorial operators which admit an $\mathcal{H}^\infty$-calculus by $\mathcal{H}^\infty(X)$. The infimum ϕ_A^∞ of all angles ϕ such that (2.1) holds for some constant $M > 0$ is called the $\mathcal{H}^\infty$-*angle* of A.

Let $\mathcal{T} \subset \mathcal{B}(X)$ be an arbitrary set of bounded linear operators on X. Then $\mathcal{T}$ is called $\mathcal{R}$-*bounded* if there is a constant $M > 0$ such that the inequality

$$\mathbb{E}(|\sum_{i=1}^N \varepsilon_i T_i x_i|) \leq M\mathbb{E}(|\sum_{i=1}^N \varepsilon_i x_i|) \tag{2.2}$$

is valid for every $N \in \mathbb{N}$, $T_i \in \mathcal{T}$, $x_i \in X$, and all independent, symmetric $\{\pm 1\}$-valued random variables ε_i on a probability space $(\Omega, \mathcal{A}, P)$ with expectation $\mathbb{E}$. The smallest constant M in (2.2) is called the $\mathcal{R}$-*bound* of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$. A sectorial operator A is called $\mathcal{R}$-*sectorial* if the set

$$\{\lambda(\lambda+A)^{-1} : \ \lambda \in \Sigma_{\pi-\phi}\} \text{ is } \mathcal{R}\text{-bounded for some } \phi \in (0, \pi).$$

The infimum ϕ_A^R of such angles ϕ is called the $\mathcal{R}$-*angle* of A. We denote the class of $\mathcal{R}$-sectorial operators by $\mathcal{RS}(X)$. The relation $\phi_A^R \geq \phi_A$ is clear. If X is a space of class $\mathcal{HT}$ and $A \in \mathcal{H}^\infty(X)$ then it follows from a result of Clément and Prüss [3] that $A \in \mathcal{RS}(X)$ with $\phi_A^R \leq \phi_A^\infty$.

Finally, an operator $A \in \mathcal{H}^\infty(X)$ is said to admit an $\mathcal{R}$-*bounded* $\mathcal{H}^\infty$-calculus if the set

$$\{f(A) : \ f \in \mathcal{H}^\infty(\Sigma_\phi), |f|_{\mathcal{H}^\infty(\Sigma_\phi)} \leq 1\}$$

is $\mathcal{R}$-bounded for some $\phi \in (0, \pi)$. Again, the infimum $\phi_A^{R\infty}$ of such ϕ is called the $\mathcal{RH}^\infty$-angle of A, and the class of such operators is denoted by $\mathcal{RH}^\infty(X)$.

If X enjoys the so-called *property* (α), see [2, Definition 3.11], then every operator $A \in \mathcal{H}^\infty(X)$ already has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, that is, $\mathcal{H}^\infty(X) = \mathcal{RH}^\infty(X)$ and $\phi_A^{R\infty} = \phi_A^\infty$, see [12, Theorem 5.3]. In particular, the L_p-spaces with $1 < p < \infty$ have property (α), see [2].

We refer to the monograph of Denk, Hieber, and Prüss [6] for further information and background material.

We now state a recent result on the existence of an operator-valued $\mathcal{H}^\infty$-calculus proved in [12], the general Kalton-Weis theorem.

Theorem 2.1. *Let X be a Banach space, $A \in \mathcal{H}^\infty(X)$, $F \in \mathcal{H}^\infty(\Sigma_\phi; \mathcal{B}(X))$ such that*

$$F(\lambda)(\mu - A)^{-1} = (\mu - A)^{-1}F(\lambda), \quad \mu \in \rho(A), \ \lambda \in \Sigma_\phi.$$

Suppose $\phi > \phi_A^\infty$ and $\mathcal{R}(F(\Sigma_\phi)) < \infty$. Then

$$F(A) \in \mathcal{B}(X) \quad \textit{and} \quad |F(A)|_{\mathcal{B}(X)} \leq C_A \mathcal{R}(F(\Sigma_\phi))$$

where C_A denotes a constant depending on A but not on F.

Given two sectorial operators A and B we define

$$(A + B)x := Ax + Bx, \quad x \in \mathsf{D}(A + B) := \mathsf{D}(A) \cap \mathsf{D}(B).$$

A and B are said to *commute* if there are numbers $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ such that

$$(\lambda - A)^{-1}(\mu - B)^{-1} = (\mu - B)^{-1}(\lambda - A)^{-1}.$$

In this case, the commutativity relation holds for all $\lambda \in \rho(A)$ and $\mu \in \rho(B)$.

In their famous paper [5] Da Prato and Grisvard proved the following result: suppose $A, B \in \mathcal{S}(X)$ commute and the *parabolicity condition* $\phi_A + \phi_B < \pi$ holds true. Then $A + B$ is closable and its closure $L := \overline{A + B}$ is again sectorial with spectral angle $\phi_L \leq \max\{\phi_A, \phi_B\}$.

The natural question in this context then is whether $A+B$ is already closed, i.e., if *maximal regularity* holds. There are many contributions to this question, see for instance the discussion in Prüss and Simonett [17].

Applied to the functions $F(z) = z(z+B)^{-1}$ and $F(z) = f(z+B)$, Theorem 2.1 implies the following result on sums of commuting operators: suppose $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$, A, B commute, and $\phi_A^\infty + \phi_B^R < \pi$. Then $A + B$ is closed and sectorial. If in addition $B \in \mathcal{RH}^\infty(X)$ and $\phi_A^\infty + \phi_B^{R\infty} < \pi$, then $A + B$ has an $\mathcal{H}^\infty$-calculus as well.

We now turn to the non-commuting case and we assume that A and B satisfy the Labbas-Terreni commutator condition, which reads as follows.

$$\left\{\begin{array}{l} 0\in\rho(A).\text{ There are constants } c>0,\ \ 0\le\alpha<\beta<1,\\ \psi_A>\phi_A,\ \psi_B>\phi_B,\ \psi_A+\psi_B<\pi,\\ \text{ such that for all } \lambda\in\Sigma_{\pi-\psi_A},\ \mu\in\Sigma_{\pi-\psi_B}\\ |A(\lambda+A)^{-1}[A^{-1}(\mu+B)^{-1}-(\mu+B)^{-1}A^{-1}]|\le c/(1+|\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{array}\right. \tag{2.3}$$

Assuming this condition we have the following generalization of the Kalton-Weis theorem on sums of operators to the non-commuting case proved recently by the authors in [17].

Theorem 2.2. *Suppose $A\in\mathcal{H}^\infty(X)$, $B\in\mathcal{RS}(X)$ and suppose that (2.3) holds for some angles $\psi_A>\phi_A^\infty$, $\psi_B>\phi_B^R$ with $\psi_A+\psi_B<\pi$.*
Then there is a number $\omega_0\ge 0$ such that ω_0+A+B is invertible and sectorial with angle $\phi_{\omega_0+A+B}\le\max\{\psi_A,\psi_B\}$. Moreover, if in addition $B\in\mathcal{RH}^\infty(X)$ and $\psi_B>\phi_B^{R\infty}$, then

$$\omega_0+A+B\in\mathcal{H}^\infty(X)\quad\textit{and}\quad \phi^\infty_{\omega_0+A+B}\le\max\{\psi_A,\psi_B\}.$$

This result will be one of the main tools for the theory developed below.

If A and B are sectorial operators on X, their product is defined by

$$(AB)x:=ABx,\quad \mathsf{D}(AB):=\{x\in\mathsf{D}(B):\ Bx\in\mathsf{D}(A)\}.$$

Closedness of the product is easier to obtain than for sums, as AB is closed as soon as A is invertible or B is bounded. On the other hand, it is in general not a simple task to prove that AB is again sectorial. The following assertive result is a consequence of the general Kalton-Weis theorem: suppose that A and B are sectorial commuting operators in X, that A is invertible, and suppose that $A\in\mathcal{H}^\infty(X)$, $B\in\mathcal{RS}(X)$ such that $\phi_A^\infty+\phi_B^R<\pi$. Then AB is sectorial with angle $\phi_{AB}\le\phi_A^\infty+\phi_B^R$. If moreover $B\in\mathcal{RH}^\infty(X)$ and $\phi_A^\infty+\phi_B^{R\infty}<\pi$, then

$$AB\in\mathcal{H}^\infty(X)\quad\textit{and}\quad \phi^\infty_{AB}\le\phi^\infty_A+\phi_B^{R\infty}.$$

This result was recently extended in [11] to the case of non-commuting operators.

3. Parametric second-order symbols

In this section we consider the case where h is a second-order polynomial, that is, we consider the parametric problem

$$\mu u+\lambda e^{2sx}u-(\partial_x+\beta)^2u=f,\quad x\in\mathbb{R}, \tag{3.1}$$

where $\lambda\in\Sigma_\pi$, $\mu\in\mathbb{C}$, $\beta\in\mathbb{R}$, $s\in\mathbb{R}\setminus\{0\}$ are fixed parameters. In this case we have

$$h(i\xi)=-(i\xi+\beta)^2=\xi^2-\beta^2-2i\beta\xi,\quad \xi\in\mathbb{R}.$$

The spectrum $\sigma(h(\partial_x))$ of the operator $h(\partial_x)$ in $L_p(\mathbb{R})$ is given by

$$\sigma(h(\partial_x))=P_\beta:=h(i\mathbb{R})=\{\xi^2-\beta^2-2i\beta\xi:\xi\in\mathbb{R}\},$$

a parabola with vertex in $-\beta^2$ opening to the right symmetric with respect to the real axis. We are going to derive an explicit characterization of the spectrum of $\lambda e^{2sx} + h(\partial_x)$ as well as an integral representation of the solutions of (3.1). Note that the change of variables $y = -x$ transforms (3.1) into itself, when replacing s by $-s$ and β by $-\beta$. Therefore we assume $s > 0$ in the sequel.

We introduce the variable transformation

$$u(x) = e^{-\beta x} w(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R}.$$

Setting $z = \sqrt{\lambda} e^{sx}/s$ and $\nu^2 = \mu/s^2$, problem (3.1) transforms to

$$(z^2 + \nu^2) w(z) - z^2 w''(z) - z w'(z) = g(z), \quad \operatorname{Re} z > 0, \tag{3.2}$$

where

$$g(z) = z^{\beta/s} s^{\beta/s-2} \lambda^{-\beta/2s} f(s^{-1} \ln(sz/\sqrt{\lambda})).$$

(3.2) is nothing but the modified Bessel equation with parameter ν. A fundamental system of this equation is given by the modified Bessel function $I_\nu(z)$ and the McDonald function $K_\nu(z)$. The general solution of (3.2) is given by

$$w(z) = aI_\nu(z) + bK_\nu(z) + K_\nu(z) \int_0^z I_\nu(\sigma) g(\sigma) d\sigma/\sigma + I_\nu(z) \int_z^\infty K_\nu(\sigma) g(\sigma) d\sigma/\sigma, \tag{3.3}$$

where so far a, b are arbitrary numbers.

We are going to compute the spectrum of $A_{\lambda,\beta} = \lambda e^{2sx} - (\partial_x + \beta)^2$. Let us first look for eigenfunctions of $A_{\lambda,\beta}$, i.e., for the point spectrum $\sigma_p(A_{\lambda,\beta})$. So we have from (3.3) that $-\mu \in \sigma_p(A_{\lambda,\beta})$ if and only if a function of the form

$$u(x) = e^{-\beta x}[aI_\nu(\sqrt{\lambda} e^{sx}/s) + bK_\nu(\sqrt{\lambda} e^{sx}/s)], \quad x \in \mathbb{R},$$

belongs to $L_p(\mathbb{R})$ and $e^{2sx}u$ does so as well, since then we also have $u \in H_p^2(\mathbb{R})$ since $\omega + h(\partial_x) \in \mathcal{B}(H_p^2(\mathbb{R}), L_p(\mathbb{R}))$ is an isomorphism for $\omega > \beta^2$. In the sequel we are employing the asymptotics for $I_\nu(z)$ and $K_\nu(z)$ near zero and infinity which, for example, may be found in Abramowitz and Stegun [1]. For $Re\, \nu > 0$ these read

$$I_\nu(z) \sim e^z/\sqrt{2\pi z}, \quad K_\nu(z) \sim e^{-z}\sqrt{\pi/2z}, \quad \operatorname{Re} z > 0,\ |z| \to \infty, \tag{3.4}$$

and

$$I_\nu(z) \sim 2^{-\nu} z^\nu/\Gamma(\nu+1), \quad K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu) z^{-\nu}, \quad \operatorname{Re} z > 0,\ z \to 0. \tag{3.5}$$

The estimates are uniform for $|\arg(z)| \le \theta < \pi/2$. From these asymptotics we may conclude $a = 0$, and set $b = 1$. Since $K_\nu(z)$ is exponentially decaying at infinity,

$$\int_0^\infty e^{p\gamma x} |u(x)|^p dx = c \int_1^\infty t^{(\gamma-\beta)p/s} |K_\nu(e^{i\varphi} t)|^p dt/t < \infty,$$

for each $p \in (1, \infty)$, $\beta, \gamma \in \mathbb{R}$ and $s > 0$ since $|\varphi| = |\arg \lambda/2| < \pi/2$. On the other hand,

$$\int_{-\infty}^0 e^{p\gamma x} |u(x)|^p dx = c \int_0^1 t^{(\gamma-\beta)p/s} |K_\nu(e^{i\varphi} t)|^p dt/t$$

is finite if and only if

$$\mathrm{Re}\,\nu p + (\beta - \gamma)p/s < 0, \quad \text{that is, iff} \quad \mathrm{Re}\,\sqrt{\mu} < \gamma - \beta.$$

Choosing $\gamma = 0$ and $\gamma = 2s$ we see that the function

$$u_\mu(x) = e^{-\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R},$$

is an eigenfunction corresponding to the eigenvalue $-\mu$ of $A_{\lambda,\beta}$ if and only if $\mathrm{Re}\,\sqrt{\mu} < -\beta$. Note that this property is independent of $\lambda \in \Sigma_\pi$, $s > 0$ and $p \in (1, \infty)$.

Next observe that the dual operator $A^*_{\lambda,\beta}$ of $A_{\lambda,\beta}$ in $L_p(\mathbb{R})^* = L_{p'}(\mathbb{R})$ is precisely $A^*_{\lambda,\beta} = A_{\lambda,-\beta}$. Therefore we see from the above argument that $-\mu$ is an eigenvalue of $A^*_{\lambda,\beta}$ if and only if $\mathrm{Re}\,\sqrt{\mu} < \beta$. A dual eigenfunction is

$$u^*_\mu(x) = e^{\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad x \in \mathbb{R}.$$

Thus the set $\mathrm{Re}\,\sqrt{-\mu} \le |\beta|$ belongs to the spectrum of $A_{\lambda,\beta}$ for all $s > 0$, $1 < p < \infty$, and $\lambda \in \Sigma_\pi$, as long as $\beta \ne 0$. For $\beta = 0$ the point spectra of $A_{\lambda,\beta}$ and of $A^*_{\lambda,\beta}$ are both empty. We still claim that the set $\mathrm{Re}\,\sqrt{-\mu} = 0$, i.e., the set $\mu \in \mathbb{R}_+$, belongs to the spectrum of $A_{\lambda,\beta}$. In fact these values of $-\mu$ are approximate eigenvalues. Suppose on the contrary that $\mu + A_{\lambda,0}$ is invertible. Then $\mu + A_{\lambda,\beta}$ must be invertible for small $|\beta|$ as well, since $A_{\lambda,\beta}$ is a small perturbation of $A_{\lambda,0}$.

It is easy to identify the set $\mathrm{Re}\,\sqrt{-\mu} \le |\beta|$. It is the filled up parabola $\mathbb{P}_\beta$, defined by

$$\mathbb{P}_\beta = \{\xi^2 - \beta^2 + \sigma - 2i\beta\xi : \ \xi \in \mathbb{R},\ \sigma \ge 0\}.$$

The main result of this section is the following theorem which states that $\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta$, for all $p \in (1, \infty)$, $\lambda \in \Sigma_\pi$, $s > 0$, and $\beta \in \mathbb{R}$.

This is truly a remarkable result since the spectrum of $\lambda - (\partial_x + \beta)^2$ is the shifted parabola $\lambda + P_\beta$ instead!

Theorem 3.1. *Let $\lambda \in \Sigma_\pi$, $\beta \in \mathbb{R}$, $s > 0$, $1 < p < \infty$, and let $A_{\lambda,\beta}$ in $L_p(\mathbb{R})$ be defined according to*

$$A_{\lambda,\beta} = \lambda e^{2sx} - (\partial_x + \beta)^2,$$

with domain $\mathsf{D}(A_{\lambda,\beta}) = H^2_p(\mathbb{R}) \cap L_p(\mathbb{R}; e^{2sxp}dx)$. Then we have

$$\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta = \{\xi^2 - \beta^2 + \sigma - 2i\beta\xi : \ \xi \in \mathbb{R},\ \sigma \ge 0\}.$$

If $\beta < 0$ then $\sigma_p(A_{\lambda,\beta}) = \mathring{\mathbb{P}}_\beta$ and $\sigma_r(A_{\lambda,\beta}) = \emptyset$, while for $\beta > 0$ we have $\sigma_r(A_{\lambda,\beta}) = \mathring{\mathbb{P}}_\beta$ and $\sigma_p(A_{\lambda,\beta}) = \emptyset$. The resolvent of $A_{\lambda,\beta}$ has the integral representation

$$(\mu + A_{\lambda,\beta})^{-1} f(x) = \int_{\mathbb{R}} k_{\lambda,\beta,\mu}(x, y) f(y)dy, \quad x \in \mathbb{R},\ f \in L_p(\mathbb{R}),$$

where the kernel $k_{\lambda,\beta,\mu}$ is given by

$$k_{\lambda,\beta,\mu}(x, y) = s^{-1} e^{-\beta(x-y)} \begin{cases} K_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sx}/s) I_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sy}/s) & \text{for} \quad x > y \\ I_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sx}/s) K_{\sqrt{\mu}/s}(\sqrt{\lambda} e^{sy}/s) & \text{for} \quad x < y. \end{cases}$$

Finally, for every fixed $\phi \in (0,\pi)$ *and* $\mu \notin -\mathbb{P}_\beta$ *there exists a positive constant* M *such that*

$$|(\mu + A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p)} + |\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p)} \le M, \qquad \lambda \in \Sigma_\phi.$$

Proof. We first derive estimates for the kernel $k_{\lambda,\beta,\mu}$ for fixed $\mu \notin -\mathbb{P}_\beta$. From the asymptotics of I_ν and K_ν given in (3.4) and (3.5) we obtain for each $\theta < \pi/2$ a constant $c_{\nu,\theta} > 0$ such that

$$|I_\nu(z)| \le c_{\nu,\theta} \frac{|z|^{\mathrm{Re}\,\nu} e^{\mathrm{Re}\,z}}{(1+|z|)^{\mathrm{Re}\,\nu+1/2}}, \quad |\arg(z)| \le \theta,$$

and

$$|K_\nu(z)| \le c_{\nu,\theta} \frac{|z|^{-\mathrm{Re}\,\nu} e^{-\mathrm{Re}\,z}}{(1+|z|)^{-\mathrm{Re}\,\nu+1/2}}, \quad |\arg(z)| \le \theta.$$

With $\nu^2 = \mu/s^2$ these estimates yield for $x > y$

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(\beta+s\mathrm{Re}\,\nu)(x-y)}}{1+|\sqrt{\lambda}|e^{sx}/s} \cdot \Big(\frac{1+|\sqrt{\lambda}|e^{sx}/s}{1+|\sqrt{\lambda}|e^{sy}/s}\Big)^{\mathrm{Re}\,\nu+1/2} e^{-s^{-1}\mathrm{Re}\,\sqrt{\lambda}(e^{sx}-e^{sy})}.$$

The elementary inequality

$$\Big(\frac{1+a}{1+b}\Big)^\gamma \le c e^{\varepsilon(a-b)},$$

with $c > 0$ depending on $\varepsilon > 0$ and $\gamma > 0$ but not on $a > b > 0$, and the relation $\mathrm{Re}\,\sqrt{\lambda} \ge c'|\sqrt{\lambda}|$, valid for $|\arg\lambda| \le \phi < \pi$, then imply

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(\beta+s\mathrm{Re}\,\nu)(x-y)}}{1+|\sqrt{\lambda}|e^{sx}} \cdot e^{-c_1|\sqrt{\lambda}|(e^{sx}-e^{sy})}, \quad x > y,$$

with some constants $C = C(\nu,\phi) > 0$ and $c_1 = c_1(\phi) > 0$. In a similar way we obtain for $x < y$

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(-\beta+s\mathrm{Re}\,\nu)(y-x)}}{1+|\sqrt{\lambda}|e^{sy}} \cdot e^{-c_1|\sqrt{\lambda}|(e^{sy}-e^{sx})}, \quad x < y.$$

Combining these estimates leads to

$$|k_{\lambda,\beta,\mu}(x,y)| \le C \frac{e^{-(s\mathrm{Re}\,\nu-|\beta|)|x-y|}}{1+|\sqrt{\lambda}|e^{s\max\{x,y\}}} \cdot e^{-c_1|\sqrt{\lambda}||e^{sx}-e^{sy}|}, \quad x, y \in \mathbb{R}.$$

This shows $|k_{\lambda,\beta,\mu}(x,y)| \le C\kappa(x-y)$ with $\kappa(x) = e^{-(s\mathrm{Re}\,\nu-|\beta|)|x|}$, that is, the kernel $k_{\lambda,\beta,\mu}$ is dominated by a convolution kernel. Convolution with κ is L_p-bounded if $\kappa \in L_1(\mathbb{R})$, which is precisely the condition $\mathrm{Re}\,\nu s > |\beta|$, which in turn is equivalent to $\mu \notin -\mathbb{P}_\beta$. This shows that the resolvent of $A_{\lambda,\beta}$ is well-defined and L_p-bounded for all $\mu \notin \mathbb{P}_\beta$.

It remains to estimate $\lambda e^{2sx} k_{\lambda,\beta,\mu}$ to conclude $\sigma(A_{\lambda,\beta}) = \mathbb{P}_\beta$. Let

$$\gamma := \mathrm{Re}\,\nu s - |\beta| > 0 \quad \text{and} \quad \alpha := c_1|\sqrt{\lambda}| > 0.$$

Then we have

$$|\lambda e^{2sx} k_{\lambda,\beta,\mu}(x,y)| \le C e^{-\gamma|x-y|}[1 + \alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|}].$$

This is clear in case $x < y$ and for $x > y$ it follows from

$$\alpha e^{sx} e^{-\alpha(e^{sx}-e^{sy})} = \alpha e^{sy} e^{-\alpha(e^{sx}-e^{sy})} + \alpha(e^{sx} - e^{sy}) e^{-\alpha(e^{sx}-e^{sy})} \leq \alpha e^{sy} e^{-\alpha(e^{sx}-e^{sy})} + 1.$$

Thus we obtain

$$|\lambda e^{2sx} k_{\lambda,\beta,\mu}(x,y)| \leq Ce^{-\gamma|x-y|} + C\alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|}, \quad x,y \in \mathbb{R}.$$

The first kernel on the right yields convolution with an L_1-function which is L_p-bounded, as before. The second one is symmetric, and we have

$$\int_{\mathbb{R}} s\alpha e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|} dy \leq e^{-\alpha e^{sx}} \int_{-\infty}^{x} s\alpha e^{sy} e^{\alpha e^{sy}} dy + e^{\alpha e^{sx}} \int_{x}^{\infty} s\alpha e^{sy} e^{-\alpha e^{sy}} dy \leq 2.$$

This shows that the second kernel defines an integral operator which is simultaneously bounded in $L_1(\mathbb{R})$ and in $L_\infty(\mathbb{R})$, with bound independent of α. By the Riesz-Thorin interpolation theorem it is also bounded in $L_p(\mathbb{R})$, uniformly in $\alpha > 0$, i.e., in $\lambda \in \Sigma_\phi$ with $\phi < \pi$.

These arguments show that

$$(\mu + A_{\lambda,\beta})^{-1} L_p(\mathbb{R}) \subset L_p(\mathbb{R}; (1 + e^{2sxp})dx).$$

Since we know that the spectrum of $-(\partial_x + \beta)^2$ is the parabola P_β, we may conclude that $(\mu + A_{\lambda,\beta})^{-1} L_p(\mathbb{R}) \subset H_p^2(\mathbb{R})$, again with bound uniform in $\lambda \in \Sigma_\phi$. This concludes the proof of the theorem. $\square$

One should observe that our estimates are not uniform in μ, but they are of course so when μ is bounded. This then implies that near a point $\mu_0 \in P_\beta$ we have an estimate of the form

$$|(\mu - A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p(\mathbb{R}))} \leq C/\mathrm{dist}(\mu, P_\beta), \quad \mu \notin \mathbb{P}_\beta,$$

which is optimal besides the constant C. However, we do not know whether a resolvent estimate of the form

$$|\mu(\mu - A_{\lambda,\beta})^{-1}|_{\mathcal{B}(L_p(\mathbb{R})} \leq C_\rho, \quad \mathrm{dist}(\mu, \mathbb{P}_\beta) \geq \rho,$$

for $\rho > 0$, uniformly in $\lambda \in \Sigma_\phi$, is valid. We are able to show this only for $\lambda > 0$.

4. Operator-valued symbols of second order

Having established uniform bounds for the operator families

$$\mathcal{K}_0 = \{(\mu + A_{\lambda,\beta})^{-1} : \lambda \in \Sigma_\phi\}, \quad \mathcal{K}_1 = \{\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1} : \lambda \in \Sigma_\phi\} \subset \mathcal{B}(L_p(\mathbb{R}))$$

for every fixed $\phi \in (0, \pi)$, let us consider $\mathcal{R}$-boundedness of $\mathcal{K}_0$ and $\mathcal{K}_1$. Suppose we have established this property. Then holomorphy in λ permits to employ Theorem 2.1. We may replace λ by any sectorial operator D with bounded $\mathcal{H}^\infty$-calculus and angle $\phi_D^\infty < \pi$ which commutes with e^{2sx} and ∂_x. In particular, we are allowed to plug in $D = \partial_t$, or $D = \partial_t^\alpha$ for $\alpha \in (0, 2)$ or, more generally, an anomalous diffusion operator $D = \partial_t^\alpha - \Delta_y$. The result then is that for all these different operators the spectrum of $L = De^{2sx} - (\partial_x + \beta)^2$ is contained in the parabola $\mathbb{P}_\beta$.

To prove $\mathcal{R}$-boundedness of $\mathcal{K}_0$ and $\mathcal{K}_1$ we use the domination theorem for kernel operators in $L_p(\mathbb{R})$ which says the following: if each of the kernels $k_{\lambda,\beta,\mu}$ for $\lambda \in \Sigma_\phi$, $\beta \in \mathbb{R}$ and $\mu \notin -\mathbb{P}_\beta$ fixed is pointwise bounded by a kernel κ_α and the set of kernel operators K_α, for α belonging to some index set, is $\mathcal{R}$-bounded then the family $\mathcal{K} = \{K_\alpha\}$ is so as well, see [6] for instance. Recall also that $\mathcal{R}$-boundedness is additive.

In our situation we have bounds for $\mathcal{K}_1$ in terms of a single convolution kernel $e^{-\gamma|x|}$, for some $\gamma > 0$, plus the family $\mathcal{L} := \{\alpha s e^{s\min\{x,y\}} e^{-\alpha|e^{sx}-e^{sy}|} :\ \alpha > 0\}$. Therefore, it is enough to show $\mathcal{R}$-boundedness in $L_p(\mathbb{R})$ of the family of kernel operators $\mathcal{L}$. To prove this the following lemma will be useful. We first introduce some more notation. Given any function $f \in L_{1,loc}(\mathbb{R})$, the maximal function Mf of f is defined by

$$(Mf)(x) := \sup_{a>0} \frac{1}{2a} \int_{x-a}^{x+a} |f(s)|\, ds, \quad x \in \mathbb{R}.$$

The result now reads as

Lemma 4.1.

(a) *Suppose $\phi \in C^2(\mathbb{R})$ is positive, nondecreasing and convex with $x\phi'(x) \to 0$ as $x \to -\infty$. Then the integral operator T_ϕ defined on $\mathcal{D}(\mathbb{R})$ by*

$$T_\phi f(x) = \phi(x)^{-1} \int_{-\infty}^{x} \phi'(y) f(y) dy, \quad x \in \mathbb{R},$$

satisfies $|T_\phi f(x)| \leq 2(Mf)(x)$ for all $x \in \mathbb{R}$.

(b) *Suppose $\psi \in C^2(\mathbb{R})$ is positive, nonincreasing and $\ln\psi$ is concave with $x\psi(x) \to 0$ as $x \to \infty$. Then the integral operator S_ψ defined on $\mathcal{D}(\mathbb{R})$ by*

$$S_\psi f(x) = \frac{-\psi'(x)}{\psi(x)^2} \int_x^\infty \psi(y) f(y) dy, \quad x \in \mathbb{R},$$

satisfies $|S_\psi f(x)| \leq 2(Mf)(x)$ for all $x \in \mathbb{R}$.

Proof. (a) Let $f \in \mathcal{D}(\mathbb{R})$ be given. Then we have

$$\begin{aligned} |T_\phi f(x)| &\leq \phi(x)^{-1} \int_{-\infty}^x \phi'(y)|f(y)|dy = \phi(x)^{-1} \int_{-\infty}^x \phi''(y) \int_y^x |f(s)| ds dy \\ &\leq 2(Mf)(x)\phi(x)^{-1} \int_{-\infty}^x \phi''(y)(x-y)dy \\ &= 2(Mf)(x)\phi(x)^{-1} \int_{-\infty}^x \phi'(y)dy \leq 2(Mf)(x). \end{aligned}$$

These estimates can be justified by first integrating over a finite interval and then letting the lower limit go to $-\infty$. It should be noted that $\int_{-\infty}^x \phi'(y)\, dy \leq \phi(x)$ since ϕ is positive and nondecreasing.

(b) Similarly we obtain

$$\begin{aligned}
|S_\psi f(x)| &\le \frac{-\psi'(x)}{\psi(x)^2}\int_x^\infty \psi(y)|f(y)|dy = \frac{\psi'(x)}{\psi(x)^2}\int_x^\infty \psi'(y)\int_x^y |f(s)|dsdy\\
&\le 2(Mf)(x)\frac{\psi'(x)}{\psi(x)^2}\int_x^\infty \psi'(y)(y-x)dy\\
&= 2(Mf)(x)\frac{-\psi'(x)}{\psi(x)^2}\int_x^\infty \psi(y)dy\\
&\le 2(Mf)(x)\psi(x)^{-1}\int_x^\infty -\psi'(y)dy = 2(Mf)(x),
\end{aligned}$$

where in the last inequality we used that $-\psi'(x)/\psi(x) = -(d/dx)\ln\psi(x)$ is increasing, since $\ln\psi(x)$ is concave by assumption. □

Now we can prove the following result.

Proposition 4.2. *Let $\beta\in\mathbb{R}$, $s>0$, $1<p<\infty$, and suppose $\mu\notin -\mathbb{P}_\beta$. Then the operator families*

$$\mathcal{K}_0 = \{(\mu + A_{\lambda,\beta})^{-1} :\ \lambda\in\Sigma_\phi\}\subset\mathcal{B}(L_p(\mathbb{R}))$$

and

$$\mathcal{K}_1 = \{\lambda e^{2sx}(\mu + A_{\lambda,\beta})^{-1} :\ \lambda\in\Sigma_\phi\}\subset\mathcal{B}(L_p(\mathbb{R}))$$

are $\mathcal{R}$-bounded for every fixed $\phi\in(0,\pi)$.

Proof. The proof uses the characterization of $\mathcal{R}$-boundedness of integral operators in L_p-spaces by means of square function estimates and proceeds like the proof of Theorem 4.8 in Denk, Hieber and Prüss [6]; see also Clément and Prüss [3] where this argument appears for the first time.

Take any functions $f_i\in L_p(\mathbb{R})$ and any numbers $\lambda_i\in\Sigma_\phi$ and let K_i denote the integral operator with kernel $\lambda_i e^{2sx}k_{\lambda_i,\beta,\mu}(x,y)$. Then with $\phi_i(x) = e^{\alpha_i e^{sx}}$, $\psi_i(x) = e^{-\alpha_i e^{sx}}$, $\alpha_i = c_1|\sqrt{\lambda_i}|>0$, Lemma 4.1 yields

$$|K_i f_i(x)| \le C(Mf_i)(x), \qquad \text{for a.a. } x\in\mathbb{R},$$

hence

$$\sum_i |K_i f_i(x)|^2 \le C\sum_i |Mf_i(x)|^2, \qquad \text{for a.a. } x\in\mathbb{R},$$

where $C>0$ denotes a constant that is independent of f_i and λ_i. Since the maximal operator M is bounded in $L_p(\mathbb{R};l_2)$ for $1<p<\infty$, see [19, Theorem II.1.1], we may continue

$$|(\sum_i |K_i f_i|^2)^{1/2}|_{L_p(\mathbb{R})} \le C|(\sum_i |Mf_i|^2)^{1/2}|_{L_p(\mathbb{R})} \le C|(\sum_i |f_i|^2)^{1/2}|_{L_p(\mathbb{R})}.$$

This implies that the family $\mathcal{K}_1$ satisfies a square function estimate, and it follows from [6, Remark 3.2(4)] that $\mathcal{K}_1$ is $\mathcal{R}$-bounded. $\mathcal{K}_0$ is $\mathcal{R}$-bounded since it is dominated by a single L_p-bounded kernel operator, see [6, Proposition 3.5] for instance. □

We can now state the following result, alluded to in the introduction to this section.

Theorem 4.3. *Let $\beta \in \mathbb{R}$, $s > 0$, $1<p<\infty$, and let D denote an invertible sectorial operator on a Banach space X with bounded $\mathcal{H}^\infty$-calculus and angle $\phi_D^\infty < \pi$. Let L on $L_p(\mathbb{R}; X)$ be defined by means of*

$$Lu = De^{2sx} - (\partial_x + \beta)^2,$$

with natural domain

$$\mathsf{D}(L) = \{u \in H_p^2(\mathbb{R}; X) : \; e^{2sx}u \in L_p(\mathbb{R}; \mathsf{D}(D))\}.$$

Then $\sigma(L) \subset \mathbb{P}_\beta$, and for any $\omega > \beta^2$, $\omega + L$ is invertible, sectorial, and admits a bounded $\mathcal{H}^\infty$-calculus in $L_p(\mathbb{R}; X)$.

Proof. We may conclude from Theorem 3.1 and Proposition 4.2 that $\sigma(L) \subset \mathbb{P}_\beta$, see the arguments given at the beginning of the current section. Consequently

$$\mu + L \quad \text{is invertible for every } \mu \notin -\mathbb{P}_\beta. \tag{4.1}$$

We now consider the operator A given by

$$A := De^{2sx}, \quad \mathsf{D}(A) := \{u \in L_p(\mathbb{R}, X) : e^{2sx}u \in L_p(\mathbb{R}, \mathsf{D}(D)\}.$$

A is the product of D with the multiplication operator $M := e^{2sx}$, which has an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi_M^{R\infty} = 0$ on $L_p(\mathbb{R})$, and hence also on $L_p(\mathbb{R}, X)$. It follows from the remarks at the end of Section 2 that $A \in \mathcal{H}^\infty(\mathbb{R}, X)$ with $\phi_A^\infty \leq \phi_D^\infty$. Next we consider the operator B, defined by

$$B := \delta_0 - (\partial_x - \beta)^2, \quad \mathsf{D}(B) = H_p^2(\mathbb{R}),$$

where $\delta_0 > \beta^2$. It follows from Mikhlin's theorem that B is sectorial and admits an $\mathcal{H}^\infty$-calculus on $L_p(\mathbb{R})$. Moreover, we can also conclude that $\delta_0 > \beta^2$ can be chosen in such a way that $\phi_D^\infty + \phi_B^\infty < \pi$. Since $L_p(\mathbb{R})$ has property α we obtain, in addition, that $\phi_D^\infty + \phi_B^{R\infty} < \pi$. This result also holds for the canonical extension of B to $L_p(\mathbb{R}, X)$. The same arguments as in [17, Section 5] now show that there exists a number $\omega_0 \geq 0$ such that

$$\omega_0 + A + B \quad \text{is sectorial and admits an } \mathcal{H}^\infty\text{-calculus on } L_p(\mathbb{R}, X). \tag{4.2}$$

The remaining assertions of Theorem 4.3 are now a consequence of (4.1)–(4.2) and [6, Proposition 2.7]. □

Remarks 4.4. (a) Suppose $D = \partial_t$, with domain $\mathsf{D}(\partial_t) = \{u \in W_p^1(J) : u(0) = 0\}$. Then it can be shown that, in fact, $\sigma(L) = \mathbb{P}_\beta$. This follows from the property that every number $-\mu \in \mathring{\mathbb{P}}_\beta$ is an eigenvalue of either $A_{\lambda,\beta}$ or its dual, with eigenfunction

$$u_\mu(\lambda, x) = e^{\pm\beta x} K_\nu(\sqrt{\lambda} e^{sx}/s), \quad \lambda \in \Sigma_\pi, \; x \in \mathbb{R},$$

see Section 3. Taking the inverse Laplace transform with respect to λ over an appropriate contour will provide an eigenfunction of L, or its dual, with eigenvalue $-\mu$.

(b) A correspondig result to Theorem 4.3 can also be stated for operators of the form $D + (\omega - (\partial_x + \beta)^2)e^{-2sx}$, but we leave this to the interested reader.

5. Parabolic equations with dynamic boundary conditions on wedges and angles

In this section we consider an application of our main results to the diffusion equation on a domain of wedge or angle type, that is, on the domain $G = \mathbb{R}^m \times C_\alpha$, where $m \in \mathbb{N}_0$, and for $\alpha \in (0, 2\pi)$, C_α denotes the angle

$$C_\alpha = \{x = (r\cos\phi, r\sin\phi) : \ r > 0, \ \phi \in (0, \alpha)\}.$$

The boundary $\Gamma = \partial G$ then consists of two faces

$$\Gamma_0 = \{(y, r, 0); \ y \in \mathbb{R}^m, \ r > 0\}, \quad \Gamma_\alpha = \{(y, r\cos\alpha, r\sin\alpha) : \ y \in \mathbb{R}^m, \ r > 0\}.$$

We consider the problem

$$\left\{\begin{aligned} \partial_t u - \Delta u &= f_1 && \text{in } G \times (0,T)\\ u &= f_2 && \text{on } \Gamma_\alpha \times (0,T)\\ \partial_t u + \partial_\nu u &= f_3 && \text{on } \Gamma_0 \times (0,T)\\ u|_{t=0} &= u_0 && \text{on } G. \end{aligned}\right. \tag{5.1}$$

Here $m \in \mathbb{N}_0$ and ν denotes the outer normal for G at Γ. The function f_1 is given in a weighted L_p-space, i.e.,

$$f_1 \in L_p(J \times \mathbb{R}^m; L_p(C_\alpha; |x|^\gamma dx)),$$

where $\gamma \in \mathbb{R}$ will be chosen appropriately, and $J = (0, T)$. The functions f_2 and f_3 are supposed to belong to certain trace spaces.
It is natural to introduce polar coordinates in the x-variables, $x = (r\cos\phi, r\sin\phi)$ where $\phi \in (0, \alpha)$ and $r > 0$. Then the diffusion operator $\partial_t - \Delta$ transforms into

$$\partial_t - \Delta_y - [\partial_r^2 + \frac{1}{r}\partial_r] - \frac{\partial_\phi^2}{r^2},$$

where y denotes the variable in $\mathbb{R}^m$, and Δ_y is the Laplacian in the y-variables. The underlying space for the function f_1 now is

$$f_1 \in L_p(J \times \mathbb{R}^m \times (0,\alpha); L_p(\mathbb{R}_+; r^{\gamma+1} dr)).$$

It is also natural to employ the Euler transformation $r = e^x$ where now $x \in \mathbb{R}$. Setting

$$u(t, y, \phi, r) = r^\beta v(t, y, \phi, \ln r),$$

we arrive at the following problem for the unknown function v.

$$\left\{\begin{aligned} e^{2x}(\partial_t - \Delta_y)v + P(\partial_x)v - \partial_\phi^2 v &= g_1 && \text{in } (0,T) \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R}\\ v &= g_2 && \text{on } (0,T) \times \mathbb{R}^m \times \{\alpha\} \times \mathbb{R}\\ e^x \partial_t v - \partial_\phi v &= g_3 && \text{on } (0,T) \times \mathbb{R}^m \times \{0\} \times \mathbb{R}\\ v|_{t=0} &= v_0 && \text{on } \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \end{aligned}\right. \tag{5.2}$$

where $g_1(t,y,\phi,x) = e^{(2-\beta)x} f_1(t,y,\phi,e^x)$ and

$$g_2(t,y,x) = e^{-\beta x} f_2(t,y,e^x), \quad g_3(t,y,x) = e^{(1-\beta)x} f_3(t,y,e^x).$$

The differential operator $P(\partial_x)$ is given by the polynomial $P(z) = -(z+\beta)^2$, as a simple computation shows. The resulting equations are now defined in a smooth domain, but they contain the (non-standard) differential operators $e^{sx}\partial_t$, $s = 1,2$, and $e^{2x}\Delta_y$. We observe that these operators do not commute with $P(\partial_x)$.

Next we note that

$$\int_{\mathbb{R}} |g_1(t,y,\phi,x)|^p dx = \int_0^\infty |r^{2-\beta} f_1(t,y,\phi,r)|^p dr/r < \infty,$$

in case we choose $p(2-\beta) = \gamma + 2$, that is, $\beta = 2 - (\gamma+2)/p$. Making this choice of β, we can remove the weight and work in the unweighted base space

$$X := L_p(J \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R}).$$

We want to extract the boundary symbol for problem (5.2). For this purpose we define an operator A_β in $X = L_p(J \times \mathbb{R}^m \times \mathbb{R})$, $J = (0,T)$, by means of

$$(A_\beta u)(t,y,x) = ((\partial_t - \Delta_y)e^{2x} - (\partial_x + \beta)^2)u, \quad (t,y,x) \in J \times \mathbb{R}^m \times \mathbb{R},$$

with domain

$$\begin{aligned}\mathsf{D}(A_\beta) \;=\; & L_p(J \times \mathbb{R}^m; H_p^2(\mathbb{R})) \cap {}_0H_p^1(J; L_p(\mathbb{R}^m; L_p(\mathbb{R}; e^{2xp}dx))) \\ & \cap H_p^2(\mathbb{R}^m; L_p(J; L_p(\mathbb{R}; e^{2xp}dx))).\end{aligned}$$

Then the solution of the homogeneous problem

$$\left\{\begin{aligned} e^{2x}(\partial_t - \Delta_y)v + P(\partial_x)v - \partial_\phi^2 v = 0 &\quad \text{in } (0,T) \times \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \\ v = 0 &\quad \text{on } (0,T) \times \mathbb{R}^m \times \{\alpha\} \times \mathbb{R} \\ v = \rho &\quad \text{on } (0,T) \times \mathbb{R}^m \times \{0\} \times \mathbb{R} \\ v|_{t=0} = 0 &\quad \text{on } \mathbb{R}^m \times (0,\alpha) \times \mathbb{R} \end{aligned}\right.$$

with Dirichlet datum $v = \rho$ on $J \times \mathbb{R}^m \times \{0\} \times \mathbb{R}$ is given by

$$v(\phi) = \varphi(A_\beta,\phi)\rho, \quad \varphi(z,\phi) = \sinh((\alpha-\phi)\sqrt{z})/\sinh(\alpha\sqrt{z}), \quad \phi \in (0,\alpha).$$

Evaluating the normal derivative and inserting into the dynamic boundary condition yields

$$e^x\partial_t\rho + \psi(A_\beta)\rho = g, \quad \rho|_{t=0} = 0, \quad (t,y,x) \in J \times \mathbb{R}^m \times \mathbb{R} \tag{5.3}$$

where $\psi(z) = \sqrt{z}\coth(\alpha\sqrt{z})$.

We want to study the boundary symbol $e^x\partial_t + \psi(A_\beta)$ in the base space $X := L_p(J \times \mathbb{R}^m \times \mathbb{R})$. For this purpose we note that according to Theorem 4.3 the operator $A_\beta + \delta$ admits a bounded $\mathcal{H}^\infty$-calculus for each $\delta > \beta^2$. Moreover, it is not difficult to see that $A_\beta + \delta$ is m-accretive on X whenever $\delta > \beta^2$. The function ψ is meromorphic on $\mathbb{C}$ with poles in

$$\{z_k = -r_k^2 := -k^2(\pi/\alpha)^2 : k \in \mathbb{N}\}$$

and $\psi(z) \sim \sqrt{z}$ as $z \to \infty$, provided $|\arg z| \le \theta < \pi$. It follows that $\psi(A_\beta)$ is a well-defined, closed linear operator with $\mathsf{D}(\psi(A_\beta)) = \mathsf{D}((A_\beta + \delta)^{1/2})$, see [6, Section 2.1] for more details.
The function ψ admits the following representation as a series

$$\psi(z) = \frac{1}{\alpha}\Big[1 + 2z\sum_{k=1}^{\infty}\frac{1}{z + r_k^2}\Big], \quad z \notin \{-r_j^2 : j \in \mathbb{N}\}.$$

Inserting A_β into this representation we obtain

$$\begin{aligned}\alpha\,\psi(A_\beta) &= 1 + 2\sum_{k=1}^{\infty} A_\beta(A_\beta + r_k^2)^{-1}\\ &= 1 + 2\sum_{k=1}^{\infty}(A_\beta + \beta^2)(A_\beta + r_k^2)^{-1} - 2\beta^2\sum_{k=1}^{\infty}(A_\beta + r_k^2)^{-1}.\end{aligned}$$

Employing the semi-inner product $(\cdot\,,\,\cdot)$ in X we estimate as follows.

$$\begin{aligned}\alpha\big(\psi(A_\beta)u, u\big) &\ge |u|^2 + 2\sum_{k=1}^{\infty}\big((A_\beta + \beta^2)(A_\beta + \beta^2 + (r_k^2 - \beta^2))^{-1}u, u\big)\\ &\qquad - 2\beta^2\sum_{k=1}^{\infty}|(A_\beta + \beta^2 + (r_k^2 - \beta^2))^{-1}u||u|\\ &\ge \Big[1 - 2\beta^2\sum_{k=1}^{\infty}(r_k^2 - \beta^2)^{-1}\Big]|u|^2\\ &\ge \Big[1 - \frac{\alpha|\beta|}{\pi - \alpha|\beta|}\Big]|u|^2 = \frac{\pi - 2\alpha|\beta|}{\pi - \alpha|\beta|}|u|^2,\end{aligned}$$

provided $\frac{1}{(k+1)\pi - \alpha|\beta|} \le \frac{1}{k\pi + \alpha|\beta|}$ for all $k \in \mathbb{N}$, which is equivalent to $|\beta| \le \pi/2\alpha$.

Thus if β is restricted to the range $|\beta| < \pi/2\alpha$ then $\psi(A_\beta)$ is strictly accretive in X.

We remark that the condition $|\beta| < \pi/2\alpha$ is also necessary for $\psi(A_\beta)$ to be strictly accretive. In order to see this, recall that $\psi(\sigma(A_\beta)) \subset \sigma(\psi(A_\beta))$ according to the (weak) spectral mapping theorem. Thus for $\psi(A_\beta)$ to be strictly accretive, $\psi(\sigma(A_\beta))$ must be contained in the set $[\mathrm{Re}\, z \ge c]$ with $c > 0$. Note that the curves $-(\eta + i\xi)^2$ with $0 \le \eta \le |\beta|$ fill up the parabola $\mathbb{P}_\beta$, the spectrum of A_β. The function

$$\mathrm{Re}\,\psi(-(\eta + i\xi)^2) = \frac{\xi\sinh(2\alpha\xi) + \eta\sin(2\alpha\eta)}{\cosh(2\alpha\xi) - \cos(2\alpha\eta)}$$

is strictly positive for all $\xi \in \mathbb{R}$ and $0 \le |\eta| \le |\beta|$ if and only if $\sin(2\alpha|\eta|) > 0$, the latter condition being equivalent to $|\beta| < \pi/2\alpha$. Thus $\psi(\mathbb{P}_\beta) \subset [\mathrm{Re} z > 0]$ if and only if $|\beta| < \pi/2\alpha$.

It should also be observed that in case we assume a Neumann condition on $\phi = \alpha$, $\psi(z) = \sqrt{z}\tanh(\alpha\sqrt{z})$. In this case

$$\operatorname{Re}\psi(-(\eta + i\xi)^2) = \frac{\xi\sinh(2\alpha\xi) - \eta\sin(2\alpha\eta)}{\cosh(2\alpha\xi) + \cos(2\alpha\eta)},$$

which does not have positive real part for $\xi \in \mathbb{R}$, for no values of η. Thus in this case, $\psi(A_\beta)$ fails to be accretive.

We will now show that there exists a sufficiently large positive number ω_0 such that $\omega_0 + \partial_t e^x + \psi(A_\beta)$, with domain $\mathsf{D}(\partial_t e^x) \cap \mathsf{D}(\psi(A_\beta))$, is invertible, sectorial and admits a bounded $\mathcal{H}^\infty$-calculus on X. Since $e^x\partial_t + \psi(A_\beta)$ is strictly accretive, we can conclude that $e^x\partial_t + \psi(A_\beta)$ is in fact invertible and m-accretive, and hence sectorial on X. It then follows from [6, Proposition 2.7] that $\partial_t e^x + \psi(A_\beta)$ admits a bounded $\mathcal{H}^\infty$-calculus on X as well.

To prove the remaining statement we use, once more, the result on sums of non-commuting operators, Theorem 2.2. For this purpose we have to estimate the relevant commutator in the Labbas-Terreni condition (2.3). We will need the following auxiliary result for the commutator of $(z - A_\beta)^{-1}$ with the multiplication operator e^x.

Lemma 5.1. *Suppose $z \in \rho(A_\beta) \cap \rho(A_{\beta-1})$. Then*

$$[(z - A_\beta)^{-1}, e^x]e^{-x} = \big((z - A_\beta)^{-1} - (z - A_{\beta-1})^{-1}\big) \text{ on } X.$$

Proof. An easy computation shows that $(z - A_{\beta-1})e^x v = e^x(z - A_\beta)v$ for every function $v \in \mathcal{D} := \mathcal{D}((0, T] \times \mathbb{R}^m \times \mathbb{R})$. Applying $(z - A_{\beta-1})^{-1}$ to this equation gives

$$e^x v = (z - A_{\beta-1})^{-1} e^x (z - A_\beta)v, \quad v \in \mathcal{D}.$$

Substituting $v = (z - A_\beta)^{-1}(z - A_\beta)v$ on the left side and then replacing $(z - A_\beta)v$ by $e^{-x}(z - A_\beta)v$ yields

$$e^x(z - A_\beta)^{-1}e^{-x}(z - A_\beta)v = (z - A_{\beta-1})^{-1}(z - A_\beta)v, \quad v \in \mathcal{D}.$$

The assertion now follows from the fact that $\{(z - A_\beta)v : v \in \mathcal{D}\}$ is dense in X. $\square$

In order to check the commutator condition (2.3) we first remove some poles from ψ such that the remainder ψ_0 is holomorphic in a sector $-a + \Sigma_\phi$, where

$$-a < -b := \min\{-(\beta - 1)^2, -\beta^2\}, \quad \phi \in (0, \pi).$$

Observing that ψ has first-order poles at $-r_k^2$ with corresponding residues $-(2/\alpha)r_k^2$ we can take

$$\psi_0(z) := \psi(z) + \frac{2r_1^2}{\alpha} \cdot \frac{1}{z + r_1^2} + c\,,$$

with c a constant. The condition $|\beta| < \pi/\alpha$ ensures that ψ_0 is in fact holomorphic on an open set containing the closure of $-a + \Sigma_\phi$ for $-a \in (-r_2^2, -b)$ and any $\phi \in (0, \pi)$. By choosing c big enough we can, moreover, arrange that ψ_0 maps the sector $-a+\Sigma_\phi$, with $\phi<\pi$, into a sector Σ_θ with $\theta<\pi/2$. The difference $\psi(A_\beta)-\psi_0(A_\beta)$ is bounded, as one easily verifies. Then we fix $\eta>0$ and compute

$$\begin{aligned}&(\eta+\partial_t e^x)(\lambda+\eta+\partial_t e^x)^{-1}[(\eta+\partial_t e^x)^{-1},(\mu+\psi_0(A_\beta))^{-1}]\\ &=(\lambda+\eta+\partial_t e^x)^{-1}[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\cdot\partial_t e^x(\eta+\partial_t e^x)^{-1},\end{aligned}$$

which gives

$$\begin{aligned}&\big|(\eta+\partial_t e^x)(\lambda+\eta+\partial_t e^x)^{-1}[(\eta+\partial_t e^x)^{-1},(\mu+\psi_0(A_\beta))^{-1}]\big|_{\mathcal{B}(X)}\\ &\le C_\eta\,(1+|\lambda|)^{-1}\cdot\big|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\big|_{\mathcal{B}(X)}.\end{aligned}$$

Next taking $\Gamma=-a+\partial\Sigma_\phi$, the boundary of the sector $-a+\Sigma_\phi$, with ϕ appropriate, and using Lemma 5.1 we get

$$\begin{aligned}&\big|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\big|_{\mathcal{B}(X)}=\Big|\frac{1}{2\pi i\mu}\int_\Gamma\frac{\psi_0(z)}{\mu+\psi_0(z)}[(z-A_\beta)^{-1},e^x]e^{-x}\,dz\Big|_{\mathcal{B}(X)}\\ &\le\frac{C}{|\mu|}\int_\Gamma\Big|\frac{\psi_0(z)}{\mu+\psi_0(z)}\Big|\cdot\big|(z-A_\beta)^{-1}-(z-A_{\beta-1})^{-1}\big|_{\mathcal{B}(X)}\,|dz|.\end{aligned}$$

Since $A_\beta-A_{\beta-1}=-2\partial_x-2\beta+1$ we obtain

$$\begin{aligned}\big|(z-A_\beta)^{-1}-(z-A_{\beta-1})^{-1}\big|_{\mathcal{B}(X)}&\le\big|(z-A_\beta)^{-1}(A_\beta-A_{\beta-1})(z-A_{\beta-1})^{-1}\big|_{\mathcal{B}(X)}\\ &\le C(1+|z|)^{-3/2}.\end{aligned}$$

This implies

$$\begin{aligned}&\big|[(\mu+\psi_0(A_\beta))^{-1},e^x]e^{-x}\big|_{\mathcal{B}(X)}\le\frac{C}{|\mu|}\int_\Gamma\Big|\frac{\psi_0(z)}{\mu+\psi_0(z)}\Big|\frac{|dz|}{(1+|z|)^{3/2}}\\ &\le\frac{C}{|\mu|}\int_\Gamma\frac{|dz|}{(|\mu|+|z|^{1/2})(1+|z|)}\le\frac{C_\varepsilon}{|\mu|^{2-\varepsilon}}\,,\end{aligned}$$

since $\psi_0(z)\sim\sqrt{z}$. Thus the assumptions of Theorem 2.2 are satisfied for

$$A:=\eta+\partial_t e^x\quad\text{and}\quad B:=\psi_0(A_\beta),$$

with $\alpha=0$ and $\beta=1-\varepsilon$, for each $\varepsilon>0$. Indeed, observe that $\psi_0(A_\beta)$ is sectorial with angle strictly smaller than $\pi/2$. This follows from the fact that ψ_0 maps the sector $-a+\Sigma_\phi$ into a sector Σ_θ with $\theta<\pi/2$. Hence the parabolicity condition holds. Moreover, it is clear that $\psi_0(A_\beta)$ admits a bounded $\mathcal{H}^\infty$-calculus on X since $A_\beta+\delta$ does so according to Theorem 4.3. We may conclude that there is a number $\omega_0>0$ such that $\omega_0+\partial_t e^x+\psi_0(A_\beta)$ with natural domain is closed, invertible, sectorial, and admits an $\mathcal{H}^\infty$-calculus on X. Perturbing by the bounded linear operator $\psi(A_\beta)-\psi_0(A_\beta)$ we see that the same result holds for $\omega_0+\partial_t e^x+\psi(A_\beta)$, possibly after choosing a larger number ω_0.

We summarize our considerations in

Theorem 5.2. *Let $1<p<\infty$ and assume $|\beta|<\pi/2\alpha$. Then for each $g\in X:=L_p(J\times\mathbb{R}^m\times\mathbb{R})$,* (5.3) *admits a unique solution ρ in*

$$L_p(J\times\mathbb{R}^m;H^r_p(\mathbb{R}))\cap L_p(J;H^1_p(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx)))\cap H^1_p(J;L_p(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx))).$$

There is a constant $M > 0$, independent of g, such that

$$|e^{sx}\partial_t\rho|_X + |\rho|_{L_p(J\times\mathbb{R}^m;H_p^r(\mathbb{R}))} + |\rho|_{L_p(J;H_p^1(\mathbb{R}^m;L_p(\mathbb{R};e^{xp}dx)))} \leq M|g|_X.$$

The operator $L = \partial_t e^{sx} + \psi((\partial_t - \Delta_y)e^{2x} - (\partial_x + \beta)^2)$ admits a bounded $\mathcal{H}^\infty$-calculus on X.

Acknowledgement

This paper was initiated while the first author was visiting the Department of Mathematics at Vanderbilt University in the spring of 2004. He expresses his thanks to the department for their kind hospitality and for financial support.

References

[1] M. Abramowitz, I. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series 55, 1964.

[2] Ph. Clément, B. de Pagter, F.A. Sukochev, H. Witvilet. *Schauder decompositions and multiplier theorems.* Studia Math. **138** (2000), 135–163.

[3] Ph. Clément, J. Prüss. *An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces.* Evolution equations and their applications in physical and life sciences. Lecture Notes in Pure and Applied Mathematics 215, Marcel Dekker, New York (2001), 67–87.

[4] Ph. Clément, J. Prüss. *Some remarks on maximal regularity of parabolic problems.* Evolution equations: applications to physics, industry, life sciences and economics (Levico Terme, 2000). Progr. Nonlinear Differential Equations Appl. 55, Birkhäuser, Basel (2003), 101–111.

[5] G. Da Prato, P. Grisvard. *Sommes d'opératours linéaires et équations différentielles opérationelles*, J. Math. Pures Appl. **54** (1975), 305–387.

[6] R. Denk, M. Hieber, and J. Prüss. *$\mathcal{R}$-boundedness and problems of elliptic and parabolic type.* Memoirs of the AMS vol. 166, No. 788 (2003).

[7] J. Escher, J. Prüss, G. Simonett. *Analytic solutions for a Stefan problem with Gibbs-Thomson correction.* J. Reine Angew. Math. **563** (2003), 1–52.

[8] J. Escher, J. Prüss, G. Simonett. *Well-posedness and analyticity for the Stefan problem with surface tension.* In preparation.

[9] J. Escher, J. Prüss, G. Simonett. *Maximal regularity for the Navier-Stokes equations with surface tension on the free boundary.* In preparation.

[10] E.V. Frolova. *An initial-boundary value problem with a noncoercive boundary condition in a domain with edges.* (Russian) Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI) **213** (1994). Translation in J. Math. Sci. (New York) **84** (1997), 948–959.

[11] F. Haller, M. Hieber. *H^∞-calculus for products of non-commuting operators.* Preprint.

[12] N.J. Kalton, L. Weis. *The H^∞-calculus and sums of closed operators.* Math. Ann. **321** (2001), 319–345.

[13] R. Labbas, B. Terreni. *Somme d'opérateurs linéaires de type parabolique.* Boll. Un. Mat. Ital. **7** (1987), 545–569.

[14] J. Prüss. *Evolutionary Integral Equations and Applications.* Volume 87 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1993.

[15] J. Prüss. *Maximal regularity for abstract parabolic problems with inhomogeneous boundary data.* Math. Bohemica **127** (2002), 311–317.

[16] J. Prüss. *Maximal regularity for evolution equations in L_p-spaces.* Conf. Semin. Mat. Univ. Bari **285** (2003), 1–39.

[17] J. Prüss, G.Simonett. *$\mathcal{H}^\infty$-calculus for noncommuting operators.* Preprint 2003 (submitted).

[18] V.A. Solonnikov, E.V. Frolova. *A problem with the third boundary condition for the Laplace equation in a plane angle and its applications to parabolic problems.* (Russian) Algebra i Analiz **2** (1990), 213–241. Translation in Leningrad Math. J. **2** (1991), 891–916.

[19] E. Stein. *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.

[20] L. Weis. *A new approach to maximal L_p-regularity.* In Evolution Equ. and Appl. Physical Life Sci. Lecture Notes in Pure and Applied Mathematics 215, Marcel Dekker, New York (2001), 195–214.

Jan Prüss
Fachbereich Mathematik und Informatik
Martin-Luther-Universität Halle-Wittenberg
D-60120 Halle, Germany
e-mail: `pruess@mathematik.uni-halle.de`

Gieri Simonett
Department of Mathematics
Vanderbilt University
Nashville, TN 37240, USA
e-mail: `simonett@math.vanderbilt.edu`