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THE VOLUME PRESERVING MEAN CURVATURE FLOW NEAR SPHERES

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ABSTRACT. By means of a center manifold analysis we investigate the averaged mean curvature flow near spheres. In particular, we show that there exist global solutions to this flow starting from non-convex initial hypersurfaces.

1. INTRODUCTION

Let $\mathcal{G}$ be a compact, closed, connected, embedded hypersurface in $\mathbb{R}^n$ of class $C^{1+\beta}$. We are interested in the **averaged mean curvature flow**, i.e., in finding a family $M = \{M_t\ ;\ t \geq 0\}$ of smooth hypersurfaces in $\mathbb{R}^n$ satisfying the following evolution equation:

$$V = h - H, \qquad M_0 = \mathcal{G}, \tag{1.1}$$

where $V(t)$ denotes the normal velocity of M at time t and $H(t)$ stands for the mean curvature of M_t. Finally, $h(t)$ is the average of the mean curvature on M_t, i.e.,

$$h(t) := \frac{\int_{M_t} H\, d\sigma}{\int_{M_t} d\sigma}, \quad t \geq 0.$$

The averaged mean curvature flow has some interesting geometrical features. Suppose that $M = \{M_t\ ;\ t \geq 0\}$ is a smooth solution to (1.1) and let $Vol(t)$ and $A(t)$ denote the volume enclosed by M_t and the area of M_t, respectively. Then these functions are smooth, and we find for their derivatives

$$\frac{d}{dt} Vol(t) = \int_{M_t} V\, d\sigma = \int_{M_t} (h - H)\, d\sigma = 0,$$

and, see e.g., [11] Theorem 4 or [9] p. 70,

$$\frac{1}{n-1}\frac{d}{dt}A(t) = \int_{M_t} HV\, d\sigma = \int_{M_t} (hH - H^2)\, d\sigma = -\int_{M_t} (h-H)^2 d\sigma \leq 0,$$

since obviously $\int h(h-H)\, d\sigma = 0$. Hence the averaged mean curvature flow is **volume preserving** and **area shrinking**. Moreover, observe that every Euclidean sphere is an equilibrium for (1.1). These simple observations form in fact the starting point of our investigations. More precisely, the isoperimetric inequality suggests

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analyzing the infinite-dimensional semiflow generated by (1.1) near spheres. In order to formulate our main result, let us introduce the following notation. Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $C^\infty(U)$ in the usual Hölder norm of $C^s(U)$. If Γ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Gamma)$ are defined by means of a smooth atlas for Γ. We have the following:

Main Result. *Assume that $0 < \beta < 1$ and let $\mathcal{G}$ be a compact, closed, connected, embedded hypersurface in $\mathbb{R}^n$ of class $h^{1+\beta}$. Then:*

a) The averaged mean curvature flow (1.1) *has a unique local classical solution $M = \{M_t\ ;\ t \in [0,T)\}$ for some $T > 0$. Each hypersurface M_t is of class C^∞ for $t \in (0,T)$. Moreover, the mapping $[t \mapsto M_t]$ is continuous on $[0,T)$ with respect to the $h^{1+\beta}$-topology and smooth on $(0,T)$ with respect to the C^∞-topology.*

b) Suppose that the initial hypersurface $\mathcal{G}$ is an $h^{1+\beta}$-graph in normal direction over some smooth hypersurface Γ. Then the mapping $\varphi := [(t,\mathcal{G}) \mapsto M_t]$ induces a local smooth semiflow on $h^{1+\beta}(\Gamma)$.

c) Let $\mathcal{S}$ be a fixed Euclidean sphere and let $\mathcal{M}$ denote the set of all spheres which are sufficiently $h^{1+\beta}(\mathcal{S})$-close to $\mathcal{S}$. Then $\mathcal{M}$ attracts at an exponential rate all solutions which are $h^{1+\beta}(\mathcal{S})$-close to $\mathcal{M}$. In particular, if $\mathcal{G}$ is sufficiently close to $\mathcal{S}$ in $h^{1+\beta}(\mathcal{S})$, then M_t exists globally and converges exponentially fast to some sphere as $t \to \infty$. The convergence is in the C^k-topology for any fixed $k \in \mathbb{N}$.

Remarks. a) The existence part of the above result is well-known for smooth initial hypersurfaces $\mathcal{G}$, cf. [10] Theorem 0.1 and [8] Theorem 4.1. However, it is important to be able to allow $h^{1+\beta}$-hypersurfaces as initial data in order to get the semiflow property stated in b) on the large space $h^{1+\beta}$ (cf. also Remark e) below).

b) The proof of part c) of the Main Result consists of two steps. We first show that the semiflow φ admits a stable $(n+1)$-dimensional *local center manifold* $\mathcal{M}^c$. In particular, this means that $\mathcal{M}^c$ is a locally invariant manifold and that $\mathcal{M}^c$ contains all small global solutions of φ. Additionally, there is an $(n+1)$-dimensional linear subspace N of $h^{1+\beta}(\mathcal{S})$, which is invariant under the linearization of φ and to which $\mathcal{M}^c$ is tangential at 0. In a second step we then prove that $\mathcal{M}^c$ and $\mathcal{M}$ coincide.

c) Under suitable spectral assumptions for the linearization, the existence of center manifolds is well-known for finite-dimensional dynamical systems. The corresponding construction for quasilinear infinite-dimensional semiflows (e.g. for φ) is considerable more involved. The basic technical tool here is the theory of maximal regularity, due to G. Da Prato and P. Grisvard [3]. These results particularly allow us to treat (1.1) as a fully nonlinear perturbed linear evolution equation; see [4, 12, 13].

d) It is well-known that local stable center manifolds, which do not consist of equilibria only, are not unique, in general.

e) It is important to note that the exponential attractivity of $\mathcal{M}$ holds for initial data $\mathcal{G}$ which are $h^{1+\beta}$-close to $\mathcal{S}$. This result is close to optimal and has a nice application to non-convex initial data; see the Corollary below.

f) Observe that Remark b) yields the fact that $\mathcal{G}$ is an equilibrium of (1.1) iff $\mathcal{G}$ is a Euclidean sphere. In particular, this implies the following result: Suppose that $\mathcal{G}$ is a compact, closed, connected, embedded hypersurface which is $h^{1+\beta}$-close to a sphere. Additionally, assume that the mean curvature of $\mathcal{G}$ is constant. Then $\mathcal{G}$ is a Euclidean sphere. This observation is a special case of a general result due to A. D. Alexandrov; cf. [1].

g) The volume preserving mean curvature flow shares some properties with the Mullins-Sekerka model [5, 6]—a moving boundary problem originating in the theory of phase transitions. In particular, the Mullins-Sekerka model is also volume preserving and area shrinking, and the only equilibria of this flow are spheres. As for the averaged mean curvature flow, we show in [7] that the invariant manifold $\mathcal{M}$ of Euclidean spheres is exponentially attracting.

G. Huisken [10] (and M. Gage [8] in the case of curves) proved the fundamental result that the solution to (1.1) exists globally and converges exponentially fast to a sphere, provided the initial surface $\mathcal{G}$ is uniformly convex and smooth. Moreover, it is shown in [8, 10] that M_t stays uniformly convex for all $t \geq 0$.

As an immediate consequence of part c) of our Main Result we get the following

Corollary. *Convexity is not necessary for global existence of the averaged mean curvature flow* (1.1). *More precisely, there are non-convex hypersurfaces $\mathcal{G}$ such that the solution of* (1.1) *with initial condition $M_0 = \mathcal{G}$ exists globally and converges exponentially fast to a sphere.*

Proof. Let $\mathcal{S}$ be a Euclidean sphere. Since in every $h^{1+\beta}$- neighborhood of $\mathcal{S}$ there are non-convex hypersurfaces, the assertion follows from part c) of the Main Result. □

2. Proof of the Main Result

Let $0 < \alpha < \beta_0 < \beta < 1$ be fixed and pick a compact, closed, connected, embedded hypersurface $\mathcal{G}$ of class $h^{1+\beta}$.

i) We first provide an appropriate parameterization of a small neighborhood of $\mathcal{G}$. Given $a > 0$, we find a smooth hypersurface Γ such that $\mathcal{G}$ is a C^1-close graph over Γ in normal direction, i.e., we find Γ of class C^∞ and $\rho_0 \in h^{1+\beta}(\Gamma)$ with $\|\rho_0\|_{C^1(\Gamma)} < a/2$ such that $\theta_{\rho_0} := id_\Gamma + \rho_0\nu$ is a diffeomorphism of class $h^{1+\beta}$, mapping Γ onto $\mathcal{G}$. Here, ν denotes the outer unit normal field on Γ with the sign convention that "interior" is given by the compact part of $\mathbb{R}^n$ enclosed by Γ. Let $\mathcal{V}$ be the ball in $h^{1+\beta}(\Gamma)$ with center 0 and radius a, where $a > 0$ is chosen sufficiently small such that

$$X : \Gamma \times (-a, a) \to \mathbb{R}^n, \qquad X(s, r) := s + r\nu(s)$$

is a smooth diffeomorphism onto its image $\mathcal{R} := im(X)$. It is convenient to decompose the inverse of X into $X^{-1} = (S, \Lambda)$, where

$$S \in C^\infty(\mathcal{R}, \Gamma) \quad \text{and} \quad \Lambda \in C^\infty(\mathcal{R}, (-a, a)).$$

Note that $S(x)$ is the nearest point on Γ to x and that $\Lambda(x)$ is the signed distance from x to Γ. Moreover, $\mathcal{R}$ is the neighborhood of Γ consisting of those points with distance to Γ less than a.

Now let $T > 0$ be fixed. Given any (sufficiently) smooth function $\rho : \Gamma \times [0, T] \to (-a, a)$, let

$$\Phi_\rho : \mathcal{R} \times [0, T] \to \mathbb{R}, \qquad \Phi_\rho(x, t) := \Lambda(x) - \rho(S(x), t).$$

Then for each $t \in [0, T]$, the zero-level set of $\Phi_\rho(\cdot, t)$ defines a smooth, compact, connected hypersurface $M_{\rho(t)} := \Phi_\rho^{-1}(\cdot, t)(0)$. Observe that

$$M_{\rho(t)} = \{x \in \mathbb{R}^n \,;\, x = X(s, \rho(s, t)),\ s \in \Gamma\}, \quad t \in [0, T].$$

In addition, the normal velocity of $\{M_{\rho(t)}\,;\, t\in[0,T]\}$ and the mean curvature of $M_{\rho(t)}$ (as functions parameterized over Γ) are given by

$$V(s,t)=\frac{\partial_t\rho(s,t)}{|\nabla_x\Phi(x,t)|}\Big|_{x=X(s,\rho(s,t))},\qquad (s,t)\in\Gamma\times[0,T],$$

and

$$H_\rho(s,t)=\frac{1}{n-1}\mathrm{div}_x\Big(\frac{\nabla_x\Phi_\rho(x,t)}{|\nabla_x\Phi_\rho(x,t)|}\Big)\Big|_{x=X(s,\rho(s,t))},\qquad (s,t)\in\Gamma\times(0,T),$$

respectively. Finally, we let

$$L_\rho(s,t):=|\nabla_x\Phi_\rho(x,t)|\Big|_{x=X(s,\rho(s,t))},\quad \mu_\rho:=\sqrt{\det[D_s\theta_\rho]^T[D_s\theta_\rho]},$$

and we define

$$G(\rho):=L_\rho\Big(\frac{1}{\int_\Gamma\mu_\rho\,d\sigma}\int_\Gamma H_\rho\mu_\rho\,d\sigma-H_\rho\Big),\quad \rho\in\mathcal{V}\cap h^{2+\alpha}(\Gamma).$$

Then we consider the abstract evolution equation for the distance function ρ given by

$$\partial_t\rho=G(\rho),\qquad \rho(0)=\rho_0. \tag{2.1}$$

We call a family $\rho:[0,T]\to\mathcal{V}$ a **classical** solution of (2.1) if

$$\rho\in C([0,T],\mathcal{V})\cap C^\infty((0,T),C^\infty(\Gamma))$$

and if ρ satisfies (2.1) point-wise. It is not difficult to see that the averaged mean curvature flow (1.1) and the abstract problem (2.1) are equivalent on $\mathcal{R}$. More precisely, if $M:=\{M_t\,;\,t\in[0,T]\}$ is a classical solution of (1.1) such that $M_t\subset\mathcal{R}$ for $t\in[0,T]$, then the above construction yields a classical solution of (2.1), and vice-versa; if $\rho:[0,T]\to\mathcal{V}$ is a classical solution of (2.1) then $M:=\{M_{\rho(t)}\,;\,t\in[0,T]\}$ is a classical solution of (1.1).

ii) It is known that H_ρ is a quasilinear uniformly elliptic operator of second order acting in $h^\alpha(\Gamma)$. More precisely, let $\mathcal{U}:=\{\rho\in h^{1+\beta_0}(\Gamma)\,;\,\|\rho\|_{1+\beta_0}<a\}$. Then it was shown in [7] Lemma 3.1 and [6] Lemma 3.2 that there exist functions

$$P\in C^\infty(\mathcal{U},\mathcal{L}(h^{2+\alpha}(\Gamma),h^\alpha(\Gamma)))\quad\text{and}\quad Q\in C^\infty(\mathcal{U},h^{\beta_0}(\Gamma)) \tag{2.2}$$

such that

$$H_\rho=P(\rho)\rho+Q(\rho)\quad\text{for}\quad \rho\in\mathcal{U}\cap h^{2+\alpha}(\Gamma).$$

Moreover, the linear operator $[h\mapsto P(\rho)h]$ is a uniformly elliptic operator of second order. Now let $-G^\pi_\rho$ be the linear part of the principal part of $G(\rho)$, i.e.,

$$G^\pi_\rho h=L_\rho\Big(P(\rho)h-\frac{1}{\int_\Gamma\mu_\rho\,d\sigma}\int_\Gamma P(\rho)h\mu_\rho\,d\sigma\Big),\quad h\in h^{2+\alpha}(\Gamma),$$

and fix $\rho\in\mathcal{V}$. Since L_ρ belongs to $h^\beta(\Gamma)$ and is strictly positive, the linear operator $[h\mapsto -L_\rho P(\rho)h]$ generates a strongly continuous analytic semigroup on $h^\alpha(\Gamma)$ with domain of definition equal to $h^{2+\alpha}(\Gamma)$, i.e., $L_\rho P(\rho)$ belongs to the class $\mathcal{H}(h^{2+\alpha}(\Gamma),h^\alpha(\Gamma))$ introduced by H. Amann; see, e.g. [2].

Next, let $B(\rho)$ be the linear operator in $h^\alpha(\Gamma)$ given by

$$B(\rho)h := \frac{L_\rho}{\int\limits_\Gamma \mu_\rho\, d\sigma} \int\limits_\Gamma P(\rho)h\mu_\rho\, d\sigma, \quad h \in h^{2+\alpha}(\Gamma).$$

Obviously, $\|B(\rho)h\|_{h^\alpha} = \|L_\rho\|_{h^\alpha} \big(\int_\Gamma \mu_\rho\, d\sigma\big)^{-1} |\int\limits_\Gamma P(\rho)h\mu_\rho\, d\sigma|$, and therefore

$$\|B(\rho)h\|_{h^\alpha} \le C_\rho \|h\|_{C^2}, \qquad h \in h^{2+\alpha}(\Gamma),$$

with a positive constant C_ρ depending only on $\rho \in \mathcal{V}$. Hence, given $\varepsilon > 0$, a well-known interpolation inequality shows that there exists a positive constant C_ε such that

$$\|B(\rho)h\|_{h^\alpha} \le \varepsilon \|h\|_{h^{2+\alpha}} + C_\varepsilon \|h\|_{h^\alpha}, \qquad h \in h^{2+\alpha}(\Gamma).$$

We now apply a standard perturbation argument for the class $\mathcal{H}(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma))$ in order to conclude that

$$G_\rho^\pi \in \mathcal{H}(h^{2+\alpha}(\Gamma), h^\alpha(\Gamma)), \qquad \rho \in \mathcal{V}. \tag{2.3}$$

Finally, we set $F(\rho) := G(\rho) + G_\rho^\pi \rho$ for $\rho \in \mathcal{V}$ and we rewrite problem (2.1) as

$$\partial_t \rho + G_\rho^\pi \rho = F(\rho), \qquad \rho(0) = \rho_0. \tag{2.4}$$

Observe that $F \in C^\infty(\mathcal{U}, h^{\beta_0}(\Gamma))$. Hence property (2.3) allows us to apply the general results for quasilinear parabolic problems due to H. Amann. In particular, Theorem 12.1 in [2] and the proof of Theorem 1 in [6] imply the assertions in a) and b).

iii) Now let $\mathcal{S} := \mathcal{S}_R$ be a Euclidean sphere of radius R and set $\Gamma = \mathcal{S}_R$ in the above construction. It follows from (2.2) that

$$G : \mathcal{U} \cap h^{2+\alpha}(\mathcal{S}) \to h^\alpha(\mathcal{S}), \qquad \rho \mapsto G(\rho)$$

is smooth. Our next goal is to determine the Fréchet derivative at $\rho = 0$ of the above operator. To do this, observe that $L_0 \equiv 1$. Moreover, it is shown in [7], Lemma 3.1 and its proof, that

$$\partial H_\rho\big|_{\rho=0} = -\frac{1}{n-1}\Big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\Big), \qquad \partial L_\rho\big|_{\rho=0} = 0,$$

where $\Delta_{\mathcal{S}}$ denotes the Laplace-Beltrami operator on $\mathcal{S}$. Finally, it follows from [9] p. 70 that

$$\partial \int\limits_{\mathcal{S}} \mu_\rho\, d\sigma\big|_{\rho=0} h = -\frac{n-1}{R} \int\limits_{\mathcal{S}} h\, d\sigma, \quad h \in h^{2+\alpha}(\mathcal{S}).$$

Hence for the full linearization of $G(\rho)$ at $\rho = 0$ we get the expression

$$\partial G(0)h = \frac{1}{n-1}\Big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\Big)h - \frac{1}{(n-1)|\mathcal{S}|} \int\limits_{\mathcal{S}} \Big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\Big)h\, d\sigma \tag{2.5}$$

for each $h \in h^{2+\alpha}(\mathcal{S})$; here $|\mathcal{S}|$ stands for the area of $\mathcal{S}$. Finally, note that

$$\int\limits_{\mathcal{S}} \Delta_{\mathcal{S}} h\, d\sigma = (h|\Delta_{\mathcal{S}}\mathbf{1}) = 0, \qquad h \in h^{2+\alpha}(\mathcal{S}),$$

where $(\cdot|\cdot)$ denotes the inner product in $L_2(\mathcal{S})$. So we arrive at

$$\partial G(0)h = \frac{1}{n-1}\Big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\Big)h - \frac{1}{|\mathcal{S}|R^2}\int_{\mathcal{S}} h\,d\sigma \tag{2.6}$$

for $h \in h^{2+\alpha}(\mathcal{S})$.

iv) In our next step we determine the first eigenvalue of $A := \partial G(0)$ and locate the remainder of the spectrum. In this part of the proof we will always employ the natural complexification without distinguishing this notationally. Of course, $\sigma(A)$ consists only of eigenvalues, due to the compact embedding of $h^{2+\alpha}(\mathcal{S})$ in $h^{\alpha}(\mathcal{S})$. Furthermore, observe that $A\mathbf{1} = 0$. Moreover, it is well-known that $\lambda = (n-1)/R^2$ is an eigenvalue of $-\Delta_{\mathcal{S}}$ of multiplicity n and that the spherical harmonics $\{Y_k^R\ ;\ 1 \le k \le n\}$ of degree 1 of the R-sphere $\mathcal{S}$ span the corresponding eigenspace. Hence (2.5) shows that 0 is an eigenvalue of A of multiplicity at least $n+1$. Let $N := span\{\mathbf{1},\, Y_k^R\ ;\ 1 \le k \le n\}$ and assume that $h \in h^{2+\alpha}(\mathcal{S}) \cap N^{\perp}$ is a solution of $Ah = 0$, where the orthogonal complement has to be taken in $L_2(\mathcal{S})$. In particular, h has average 0. Consequently, we find that

$$\Big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\Big)h = 0,$$

showing that h belongs to N. Thus $h = 0$, and we conclude that the multiplicity of 0 is in fact equal to $n+1$.

Finally, assume that $\lambda \in \mathbb{C}\backslash\{0\}$ and $h \in h^{2+\alpha}(\mathcal{S})$ satisfy the equation $(\lambda - A)h = 0$. It follows from (2.5) that

$$0 = \big((\lambda - A)h|Y_k\big) = \lambda\big(h|Y_k\big), \quad k \in \{0, \cdots, n\},$$

showing that h belongs to $N^{\perp}$. Multiplying $(\lambda - A)h = 0$ with h in $L_2(\mathcal{S})$, we get

$$\lambda \int_{\mathcal{S}} |h|^2\,d\sigma = \frac{1}{n-1}\Big(\big(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\big)h\Big|h\Big).$$

But on $N^{\perp}$ the operator $(n-1)/R^2 + \Delta_{\mathcal{S}}$ is negative definite. Consequently, we see that λ belongs to $(-\infty, 0)$. In summary, the spectrum of A consists of a sequence of negative real numbers

$$\cdots < \mu_{k+1} < \mu_k < \mu_{k-1} < \cdots < \mu_1 < \mu_0 = 0$$

and μ_0 is an eigenvalue of multiplicity $n+1$.

v) In the next step we briefly sketch the construction of a center manifold $\mathcal{M}^c$ over N for φ. Let $Y_0 := |\mathcal{S}|^{-1}\mathbf{1}$ and let $Pg := \sum_{k=0}^{n}(g|Y_k)Y_k$ for $g \in h^r(\mathcal{S})$. Then P is a continuous projection of $h^r(\mathcal{S})$ onto N parallel to $\ker(P)$, and it follows from (2.5) that P commutes with A, that is, $PAg = APg = 0$ for every $g \in h^{2+\alpha}(\mathcal{S})$. Therefore, $N = \mathrm{im}(P)$ and $\ker(P)$ are complimentary subspaces of $h^{2+\alpha}(\mathcal{S})$ that reduce A. To simplify the notation we write $\pi^c = P$ and $\pi^s = (1-P)$, and we define $h_s^{2+\alpha}(\mathcal{S}) := \pi^s(h^{2+\alpha}(\mathcal{S}))$. It follows that $\sigma(\pi^c A) = \{0\}$ and $\sigma(\pi^s A) \subset (-\infty, 0)$. For this reason, N and $h_s^{2+\alpha}(\mathcal{S})$ are called the *center subspace* and the *stable subspace* of A, respectively. We are now in a position to apply Theorem 4.1 in [13] (see also [12] Theorem 9.2.2). These results imply that, given $l \in \mathbb{N}^*$, there exist an open neighborhood Λ of 0 in N and a mapping

$$\gamma \in C^l(\Lambda, h_s^{2+\alpha}(\mathcal{S})) \quad \text{with} \quad \gamma(0) = 0,\ \partial\gamma(0) = 0$$

such that $\mathcal{M}^c := \mathrm{graph}(\gamma)$ is a locally invariant manifold for the semiflow generated by the solutions of (2.1). Observe that $\mathcal{M}^c$ is an $(n+1)$-dimensional submanifold of $h^{2+\alpha}(\mathcal{S})$ with $T_0\mathcal{M}^c = N$. Moreover, the manifold $\mathcal{M}^c$ is exponentially stable and contains all small equilibria of (2.1).

vi) We show that $\mathcal{M}^c$ and $\mathcal{M}$ coincide near 0. Suppose that $\mathcal{S}'$ is a sphere which is sufficiently close to $\mathcal{S}$. Let $(z_1, \dots, z_n)$ be the coordinates of its center and R' be its radius. Recall that R is the radius of $\mathcal{S}$ and set $z_0 := R' - R$. If ρ measures the distance from $\mathcal{S}$ to $\mathcal{S}'$ in normal direction with respect to $\mathcal{S}$, we get the identity

$$(R + z_0)^2 = \sum_{k=1}^{n} \big((R+\rho)Y_k - z_k\big)^2,$$

where we write for simplicity $Y_k = Y_k^R$, $k = 1, \dots, n$. Additionally, let $Y_0 := \mathbf{1}$. Solving for ρ, we obtain that $\mathcal{S}'$ can be parameterized over $\mathcal{S}$ by the distance function

$$\rho(z) = \sum_{k=1}^{n} z_k Y_k - R + \sqrt{\big(\sum_{k=1}^{n} z_k Y_k\big)^2 + (R+z_0)^2 - \sum_{k=1}^{n} z_k^2}, \tag{2.7}$$

where $z := (z_0, \dots, z_n) \in \mathbb{R}^{n+1}$. If U is a sufficiently small neighborhood of 0 in $\mathbb{R}^{n+1}$, it is clear that any sphere $\mathcal{S}'$ which is close to $\mathcal{S}$ can be characterized by (2.7) with $z \in U$. Furthermore, the mapping $[z \mapsto \rho(z)] : U \to h^{2+\alpha}(\mathcal{S})$ is smooth and its derivative at 0 is given by

$$\partial\rho(0)h = \sum_{k=0}^{n} h_k Y_k, \qquad h \in \mathbb{R}^{n+1}. \tag{2.8}$$

Now let $\{F_0(z), \dots, F_n(z)\}$ be the coordinates of $\pi^c\rho(z)$ with respect to the basis $\{Y_0, \dots, Y_n\}$ of N. Then (2.8) yields that $\partial F(0) = id_{\mathbb{R}^{n+1}}$. Consequently, the inverse function theorem implies that F is a smooth diffeomorphism from U onto its image $V := im(F)$, provided U is small enough. Let $\mathcal{M} := \{\rho(z)\,;\, z \in U\}$. Then it follows that $\pi^c\mathcal{M}$ is an open neighborhood of 0 in N, which can be assumed to coincide with the open neighborhood Λ of 0 in N constructed in step (v). By Remark b) we know that $\mathcal{M} \subset \mathcal{M}^c$. Hence we conclude that $\mathcal{M} = \mathcal{M}^c$

vii) As is in [7], Theorem 6.5 and Proposition 6.6, one shows the following result. Given $k \in \mathbb{N}$ and $\omega \in (0, -\mu_1)$, there exists a neighborhood $W = W(k, \omega)$ of 0 in $h^{1+\beta}(\mathcal{S})$ with the following property: Given $\rho \in W$, the solution $\rho(\cdot, \rho_0)$ of (2.4) exists globally and there exist $c = c(k, \omega) > 0$, $T = T(k, \omega) > 0$, and a unique $z_0 = z_0(\rho_0) \in \Lambda$ such that

$$\|\big(\pi^c\rho(t,\rho_0), \pi^s\rho(t,\rho_0)\big) - \big(z_0, \gamma(z_0)\big)\|_{C^k} \le ce^{-\omega t}\|\pi^s\rho_0 - \gamma(\pi^c\rho_0)\|_{h^{1+\beta}}$$

for $t > T$. According to step (vi), $(z_0, \gamma(z_0))$ is a sphere and the proof is complete.

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