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# STABILITY OF EQUILIBRIA FOR THE STEFAN PROBLEM WITH SURFACE TENSION*

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**Abstract.** We characterize the equilibrium states for the two-phase Stefan problem with surface tension and with or without kinetic undercooling, and we analyze their stability in terms of dependence on physical and geometric quantities.

**Key words.** free boundary problem, phase transitions, surface tension, kinetic undercooling, stability, bifurcation

**AMS subject classifications.** 35R55, 35B55, 35K55, 80A22

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## 1. Introduction.

The Stefan problem is a model for phase transitions in solid-liquid systems. In this paper, we consider the two-phase Stefan problem with the modified Gibbs–Thomson law

$$u = \sigma H + \delta V \quad \text{on} \quad \Gamma(t), \quad \sigma > 0, \quad \delta \geq 0, \tag{1.1}$$

and the kinetic condition

$$[d\partial_\nu u] = (\ell - [\kappa]u)V \quad \text{on} \quad \Gamma(t). \tag{1.2}$$

Here $\Gamma(t)$ denotes the unknown moving hypersurface that separates the liquid from the solid phase, $u$ is the temperature, $H$ the mean curvature of $\Gamma(t)$, $\sigma$ the surface tension coefficient, $\delta$ the coefficient of kinetic undercooling, $V$ the normal velocity of $\Gamma(t)$, $\ell$ the latent heat, $[\kappa]$ the jump of the heat capacities across $\Gamma(t)$, and $[d\partial_\nu u]$ the jump of the heat fluxes across $\Gamma(t)$. Note that in case $\sigma = \delta = 0$, i.e., for the classical Stefan problem, we have $u = 0$ at the interface, and then the kinetic condition becomes the classical Stefan condition.

Under appropriate boundary conditions we will show that spheres (together with constant temperature distributions) are the only equilibrium states for this system, and we will characterize the stability of these equilibria in terms of dependence on physical and geometric quantities.

In order to formulate the Stefan problem we introduce the following notation. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ consists of two disjoint components, an "interior" part $J_1$ and an "exterior" part $J_2$. We think of $\Omega$ as a homogeneous medium which is occupied by a liquid and a solid phase, say water and ice, that initially occupy the regions $\Omega_0^1$ and $\Omega_0^2$, and that are separated by a sharp interface $\Gamma_0$. More precisely, we assume that $\Gamma_0 \subset \Omega$ is a compact closed hypersurface, and that $\Omega_0^1$ and $\Omega_0^2$ are disjoint open sets such that $\overline{\Omega} = \overline{\Omega_0^1} \cup \overline{\Omega_0^2}$, and such that $\partial\Omega_0^i = J_i \cup \Gamma_0$ for $i = 1, 2$. For the sake of definiteness we consider the open set $\Omega_0^1$ as

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the region occupied by the liquid phase. Consequently, the component $J_1$ is in contact with the liquid phase, and $J_2$ is in contact with the solid phase. The boundaries $J_1$ and $J_2$, corresponding for instance to the walls of a container, are fixed, whereas $\Gamma_0$ will change as time evolves, due to solidification or liquidation of the two different phases.

Given $t \geq 0$, let $\Gamma(t)$ be the position of $\Gamma_0$ at time $t$, and let $V(\cdot,t)$ and $H(\cdot,t)$ be the normal velocity and the mean curvature of $\Gamma(t)$. Moreover, let $\Omega_1(t)$ and $\Omega_2(t)$ be the two regions in $\Omega$ separated by $\Gamma(t)$. According to our assumption, $\Omega_1(t)$ is the region occupied by the liquid phase, and $\Gamma(t)$ is a sharp interface which separates the liquid from the solid phase. Let $\nu(\cdot,t)$ be the outer unit normal field on $\Gamma(t)$ with respect to $\Omega_1(t)$. We shall use the convention that the normal velocity is positive if $\Omega_1(t)$ is expanding, and that the mean curvature is positive if the intersection of $\Omega_1(t)$ with a small ball centered at $\Gamma(t)$ is convex. Consequently, the normal velocity is positive if the liquid region is growing, $\nu$ points into the solid phase, and $H$ is positive for a water ball surrounded by ice, and negative for an ice ball surrounded by water.

Here we concentrate on the case $J_1 = \emptyset$. Let $\Gamma_0$ and $u_0^i : \Omega_0^i \to \mathbb{R}$ be given, where $u_0^1$ and $u_0^2$ denote the initial temperatures of the liquid and solid phase, respectively. The strong formulation of the *two-phase Stefan problem with surface tension and kinetic undercooling* consists of finding a family $\Gamma := \{\Gamma(t)\,;\, t \geq 0\}$ of hypersurfaces and functions $u_i : \cup_{t\geq 0}\, (\Omega_i(t) \times \{t\}) \to \mathbb{R}$, satisfying

$$(1.3)\qquad \begin{cases} \kappa_i \partial_t u_i - d_i \Delta u_i = 0 & \text{in} \quad \Omega_i(t), \\ \partial_\nu u_2 = 0 & \text{on} \quad J_2, \\ u_i = \sigma H_\Gamma + \delta V & \text{on} \quad \Gamma(t), \\ [d\partial_\nu u] = (\ell - [\kappa]u)V & \text{on} \quad \Gamma(t), \\ u_i(0) = u_0^i & \text{in} \quad \Omega_0^i, \\ \Gamma(0) = \Gamma_0\,, \end{cases}$$

where $\kappa_i \geq 0$ is the heat capacity of phase $i$, $d_i$ is its thermal conductivity coefficient, $\ell > 0$ is the latent heat per unit mass absorbed or released for melting or solidifying, $\sigma > 0$ is the surface tension, and $\delta \geq 0$ is the speed of kinetic undercooling. Moreover,

$$[\kappa] := \kappa_2 - \kappa_1,$$
$$[d\partial_\nu u] := d_2 \partial_\nu u_2 - d_1 \partial_\nu u_1$$

denote the jump of the heat capacities and the heat fluxes, respectively, across the interface $\Gamma(t)$. Note that $[\kappa] = \kappa_2 - \kappa_1 < 0$ is physically reasonable since in the liquid phase there are more degrees of freedom than in the solid phase; hence, the liquid phase can absorb more energy per unit mass. However, we do not assume $[\kappa] < 0$ in what follows. The condition $u_i = \sigma H_\Gamma$ on the free interface is usually called the Gibbs–Thomson law, and $u_i = \sigma H_\Gamma + \delta V$ the modified Gibbs–Thomson law, or the Gibbs–Thomson law with kinetic undercooling; see [2, 3, 16, 17, 19, 21, 24, 25, 32] for more information.

We refer to [12, 13, 14, 22, 23, 28, 29] for existence and regularity results for the Stefan problem with the Gibbs–Thomson law $u_i = \sigma H_\Gamma$ in case $\kappa_1 = \kappa_2$. The Stefan problem with surface tension and kinetic undercooling in case $\kappa_1 = \kappa_2$ has been studied in [5, 28, 29, 31]; see also [20] for the one-phase case.

It will be shown in [26] that the Stefan problem (1.3) has a unique local solution which is analytic in space and time, provided that the well-posedness condition

$$(1.4)\qquad \ell - \sigma[\kappa] H_{\Gamma_0} > 0 \quad \text{in case} \quad \delta = 0$$

is satisfied. On the other hand, if $\delta = 0$ and $\kappa_1 > \kappa_2$, problem (1.3) is not well-posed if $H_{\Gamma_0}$ is too negative, that is, in case the solid region sharply protrudes into the liquid. Associated to the Stefan problem (1.3) is the energy functional

$$E(u(t), \Gamma(t)) := \int_{\Omega} \kappa u \, dx + \ell \, |\Omega_1(t)| = \int_{\Omega_1(t)} \kappa_1 u_1 \, dx + \int_{\Omega_2(t)} \kappa_1 u_2 \, dx + \ell \, |\Omega_1(t)|, \tag{1.5}$$

where $|\Omega_1(t)|$ is the volume of the region $\Omega_1(t)$. If $(u, \Gamma)$ is a sufficiently smooth solution of (1.3), then we obtain

$$\begin{aligned} \frac{d}{dt} E(u(t), \Gamma(t)) &= \int_{\Omega_1(t)} \kappa_1 \partial_t u_1 \, dx + \int_{\Omega_2(t)} \kappa_1 \partial_t u_2 \, dx - [\kappa] \int_{\Gamma(t)} uV \, ds + \ell \int_{\Gamma(t)} V \, ds \\ &= \int_{\Gamma(t)} \Big( - [d\partial_\nu u] - [\kappa] uV + \ell V \Big) \, ds = 0, \end{aligned} \tag{1.6}$$

thus showing that energy is conserved.

If $\kappa_1 = \kappa_2 = 0$ and $\delta = 0$, then the resulting problem is the quasi-stationary Stefan problem with surface tension, which has also been termed the Mullins–Sekerka model (or the Hele–Shaw model with surface tension). Existence, uniqueness, regularity, and global existence of solutions for the quasi-stationary approximation have been investigated in [1, 4, 6, 8, 9, 10, 11, 15]. Existence and global existence of classical solutions for the quasi-stationary approximation with $\sigma > 0$ and $\delta > 0$ have been studied in [33, 20].

A major difficulty in the mathematical treatment of the Stefan problem (1.3) is due to fact that the boundary $\Gamma(t)$, and thus the geometry, is unknown and ever changing. A widely used method to overcome this inherent difficulty is to choose a fixed reference surface $\Sigma$ and then represent the moving surface $\Gamma(t)$ as the graph of a function (which we will denote by $\rho = \rho(s, t)$) in normal direction of $\Sigma$. In this way, one obtains a time-dependent (unknown) diffeomorphism from $\Sigma$ onto $\Gamma(t)$, and in the next step this diffeomorphism is extended to a diffeomorphism of fixed reference regions $D^i$ onto the unknown domains $\Omega_i(t)$. The treatment of the moving boundary problem (1.3) then proceeds by transforming the equations into a new system of equations defined on the fixed domain $D_1 \cup D_2$ from which both the solution and the parameterizing function $\rho$ have to be determined. In the context of the Stefan problem this approach has first been used by Hanzawa [18].

The same approach has also been used in [10, 11] for the quasi-stationary approximation of the Stefan problem with surface tension, and in [26] for the Stefan problem with surface tension. Once the transformed system has been obtained, one can study the mapping properties of the nonlinearities involved, and in particular, one can determine their linearizations; see [26] for more details.

In this paper, we assume that $\Gamma(t)$ does not touch the fixed boundary $J_2 = \partial\Omega$. Under this assumption, we will characterize all of the equilibrium states $(u_1, u_2, \Sigma)$ of (1.3). In fact, it is easy to see that the equilibria are precisely given by

$$\Sigma = \bigcup_{j=1}^{m} S_R(x_j), \quad u_1 = u_2 = \sigma/R,$$

where $S_R(x_j)$ denotes disjoint spheres of the same radius $R$ and centers $x_j$. This can be seen by the following arguments: the equilibria $(u_1, u_2, \Sigma)$ of the Stefan problem (1.3) are given by the system of equations

$$\begin{cases} -d_i\Delta u_i = 0 & \text{in} \quad \Omega_i, \\ \partial_\nu u_2 = 0 & \text{on} \quad \partial\Omega, \\ u_i = \sigma H_\Sigma & \text{on} \quad \Sigma, \\ [d\partial_\nu u] = 0 & \text{on} \quad \Sigma. \end{cases} \tag{1.7}$$

Taking the inner product of $(1.7)_1$ with $u_i$, the divergence theorem and condition $(1.7)_4$ yield

$$\int_{\Omega_i} |\nabla u_i|^2\, dx = 0;$$

hence $u_i$ is constant on $\Omega_i$. Equation $(1.7)_3$, in turn, shows that $u_1 = u_2$ and also that $H = u/\sigma$ is constant on $\Sigma$. But then, since $\Omega$ is bounded, $\Sigma$ must be a sphere $S_R(x_0)$ centered at some point $x_0 \in \Omega$ with radius $R > 0$, if the phases are connected. Otherwise, again due to the boundedness of $\Omega$, $\Sigma$ is the union of finitely many spheres of the same radius $R > 0$. Here we concentrate on the case of connected phases. Thus there is an $(n + 1)$-parameter family of equilibria, the parameters being the $n$ coordinates of the center $x_0$ and the radius $R$.

We want to discuss the stability of those equilibria. The linearized problem (associated to the transformed equations) at such an equilibrium state is given by

$$\begin{cases} \kappa\partial_t v - d\Delta v = f & \text{in} \quad (\Omega\setminus\Sigma)\times\mathbb{R}_+, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega\times\mathbb{R}_+, \\ v = \sigma\mathcal{A}_\Sigma\rho + \delta\partial_t\rho + g & \text{on} \quad \Sigma\times\mathbb{R}_+, \\ l\partial_t\rho - [d\partial_\nu v] = h & \text{on} \quad \Sigma\times\mathbb{R}_+, \\ v(0) = v_0 & \text{in} \quad \Omega\setminus\Sigma, \\ \rho(0) = \rho_0 & \text{on} \quad \Sigma; \end{cases} \tag{1.8}$$

see [26]. Here, $l = \ell - [\kappa]\sigma/R$, and the operator $\mathcal{A}_\Sigma$ is given by

$$\mathcal{A}_\Sigma = -\frac{1}{n-1}\left(\frac{n-1}{R^2} + \Delta_\Sigma\right),$$

where $\Delta_\Sigma$ denotes the Laplace–Beltrami operator on $\Sigma$. This is the linearization of the mean curvature $H'(0)$ at the sphere $\Sigma$; cf., e.g., Escher and Simonett [11]. Here we use the notation $v = v_1\chi_{\Omega_1} + v_2\chi_{\Omega_2}$, where $\chi_G$ denotes the characteristic function of a set $G$, and similarly $\kappa = \kappa_1\chi_{\Omega_1} + \kappa_2\chi_{\Omega_2}$ and $d = d_1\chi_{\Omega_1} + d_2\chi_{\Omega_2}$. Associated to the linearization (1.8) is the following eigenvalue problem:

$$\begin{cases} \lambda\kappa v - d\Delta v = 0 & \text{in} \quad \Omega\setminus\Sigma, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega, \\ v = \sigma\mathcal{A}_\Sigma\rho + \lambda\delta\rho & \text{on} \quad \Sigma, \\ \lambda l\rho - [d\partial_\nu v] = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.9}$$

where as before $l = \ell - [\kappa]\sigma/R$. We will now state the main results of this paper. We will formulate our results for a domain in $\mathbb{R}^n$ for $n \in \mathbb{N}$, $n > 1$, although the physically relevant dimensions, naturally, are $n = 2, 3$.

THEOREM 1.1. *Suppose that the phases in the Stefan problem are connected. Then the following assertions hold:*

(a) *The equilibrium states (without boundary contact) for problem* (1.3) *are given by*

$$(u, \Sigma), \quad \text{where} \quad \Sigma = S_R(x_0) \quad \text{and} \quad u = \sigma/R,$$

*with $S_R(x_0) \subset \Omega$ being the sphere with radius $R$ and center $x_0$.*

(b) *For $l > 0$, the eigenvalue problem* (1.9) *has countably many real eigenvalues of finite algebraic multiplicity.*

(c) $0$ *is an eigenvalue of* (1.9) *with geometric multiplicity* $(n+1)$. *The (geometric) eigenspace is spanned by*

$$(-1, Y_0), (0, Y_1), \ldots, (0, Y_n),$$

*where $Y_0 = R^2/\sigma$, and where $Y_j$, $1 \le j \le n$, are the spherical harmonics of degree* 1 *(normalized by the orthogonality condition $(Y_i|Y_j)_\Sigma = \delta_{ij}$).*

(d) *If $\sigma(\kappa|1)_\Omega \le l|\Sigma|R^2$, then* (1.9) *has no positive eigenvalues.*

(e) *If $\sigma(\kappa|1)_\Omega > l|\Sigma|R^2 > 0$, then* (1.9) *has exactly one positive, algebraically simple eigenvalue.*

(f) *If $l < 0$ and $\delta > 0$, then* (1.9) *has at least one positive eigenvalue.*

(g) *If $l < 0$ and $\delta = 0$, then the linearized problem* (1.8) *is not well-posed.*

*Proof.* The assertion in (a) has been proved above. We refer to Theorem 2.1 for a proof of assertions (b)–(e), and for additional information about the eigenvalue problem (1.9), for the case $l > 0$. The proof of (f) is given at the end of section 5, and (g) follows from [7]. □

*Remark* 1.2. (a) If $l < 0$, then all equilibrium states are linearly unstable (and the linearized problem (1.8) is not even well-posed in case $\delta = 0$). Therefore we mainly concentrate on the case $l > 0$. Define then

$$\zeta := \frac{\sigma(\kappa|1)_\Omega}{l|\Sigma|R^2}, \quad \text{where} \quad l = \ell - \frac{\sigma[\kappa]}{R}, \quad (\kappa|1)_\Omega := \int_\Omega \kappa \, dx.$$

According to Theorem 1.1.(d)–(e), we know that all eigenvalues of (1.9) are nonpositive if $\zeta \le 1$, and that there exists exactly one positive simple eigenvalue if $\zeta > 1$. We will refer to the case $\zeta \le 1$ as a *stability condition.*

Observe that neither the thermal conductivity coefficients $d_i$ nor the kinetic coefficient $\delta$ enters this stability condition, as it depends only on the heat capacities $\kappa_i$, the latent heat $\ell$, the surface tension $\sigma$, and on the geometry. In particular, decreasing the size of a ball decreases its stability, as does increasing surface tension; see also Remark 1.5(a). We also mention that the stability condition $\zeta \le 1$ is always valid in the quasi-stationary case $\kappa_i = 0$, i.e., for the Mullins–Sekerka problem.

(b) It will be shown in the forthcoming paper [27] that solutions for the Stefan problem (1.3) that start out close to an equilibrium $(u, \Sigma)$ exist globally and converge towards an equilibrium state $(u', \Sigma')$ as time goes to infinity, provided that $l > 0$ and $\zeta < 1$. This gives justice to the wording *stability condition* for the case $\zeta < 1$. We note again that $\zeta = 0$ if the heat capacities $\kappa_i$ are zero, that is, in the quasi-stationary case. In this case, global existence and convergence to equilibria were obtained in [11, 20] by using a center-manifold analysis; see also [15] for a different approach in the one-phase case.

(c) If the Gibbs–Thomson condition on the free interface $\Gamma(t)$ is replaced by $u_i = 0$, then (1.3) is called the *(classical) Stefan model*. It should be observed that, in contrast to the problem with surface tension, the classical Stefan problem does not admit nontrivial equilibrium states.

For $l > 0$, the results in Theorem 1.1 suggest that one eigenvalue, $\lambda_*$, crosses the imaginary axis at 0 as the quantity $\zeta$ increases and exceeds 1. According to part (c) of Theorem 1.1, 0 is always an eigenvalue with geometric multiplicity $(n+1)$. This suggests that as the eigenvalue $\lambda_*$ crosses through 0, the algebraic multiplicity of 0 raises by one, and then drops again as soon as the eigenvalue has crossed. This is exactly what happens, as will be proved in Theorem 2.1.

Another way to view and understand this situation can be gained from considering the following parameter-dependent eigenvalue problem:

$$\begin{cases} \lambda_* s\kappa v - d\Delta v = 0 & \text{in} \quad \Omega\setminus\Sigma, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega, \\ v = \sigma\mathcal{A}_\Sigma\rho + \lambda_*\delta\rho & \text{on} \quad \Sigma, \\ \lambda_* l\rho - [d\partial_\nu v] = 0 & \text{on} \quad \Sigma. \end{cases} \tag{1.10}$$

The following result will be proved in section 6.

Theorem 1.3. *Let $l > 0$ and set $s_0 := l|\Sigma|R^2/\sigma(\kappa|1)_\Omega$. Then the following hold:*

(a) *The eigenvalue problem* (1.10) *has an analytic curve of solutions*

$$[s \mapsto (\lambda_*(s), v(s), \rho(s))], \quad s \in (s_0 - \varepsilon_0, \infty),$$

*such that $\lambda_*(s) > 0$ iff $s > s_0$, where $\varepsilon_0$ is an appropriate positive number.*

(b) *$\lambda_*(s)$ crosses the imaginary axis with positive speed at $s = s_0$.*

(c) *$[s \mapsto \lambda_*(s)]$ is strictly increasing.*

(d) *If $\delta > 0$, then $\lambda_*(s)$ is bounded above by $\sigma/\delta R^2$.*

(e) *If $\delta = 0$, then $\lambda_*(s) \to \infty$ as $s \to \infty$.*

Clearly, the eigenvalues of the modified problem (1.10) coincide with the eigenvalues of (1.9) if $s = 1$. In case $\zeta > 1$ we have $s_0 < 1$ and see that $\lambda = \lambda_*(1)$ is a (the only) positive eigenvalue of (1.9).

According to (1.5) an equilibrium state $(\sigma/R, S_R(x_0))$ for the Stefan problem (1.3) has energy

$$\begin{aligned} \phi(R) :&= E\left(\frac{\sigma}{R}, S_R(x_0)\right) = \frac{\sigma}{R}(\kappa|1)_\Omega + \ell\,|\Omega_1| \\ &= \frac{\sigma}{R}\big(\kappa_1|\Omega_1| + \kappa_2|\Omega_2|\big) + \ell\,|\Omega_1|, \end{aligned} \tag{1.11}$$

where $|\Omega_1| = R^n|B|$ and $|\Omega_2| = |\Omega| - R^n|B|$, with $|B|$ the volume of the unit ball. A straightforward computation shows that the function $\phi$ has a unique minimum. It is attained at the point $R_*$, where $R_*$ is the unique solution of

$$\frac{\sigma(\kappa|1)_\Omega}{R^2} = \left(\ell - \frac{\sigma[\kappa]}{R}\right)|S_R|, \tag{1.12}$$

with $|S_R| = |S_R(x_0)|$ being the area of the sphere $S_R(x_0)$.

In the following, we denote by $R_*$ the point where $\phi$ attains its (unique) minimum and by $R^*$ the largest number $R$ such that $\overline{B}_R(x_0) \subset \overline{\Omega}$, and we suppose that $R_* < R^*$. Then we have the following result; see also the stability diagram in Figure 1.

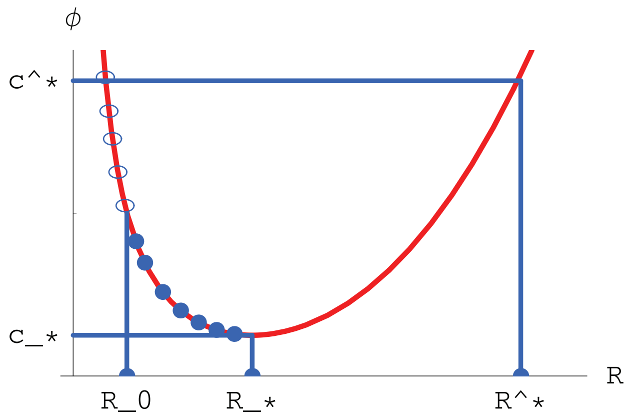


FIG. 1. *Stability diagram for $\kappa_1 < \kappa_2$ and $\delta = 0$; circled: ill-posed, dotted: unstable.*

COROLLARY 1.4. *Let $c_* = \phi(R_*)$ be the minimum value of $\phi$, and let $c^* = \phi(R^*)$. Moreover, let $c_0 = E(u_0, \Gamma_0) = \phi(R)$ be a given energy level.*

(a) *If $c_0 < c_*$, then problem (1.3) does not admit equilibrium states.*

(b) *If $c_* < c_0 < c^*$, then (1.3) admits two branches of equilibrium states. The branch of equilibria $(\sigma/R, S_R(x_0))$ with $0 < R < R_*$ is linearly unstable, whereas the branch with $R_* < R < R^*$ is linearly stable.*

(d) *In case $c_0 = c_*$ or $c_0 > c^*$, the Stefan problem (1.3) admits one family of equilibrium states. All equilibria $(\sigma/R, S_R(x_0))$ with $\phi(R) > c^*$ are linearly unstable.*

(e) *If $R_0 := [\kappa]\sigma/\ell > 0$ and $\delta = 0$, then the linearized problem is ill-posed for $R < R_0$.*

*Proof.* This follows from Remark 1.2(a), (1.12), and the fact that $\phi'(R)$ is negative for $R < R_*$ and positive for $R_* < R < R^*$. ∎

*Remark* 1.5. (a) We can show that $R_*$ is increasing with respect to $\sigma$ (and, for that matter, also with respect to $[\kappa]$). In order to see this, let $R_*(\sigma)$ denote the unique solution of (1.12). Then we have $R_*'(\sigma) > 0$. For this we note that (1.12) can be written as

$$n\ell|B|R_*^{n+1}(\sigma) - \sigma\big(\kappa_2|\Omega| + (n-1)[\kappa]|B|R_*^n(\sigma)\big) = 0. \tag{1.13}$$

Taking the derivative of this equation with respect to $\sigma$ yields

$$\begin{aligned} &n(n+1)\ell|B|R_*^{n-1}\left(R_* - \frac{(n-1)}{(n+1)}\frac{\sigma[\kappa]}{\ell}\right)R_*'(\sigma) \\ &= \kappa_2|\Omega| + (n-1)[\kappa]|B|R_*^n = (\ell/\sigma)n|B|R_*^{n+1}(\sigma) > 0. \end{aligned}$$

Note that we have used (1.13) for the last equality. It remains to be observed that the parenthetical expression in front of $R_*'(\sigma)$ is always positive. This is clear in case $[\kappa] \le 0$ and follows from the fact that $R_*$ is always greater than $R_0 = \sigma[\kappa]/l$ in case $[\kappa] > 0$. We therefore see that increasing $\sigma$ increases $R_*(\sigma)$, showing that spheres with a fixed radius $R$ can lose stability as $\sigma$ increases.

(b) If one considers the case where the domain $\Omega_1$ is occupied by the solid phase, and $\Omega_2$ by the fluid phase, then the third and fourth lines in (1.3) must be replaced by

$$\begin{aligned} u_i &= -\sigma H_\Gamma - \delta V && \text{on} \quad \Gamma(t), \\ -[d\partial_\nu u] &= (\ell + [\kappa]u)V && \text{on} \quad \Gamma(t), \end{aligned} \tag{1.14}$$

and the energy functional by

$$E(u(t), \Gamma(t)) := \int\limits_{\Omega} \kappa u \, dx + \ell\,|\Omega_2(t)| = \int\limits_{\Omega_1(t)} \kappa_1 u_1 \, dx + \int\limits_{\Omega_2(t)} \kappa_1 u_2 \, dx + \ell\,|\Omega_2(t)|,$$

while all other conventions are left unchanged. Thus, one formally has to switch signs in the normal $\nu$ and in $\ell$ and $[\kappa]$. Then all of the results and assertions stated in this paper remain valid for the equilibrium states $(-\sigma/R, S_R(x_0))$.

The plan of this paper is the following. In section 2 we will state a more general and concise version of Theorem 1.1; its proof will be given in sections 3–5. Finally, in section 6 we will prove Theorem 1.3.

**2. Main theorem.** In this section we will introduce an appropriate functional analytic setting to study the eigenvalue problem (1.9). We always assume $l > 0$ except when proving (f) of Theorem 1.1.

For the case $\delta = 0$ we define the operator $L_0$ on $E_0 := L_p(\Omega) \times W_p^{2-2/p}(\Sigma)$ by means of

$$\begin{aligned} D(L_0) :=& \big\{(v,\rho) \in W_p^2(\Omega \setminus \Sigma) \times W_p^{4-1/p}(\Sigma) : [d\partial_\nu v] \in W_p^{2-2/p}(\Sigma), \\ & \quad \partial_\nu v = 0 \text{ on } \partial\Omega,\ [v] = 0 \text{ on } \Sigma,\ \ v = \sigma \mathcal{A}_\Sigma \rho \text{ on } \Sigma\big\}, \\ L_0(v,\rho) :=& \big((d/\kappa)\Delta v, [(d/l)\partial_\nu v]\big), \quad (v,\rho) \in D(L_0). \end{aligned}$$

In case $\delta > 0$, we instead set $E_\delta := L_p(\Omega) \times W_p^{1-1/p}(\Sigma)$, and we define $L_\delta$ by

$$\begin{aligned} D(L_\delta) :=& \big\{(v,\rho) \in W_p^2(\Omega \setminus \Sigma) \times W_p^{3-1/p}(\Sigma) : \\ & \quad \partial_\nu v = 0 \text{ on } \partial\Omega,\ [v] = 0 \text{ on } \Sigma,\ v - (\delta/l)[d\partial_\nu v] = \sigma \mathcal{A}_\Sigma \rho \text{ on } \Sigma\big\}, \\ L_\delta(v,\rho) :=& \big((d/\kappa)\Delta v, [(d/l)\partial_\nu v]\big), \quad (v,\rho) \in D(L_\delta). \end{aligned}$$

We remark that $L_0$ and $L_\delta$ differ only by their respective domains of definition. It will be shown in [7] that the operators $L_\delta$ generate an analytic semigroup on $E_\delta$. This property, in conjunction with the spectral information contained in the next theorem, will be crucial in proving global existence and convergence of solutions for problem (1.3) that start out close to an equilibrium, which will be provided in [27]; see also Remark 1.2(c).

THEOREM 2.1. *Suppose* $1 < p < \infty$, *and let* $l > 0$. *For* $\delta \geq 0$ *let* $L_\delta$ *be defined as above.*

(a) *The spectrum of* $L_\delta$ *consists of countably many real eigenvalues of finite algebraic multiplicity and is independent of* $p$.

(b) $0$ *is an eigenvalue of* $L_\delta$ *with geometric multiplicity* $(n+1)$. *The null space of* $L_\delta$ *is spanned by*

$$(-1, Y_0), (0, Y_1), \ldots, (0, Y_n), \tag{2.1}$$

*where* $Y_0 = R^2/\sigma$, *and where* $Y_j$, $1 \leq j \leq n$, *are the spherical harmonics of degree* $1$ *(normalized by the orthogonality condition* $(Y_i|Y_j)_\Sigma = \delta_{ij}$*).*

(c) *Suppose that the degeneracy condition*

$$(\kappa|1)_\Omega := \kappa_1|\Omega_1| + \kappa_2|\Omega_2| = l|\Sigma|R^2/\sigma \tag{2.2}$$

*holds. Then the eigenvalue* 0 *has algebraic multiplicity* $(n+2)$.

(d) *If the degeneracy condition* (2.2) *does not hold, then* 0 *is semi-simple; that is,* $N(L_\delta^2) = N(L_\delta)$.

(e) *If* $\sigma(\kappa|1)_\Omega \le l|\Sigma|R^2$, *then* $L_\delta$ *has no positive eigenvalues.*

(f) *If* $\sigma(\kappa|1)_\Omega > l|\Sigma|R^2$, *then* $L_\delta$ *has exactly one positive simple eigenvalue.*

*Proof.* (a) By the compact embeddings $D(L_\delta) \hookrightarrow E_\delta$, the spectrum of $L_\delta$ consists of eigenvalues of finite algebraic multiplicity. The assertion that all eigenvalues are real will be proved in section 4.

Let $1 < p < \infty$ be fixed, and suppose that $\lambda$ is an eigenvalue of $L_\delta$ with a corresponding eigenfunction $(v, \rho)$. Then $v \in W_p^2(\Omega \setminus \Sigma)$, and $v$ solves the elliptic transmission problem

$$\begin{cases} \kappa\lambda v - d\Delta v = 0 & \text{in} \quad \Omega \setminus \Sigma, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega, \\ [v] = 0 & \text{on} \quad \Sigma, \\ -[d\partial_\nu v] = -\lambda l \rho & \text{on} \quad \Sigma, \end{cases}$$

with $\rho \in W_p^{4-\text{sign}(\delta)-1/p}(\Sigma)$, where $\text{sign}(\delta) = 1$ if $\delta > 0$, and $\text{sign}(\delta) = 0$ if $\delta = 0$. Due to Sobolev's imbedding theorem we have that $\rho \in W_{p_1}^{1-1/p_1}(\Sigma)$, where $p_1 \in (p, \infty)$ is appropriately chosen. Proposition 5.1 then yields $v \in W_{p_1}^2(\Omega \setminus \Sigma)$. Next, we recall that $\rho$ satisfies

$$\sigma \mathcal{A}_\Sigma \rho = v - (\delta/l)[d\partial_\nu v] =: h \quad \text{on} \quad \Sigma.$$

Since $v \in W_{p_1}^2(\Omega \setminus \Sigma)$ we see that $h \in W_{p_1}^{2-\text{sign}(\delta)-1/p_1}(\Sigma)$, and we obtain from the properties of the elliptic differential operator $\mathcal{A}_\Sigma$ that $\rho \in W_{p_1}^{4-\text{sign}(\delta)-1/p_1}(\Sigma)$. The arguments given above can now be iterated a finite number of times to show that

$$(v, \rho) \in W_q^2(\Omega \setminus \Sigma) \times W_q^{4-\text{sign}(\delta)-1/q}(\Sigma)$$

for any fixed $q > p$. Clearly, this is also true for any $q < p$. We have, thus, shown that the spectrum of $L_\delta$ is independent of $p$. The properties listed in (b)–(d) are proved in section 3, and assertion (e) is shown in section 4 while (f) is established in section 5. $\square$

PROPOSITION 2.2. *Let* $1 < p < \infty$. *Suppose that* $(\lambda, v, \rho) \in \mathbb{R} \times W_p^2(\Omega \setminus \Sigma) \times W_p^2(\Sigma)$ *solves the eigenvalue problem* (1.9). *Then the functions* $(v, \rho)$ *are smooth; that is,*

$$v|_{\Omega_i} \in C^\infty(\bar{\Omega}_i), \quad \rho \in C^\infty(\Sigma).$$

*Proof.* This follows from a similar bootstrapping argument as in the proof of Theorem 2.1(a), based on regularity properties of the elliptic transmission problems (3.4) and (5.2), and regularity properties of the differential operator $\mathcal{A}_\Sigma$. $\square$

Due to Theorem 2.1 and Proposition 2.2 we may restrict our attention to the eigenvalue problem (1.9) in the Hilbert space setting of $L_2(\Omega) \times L_2(\Sigma)$. In the following, we use the notation $(\cdot|\cdot)_\Omega$ and $\|\cdot\|_\Omega$ for the inner product and the norm in $L_2(\Omega)$, respectively, and similarly for $L_2(\Sigma)$.

**3. The trivial eigenvalue.** Let us first look at the eigenvalue problem (1.9) with $\lambda = 0$. Obviously, here $l \in \mathbb{R}$ can be arbitrary, and also $\delta \in \mathbb{R}$. For this purpose we recall some properties of the operator $\mathcal{A}_\Sigma$.

PROPOSITION 3.1. *Let $\Sigma = S_R(x_0) \subset \mathbb{R}^n$ be a sphere of radius $R$ and center $x_0$, and let*

$$A_\Sigma = -\frac{1}{n-1}\left(\frac{n-1}{R^2} + \Delta_\Sigma\right)$$

*be defined on $L_2(\Sigma)$ with domain $W_2^2(\Sigma)$. Then the following assertions hold:*

(a) *$A_\Sigma$ is self-adjoint. Its spectrum consists of countably many eigenvalues $\lambda_k = \frac{1}{(n-1)R^2}\big(k(k+n-2)-(n-1)\big)$ with $k \geq 0$. The eigenfunctions are given by the spherical harmonics of degree $k$.*

(b) *The kernel of $A_\Sigma$ is given by $N(A_\Sigma) = \mathrm{span}\{Y_1, \ldots, Y_n\}$, where $Y_j$ denotes the spherical harmonics of degree 1 on $\Sigma$, normalized by $(Y_i|Y_j)_\Sigma = \delta_{ij}$.*

(c) *The range of $A_\Sigma$, $R(A_\Sigma)$ is closed, and we have $L_2(\Sigma) = N(A_\Sigma) \oplus R(A_\Sigma)$.*

(d) *There is precisely one negative eigenvalue, namely $-1/R^2$, with eigenfunction 1, which is simple.*

(e) *$A_\Sigma$ is positive semi-definite on $L_{2,0}(\Sigma) = \{\rho \in L_2(\Sigma) : (\rho|1)_\Sigma = 0\}$ and positive definite on*

$$L_{2,0}(\Sigma) \cap R(A_\Sigma) = \{\rho \in L_2(\Sigma) : (\rho|1)_\Sigma = (\rho|Y_j)_\Sigma = 0,\ j = 1, \ldots, n\}.$$

*Proof.* We can assume, without loss of generality, that $\Sigma = S_R(0) = R\,\mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ denotes the standard unit sphere in $\mathbb{R}^n$. Let $\Phi : \Sigma \to \mathbb{S}^{n-1}$ be defined by $p \mapsto (1/R)p$. Then $\Phi$ is a smooth diffeomorphism of $\Sigma$ into $\mathbb{S}^{n-1}$, and one readily verifies that

$$(g|h)_{L_2(\Sigma)} = R^{n-1}(\Phi_* g|\Phi_* h)_{L_2(\mathbb{S}^{n-1})}, \quad \Delta_\Sigma = (1/R^2)\, \Phi^* \Delta_{\mathbb{S}^{n-1}}\, \Phi_*, \tag{3.1}$$

where $\Phi^*$ and $\Phi_*$ are the pull-back and push-forward operators, respectively. We then have

$$(\lambda - A_\Sigma)\rho = 0 \iff \left(\lambda + \frac{1}{(n-1)R^2}\big((n-1) + \Delta_{\mathbb{S}^{n-1}}\big)\right)\Phi_*\rho = 0, \tag{3.2}$$

which shows that $\lambda$ is an eigenvalue of $A_\Sigma$ iff

$$\lambda = \frac{1}{(n-1)R^2}\big(\mu - (n-1)\big) \tag{3.3}$$

with $\mu$ an eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$. The assertions in (a)–(b) and (d)–(e) follow now from (3.1)–(3.3) and well-known results for the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$; see, for instance, [30, section 31]. Since $A_\Sigma$ has compact resolvent we conclude that $R(A_\Sigma)$ is closed, and the fact that $A_\Sigma$ is self-adjoint then implies the remaining assertion in (c). ∎

Before we proceed we need the following result on the elliptic transmission problem:

$$\begin{cases} -d\Delta v = f & \text{in} \quad \Omega \setminus \Sigma, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega, \\ [v] = 0 & \text{on} \quad \Sigma, \\ -[d\partial_\nu v] = g & \text{on} \quad \Sigma. \end{cases} \tag{3.4}$$

PROPOSITION 3.2. *Let $1 < p < \infty$. Then the following hold:*

(a) *The transmission problem* (3.4) *has a solution $v \in W_p^2(\Omega \setminus \Sigma)$ if and only if $(f, g) \in L_p(\Omega) \times W_p^{1-1/p}(\Sigma)$ and the compatibility condition*

$$(f|1)_\Omega + (g|1)_\Sigma = 0$$

*is satisfied. The solution is unique with the normalization $(\kappa|v)_\Omega = 0$.*

(b) *Let $v = T_0 g$ be the unique solution of* (3.4) *with $f = 0$, $(g|1)_\Sigma = 0$, and $(\kappa|v)_\Omega = 0$. Then $T_0$ is self-adjoint and positive definite on $L_2(\Sigma)$; that is, there exists a positive constant $c = c(d_i, \Omega_i)$ such that*

$$(T_0 g|g)_{L_2(\Sigma)} \geq c\, \|g\|^2_{L_2(\Sigma)}, \qquad g \in W_2^{1/2}(\Sigma).$$

*Proof.* (a) This proof follows from known results in elliptic theory since the Lopatinskii–Shapiro conditions are satisfied.

(b) Let $g, h \in W_2^{1/2}(\Sigma)$ be given. Then we have

$$\begin{aligned}(T_0 g|h)_\Sigma = (T_0 g|[-d\partial_\nu T_0 h])_\Sigma &= (d\,\nabla T_0 g|\nabla T_0 h)_\Omega \\ &= (-[d\partial_\nu T_0 g]|T_0 h)_\Sigma = (g|T_0 h)_\Sigma,\end{aligned}$$

thus showing that $T_0$ is symmetric. For $v := T_0 g$ the computation above yields

$$(T_0 g|g)_\Sigma = (d\,\nabla v|\nabla v)_\Omega.$$

On the other hand, setting $v_i = v|_{\Omega_i}$ we obtain

$$\begin{aligned}\|g\|_{L_2(\Sigma)} &= \|d_1\partial_\nu v_1 - d_2\partial_\nu v_2\|_{L_2(\Sigma)} \leq c\big(\|v_1\|_{W_2^2(\Omega_1)} + \|v_2\|_{W_2^2(\Omega_2)}\big) \\ &\leq c\big(\|v_1\|_{L_2(\Omega_1)} + \|\Delta v_1\|_{L_2(\Omega_1)} + \|v_2\|_{L_2(\Omega_2)} + \|\Delta v_2\|_{L_2(\Omega_2)}\big) \\ &= c\|v\|_{L_2(\Omega)} \leq c\|v\|_{W_2^1(\Omega)} \leq c\|\nabla v\|_{L_2(\Omega)} \leq c(T_0 g|g)_\Sigma^{1/2}.\end{aligned}$$

Here we used the fact that

$$v = T_0 g \in W_2^1(\Omega) \cap W_2^2(\Omega \setminus \Sigma).$$

Moreover, we used that $\big(\|\cdot\|_{L_2(\Omega_i)} + \|\Delta \cdot\|_{L_2(\Omega_i)}\big)$ defines an equivalent norm on $W_2^2(\Omega_i)$, and also that $\|\nabla u\|_{L_2(\Omega)}$ defines an equivalent norm on $W_2^1(\Omega)$ for all functions $u \in W_2^1(\Omega)$ with $(\kappa|u)_\Omega = 0$. This completes the proof of Proposition 3.2. ∎

We are now ready to establish the assertions (b)–(d) of Theorem 2.1.

(b) Suppose that $(v, \rho)$ is a solution of (1.9) with $\lambda = 0$. Then, taking the inner product of $(1.9)_1$ with $v$, the divergence theorem and $(1.9)_{2,4}$ show that $v$ is constant on $\Omega \setminus \Sigma$; hence, $v$ is constant on $\Omega$ and $v = \sigma \mathcal{A}_\Sigma \rho$ due to $(1.9)_3$. A special solution of this problem is $\rho_0 = -R^2 v/\sigma$, and the solutions of the corresponding homogeneous equation are the spherical harmonics $Y_j$ on $\Sigma$ for $j = 1, \ldots, n$. Thus we obtain an $(n+1)$-dimensional null space spanned by(2.1), which proves Theorem 2.1(b).

This null space is tangent to the (n+1)-dimensional manifold of equilibria, where $(0, Y_j)$ corresponds to the center $x_0$, and $(-1, Y_0)$ corresponds to the radius $R$. Note that the null spaces of $L_0$ and $L_\delta$, $\delta > 0$, coincide.

(c) Suppose that (2.2) holds. Then there exists a pair $(v^*, \rho^*) \in N(L_\delta^2) \setminus N(L_\delta)$. Indeed, this can be seen as follows: we first solve (3.4) with $(f, g) = (-\kappa, lR^2/\sigma)$. According to Proposition 3.2, this problem has a unique solution $v_0$ with $(\kappa|v_0)_\Omega = 0$ since the necessary compatibility condition is precisely (2.2).

Set $v^* = v_0 + l\sum_{j=1}^n \alpha_j T_0 Y_j$. Then $v^*$ satisfies

$$(3.5)\qquad \begin{cases} -(d/\kappa)\Delta v^* = -1 & \text{in} \quad \Omega\setminus\Sigma, \\ \partial_\nu v^* = \ 0 & \text{on} \quad \partial\Omega, \\ [v^*] = \ 0 & \text{on} \quad \Sigma, \\ -[(d/l)\partial_\nu v^*] = \ R^2/\sigma + \Sigma_{j=1}^n \alpha_j Y_j & \text{on} \quad \Sigma. \end{cases}$$

We now want to solve $v^* = \sigma A_\Sigma \rho + (\delta/l)[d\partial_\nu v^*]$ in terms of $\rho$; that is, we consider the problem

$$\sigma A_\Sigma \rho = v^* - (\delta/l)[d\partial_\nu v^*] =: h.$$

According to Proposition 3.1(b)–(c) this problem has a solution $\rho^*$ iff $(h|Y_i)_\Sigma = 0$ for $i = 1, \ldots, n$. The conditions $(h|Y_i)_\Sigma = 0$ will then be employed to determine the coefficients $\alpha_j$. A short computation yields

$$(h|Y_i)_\Sigma = 0 \quad \Leftrightarrow \quad \delta\alpha_i + \sum_{j=1}^n l(T_0 Y_j|Y_i)_\Sigma\, \alpha_j = -(v_0|Y_i)_\Sigma, \quad i = 1, \ldots, n.$$

Since $T_0$ is self-adjoint and positive definite on $L_2(\Sigma)$, there exists a unique solution of this system, as we shall see in (6.7). Due to $\sigma A_\Sigma(R^2/\sigma + \sum_{j=1}^n \alpha_j Y_j) = -1$ (see Proposition 3.1), we conclude that $(v^*, \rho^*) \in D(L_\delta^2)$. It is then easy to see that $L_\delta(v^*, \rho^*) \neq (0,0)$ and $L_\delta^2(v^*, \rho^*) = (0,0)$. These facts in combination with part (b) show that

$$(3.6)\qquad N(L_\delta^2) = N(L_\delta) \oplus \operatorname{span}\{(v^*, \rho^*)\},$$

in the degenerate case where (2.2) holds.

Next we show, still in the degenerate case (2.2), that $N(L_\delta^3) = N(L_\delta^2)$. In fact, if $(v, \rho) \in N(L_\delta^3)$, then $L_\delta(v, \rho) = (v_N, \rho_N) + \beta(v^*, \rho^*)$ for some $(v_N, \rho_N) \in N(L_\delta)$ and some scalar $\beta$. For solvability of this equation, the compatibility condition

$$\big(\kappa(v_N + \beta v^*)|1\big)_\Omega + l\big(\rho_N + \beta\rho^*|1\big)_\Sigma = 0$$

must be valid. Due to the degeneracy condition we have $(\kappa v_N|1)_\Omega + l(\rho_N|1)_\Sigma = 0$, and the compatibility condition is reduced to $\beta\big\{(\kappa v^*)|1)_\Omega + l(\rho^*|1)_\Sigma\big\} = 0$. Using the property that $-(d/\kappa)\Delta v^* = -1$ (see (3.5)), we obtain

$$\begin{aligned} -\big\{(\kappa v^*|1)_\Omega + l(\rho^*|1)_\Sigma\big\} &= -(d\Delta v^*|v^*)_\Omega - l(\rho^*|1)_\Sigma \\ &= \|\sqrt{d}\,\nabla v^*\|_\Omega^2 + ([d\partial_\nu v^*]|v^*)_\Sigma - l(\rho^*|1)_\Sigma \\ &= \|\sqrt{d}\,\nabla v^*\|_\Omega^2 - l\left(R^2/\sigma + \sum_{j=1}^n \alpha_j Y_j|\sigma A_\Sigma \rho^*\right)_\Sigma + (\delta/l)\|[d\partial_\nu v^*]\|_\Sigma^2 - l(\rho^*|1)_\Sigma \\ &= \|\sqrt{d}\,\nabla v^*\|_\Omega^2 + (\delta/l)\|[d\partial_\nu v^*]\|_\Sigma^2 \end{aligned}$$

since $A_\Sigma$ is self-adjoint on $L_2(\Sigma)$ and $\sigma A_\Sigma(R^2/\sigma + \sum_{j=1}^n \alpha_j Y_j) = -1$; see Proposition 3.1. This implies $\beta = 0$, i.e., $(v, \rho) \in N(L_\delta^2)$, thus establishing Theorem 2.1(c).

(d) Let us examine when the eigenvalue $\lambda_0 = 0$ of $L_\delta$ is semi-simple. Assume that $(v, \rho) \in D(L_\delta^2)$ is such that $L_\delta^2(v, \rho) = 0$. Then

$$L_\delta(v, \rho) = \alpha_0(-1, Y_0) + \sum_{j=1}^n \alpha_j(0, Y_j).$$

This implies that

$$\begin{cases} -(d/\kappa)\Delta v = \alpha_0 & \text{in} \quad \Omega \setminus \Sigma, \\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega, \\ [v] = 0 & \text{on} \quad \Sigma, \\ -[(d/l)\partial_\nu v] = -\sum_{j=0}^{n} \alpha_j Y_j & \text{on} \quad \Sigma, \\ v = \sigma \mathcal{A}_\Sigma \rho + (\delta/l)[d\partial_\nu v] & \text{on} \quad \Sigma. \end{cases}$$

According to Proposition 3.2 we necessarily have

$$\alpha_0(\kappa_1|\Omega_1| + \kappa_2|\Omega_2|) = l\sum_{j=0}^{n} \alpha_j (Y_j|1)_\Sigma = l\alpha_0|\Sigma|R^2/\sigma,$$

since the mean value of $Y_j$ over $\Sigma$ is zero for $j \geq 1$. Assuming the nondegeneracy condition

$$(\kappa|1)_\Omega := \kappa_1|\Omega_1| + \kappa_2|\Omega_2| \neq l|\Sigma|R^2/\sigma, \tag{3.7}$$

we conclude that $\alpha_0 = 0$. But then

$$0 = -\int_\Omega d\Delta v v\, dx = \int_\Omega d|\nabla v|^2\, dx + \int_\Sigma [d\partial_\nu v] v\, ds,$$

which further yields

$$0 = \|\sqrt{d}\,\nabla v\|_\Omega^2 + l\sigma \sum_{j=1}^{n} \alpha_j (Y_j|\mathcal{A}_\Sigma \rho)_\Sigma + (\delta/l)\|[d\partial_\nu v]\|_\Sigma^2 = \|\sqrt{d}\,\nabla v\|_\Omega^2 + (\delta/l)\|[d\partial_\nu v]\|_\Sigma^2$$

since $\mathcal{A}_\Sigma$ is self-adjoint on $L_2(\Sigma)$ and $A_\Sigma Y_j = 0$ for $j \geq 1$. We conclude that $v$ is constant in $\Omega$ and that $0 = [d\partial_\nu v] = l\sum_{j=1}^{n} \alpha_j Y_j$; hence, $\alpha_j = 0$ for all $j$. This shows that $\lambda_0 = 0$ is a semi-simple eigenvalue of $L_\delta$, that is, $N(L_\delta^2) = N(L_\delta)$ for $\delta \geq 0$, provided the nondegeneracy condition (3.7) is valid, and this proves the assertion of Theorem 2.1(d).

**4. Nontrivial eigenvalues.** Now we consider the eigenvalue problem (1.9) for $\lambda \in \mathbb{C}$, $\lambda \neq 0$, in case $l > 0$. Suppose that $\lambda \neq 0$ is an eigenvalue with nontrivial eigenfunction $(v, \rho)$. Taking the inner product in $L_2(\Omega)$ of the first equation in (1.9) with $v$ and using the divergence theorem, we get

$$\begin{aligned} \lambda\|\sqrt{\kappa} v\|_\Omega^2 = (d\Delta v|v)_\Omega &= -\|\sqrt{d}\,\nabla v\|_\Omega^2 - ([d\partial_\nu v]|v)_\Sigma \\ &= -\|\sqrt{d}\,\nabla v\|_\Omega^2 - (\delta/l)\|[d\partial_\nu v]\|_\Sigma^2 - \lambda l\sigma(\rho|A_\Sigma\rho)_\Sigma; \end{aligned}$$

hence, we obtain the identity

$$\lambda\Big(\|\sqrt{\kappa} v\|_\Omega^2 + l\sigma(\rho|A_\Sigma\rho)_\Sigma\Big) + \|\sqrt{d}\,\nabla v\|_\Omega^2 + (\delta/l)\|[d\partial_\nu v]\|_\Sigma^2 = 0. \tag{4.1}$$

If $\operatorname{Im}\lambda \neq 0$, then $\|\sqrt{\kappa} v\|_\Omega^2 + l\sigma(\rho|A_\Sigma\rho)_\Sigma = 0$; hence, $\|\sqrt{d}\,\nabla v\|_\Omega^2 + (\delta/l)\|[d\partial_\nu]\|_\Sigma^2 = 0$. We conclude that $v$ is constant and $(1.9)_{1,4}$ now implies that $(v, \rho) = (0, 0)$ since

$\lambda \neq 0$. Therefore the eigenvalues and eigenfunctions are real, thus establishing Theorem 2.1(a).

Using $\lambda l \rho = [d\partial_\nu v]$ and the fact that $\lambda$ is real, we may rewrite (4.1) as

$$\lambda\Big(\|\sqrt{\kappa}v\|_\Omega^2 + l\sigma(\rho|A_\Sigma\rho)_\Sigma + \lambda l\delta\|\rho\|_\Sigma^2\Big) + \|\sqrt{d}\,\nabla v\,\|_\Omega^2 = 0. \tag{4.2}$$

Integrating the eigenvalue equation (1.9) we obtain

$$\lambda(v|\kappa)_\Omega = (d\Delta v|1)_\Omega = -([d\partial_\nu v]|1)_\Sigma = -\lambda l(\rho|1)_\Sigma;$$

hence, dividing by $\lambda$,

$$(v|\kappa)_\Omega + l(\rho|1)_\Sigma = 0. \tag{4.3}$$

Splitting $\rho = \rho_0 + \bar\rho$ and $v = v_0 + \bar v$, where $(\rho_0|1)_\Sigma = (v_0|\kappa)_\Omega = 0$, from (4.2) and (4.3) we derive an identity equivalent to (4.2), namely,

$$\begin{aligned}\lambda\Big(\|\sqrt{\kappa}v_0\|_\Omega^2 + l\sigma(\rho_0|A_\Sigma\rho_0)_\Sigma + \lambda l\delta\|\rho_0\|_\Sigma^2\Big) + \|\sqrt{d}\,\nabla v\|_\Omega^2 \\ +\lambda\Big(\lambda\delta + \frac{l|\Sigma|}{(\kappa|1)_\Omega} - \frac{\sigma}{R^2}\Big)l|\Sigma|\bar\rho^2 = 0.\end{aligned} \tag{4.4}$$

Since $A_\Sigma$ is positive semi-definite on $L_{2,0}(\Sigma)$, the $L_2$-functions with mean zero, we see that in case $\lambda > 0$, (4.4) implies $v =$ constant, and hence $(v,\rho) = (0,0)$, provided that

$$(\kappa|1)_\Omega \le l|\Sigma|R^2/\sigma. \tag{4.5}$$

Consequently, (1.9) cannot have positive eigenvalues if the stability condition (4.5) is satisfied, thus proving the assertion of Theorem 2.1(e).

## 5. The unstable eigenvalue.

As far as we know, for $l > 0$ and

$$\zeta := \frac{\sigma(\kappa|1)_\Omega}{l|\Sigma|R^2} \le 1 \tag{5.1}$$

there are no positive eigenvalues; however, the algebraic eigenspace of $L_\delta$ rises in dimension by one when $\zeta$ becomes 1. This indicates that for $\zeta > 1$ there is exactly one algebraically simple eigenvalue $\lambda_* > 0$. We want to prove that this is indeed the case. In order to do so, we consider the following transmission problem:

$$\begin{cases} \lambda\kappa v - d\Delta v = f & \text{in } \Omega\setminus\Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \\ [v] = 0 & \text{on } \Sigma, \\ -[d\partial_\nu v] = g & \text{on } \Sigma. \end{cases} \tag{5.2}$$

Then the following result holds.

PROPOSITION 5.1. *Let $1 < p < \infty$ and $\operatorname{Re}\lambda > 0$. Then the following hold:*

(a) *Problem (5.2) has precisely one solution $v \in W_p^1(\Omega) \cap W_p^2(\Omega\setminus\Sigma)$ iff $(f,g) \in L_p(\Omega) \times W_p^{1-1/p}(\Sigma)$.*

(b) *Let $T_\lambda$ be the solution operator for (5.2) with $f = 0$. Given any number $\theta \in (0,\pi)$, there exist positive numbers $\lambda_0 = \lambda_0(\theta, d_i, \kappa_i, \Omega_i)$ and $M_0 = M_0(\theta, d_i, \kappa_i, \Omega_i)$ such that*

$$\|T_\lambda g\|_{L_p(\Sigma)} \le M_0|\lambda|^{-1/2}\|g\|_{L_p(\Sigma)}$$

*for $g \in W_p^{1-1/p}(\Sigma)$, whenever $|\lambda| \ge \lambda_0$ and $|\arg\lambda| \le \theta$.*

(c) *For $\lambda > 0$, $T_\lambda$ is positive definite on $L_2(\Sigma)$; that is, there exists a positive constant $\beta = \beta(d_i, \kappa_i, \Omega_i)$ such that*

$$(T_\lambda g|g)_{L_2(\Sigma)} \geq \beta \frac{\sqrt{\lambda}}{1+\lambda} \|g\|^2_{L_2(\Sigma)}, \qquad g \in W_2^{1/2}(\Sigma).$$

*Proof.* (a) This proof follows from known results in elliptic theory.

(b) Suppose that $\Omega_1 = \mathbb{R}^n_-$ and $\Omega_2 = \mathbb{R}^n_+$, with $\mathbb{R}^n_\pm = \{(x', x_n) \in \mathbb{R}^n : \pm x_n < 0\}$. Then one readily obtains that

$$T_\lambda g|_{\mathbb{R}^{n-1}\times\{0\}} = \mathcal{F}^{-1}\left(m_\lambda \mathcal{F} g\right),$$

where $\mathcal{F}$ denotes the Fourier transform in the tangential variables, and where

$$m_\lambda(\xi) = \frac{1}{\sqrt{d_1}\sqrt{\kappa_1\lambda + d_1|\xi|^2} + \sqrt{d_2}\sqrt{\kappa_2\lambda + d_2|\xi|^2}}.$$

The assertion then follows from Mikhlin's multiplier theorem. The general case can be obtained by the usual procedure of localization.

(c) Let $g, h \in W_2^{1/2}(\Sigma)$ be given. Then we have

$$\begin{aligned}(T_\lambda g|h)_\Sigma = (T_\lambda g|[-d\partial_\nu T_{\bar\lambda} h])_\Sigma &= \lambda(\kappa\, T_\lambda g|T_{\bar\lambda} h)_\Omega + (d\,\nabla T_\lambda g|\nabla T_{\bar\lambda} h)_\Omega \\ &= (-[d\partial_\nu T_\lambda g]|T_{\bar\lambda} h)_\Sigma = (g|T_{\bar\lambda} h)_\Sigma,\end{aligned}$$

showing that $T_\lambda^* = T_{\bar\lambda}$, in particular, that $T_\lambda$ is symmetric for $\lambda > 0$. For $v := T_\lambda g$, $\lambda > 0$, the computation above yields

$$(T_\lambda g|g)_\Sigma = \lambda(\kappa v|v)_\Omega + (d\,\nabla v|\nabla v)_\Omega. \tag{5.3}$$

Setting $v_i = v|_{\Omega_i}$ we conclude similarly as in the proof of Proposition 3.2 that

$$\begin{aligned}\|g\|_{L_2(\Sigma)} &= \|d_1\partial_\nu v_1 - d_2\partial_\nu v_2\|_{L_2(\Sigma)} \leq c\big(\|d_1 v_1\|_{W_2^2(\Omega_1)} + \|d_2 v_2\|_{W_2^2(\Omega_2)}\big) \\ &\leq c\big(\|d_1 v_1\|_{L_2(\Omega_1)} + \|d_1\Delta v_1\|_{L_2(\Omega_1)} + \|d_2 v_2\|_{L_2(\Omega_2)} + \|d_2\Delta v_2\|_{L_2(\Omega_2)}\big) \\ &= c\big(\|d_1 v_1\|_{L_2(\Omega_1)} + \lambda\|\kappa_1 v_1\|_{L_2(\Omega_1)} + \|d_2 v_2\|_{L_2(\Omega_2)} + \lambda\|\kappa_2 v_2\|_{L_2(\Omega_2)}\big) \\ &\leq c_\lambda\sqrt{\lambda}\|v\|_{L_2(\Omega)} \leq c_\lambda\big(\sqrt{\lambda}\|v\|_{L_2(\Omega)} + \|\nabla v\|_{L_2(\Omega)}\big) \leq c_\lambda\,(T_\lambda g|g)_\Sigma^{1/2},\end{aligned}$$

where $c_\lambda = c(d_i, \kappa_i, \Omega_i)(1+\lambda)/\sqrt{\lambda}$. In the estimates above we have used that $v = T_\lambda g \in W_2^1(\Omega)\cap W_2^2(\Omega\setminus\Sigma)$, and that $\big(\|\cdot\|_{L_2(\Omega_i)} + \|\Delta\cdot\|_{L_2(\Omega_i)}\big)$ defines an equivalent norm on $W_2^2(\Omega_i)$ and, lastly, we employed (5.3). This completes the proof of Proposition 5.1. ∎

We assume now that $\lambda > 0$ is a fixed number. For given $\rho \in W_2^{1/2}(\Sigma)$, let $v$ be the solution of the transmission problem

$$\begin{cases} \lambda\kappa v - d\Delta v = 0 & \text{in } \Omega\setminus\Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \\ [v] = 0 & \text{on } \Sigma, \\ -[d\partial_\nu v] = -\lambda l\rho & \text{on } \Sigma. \end{cases} \tag{5.4}$$

Then $v = T_\lambda(-\lambda l\rho) = -\lambda l T_\lambda\rho$ with $T_\lambda$ being the solution operator introduced above. By inserting this representation of $v$ into the equation $v = \sigma A_\Sigma\rho + \lambda\delta\rho$, we obtain the problem

$$\lambda\delta\rho + \lambda l T_\lambda\rho + \sigma A_\Sigma\rho = 0, \tag{5.5}$$

which is equivalent to the eigenvalue problem.
Setting $B_\lambda(s) := \lambda\delta I + \lambda l T_\lambda + s\sigma A_\Sigma$ for $s > 0$ and employing Proposition 5.1(c), we obtain the estimate

$$\begin{aligned}(B_\lambda(s)\rho|\rho)_\Sigma &\geq \lambda(\delta+\gamma l)\|\rho\|_\Sigma^2 + s\sigma(A_\Sigma\rho|\rho)_\Sigma \\ &= \lambda(\delta+\gamma l)\|\rho_0\|_\Sigma^2 + s\sigma(A_\Sigma\rho_0|\rho_0)_\Sigma + \big\{\lambda(\delta+\gamma l) - s\sigma/R^2\big\}|\Sigma|\bar\rho^2,\end{aligned}$$

where $\gamma = \beta\sqrt{\lambda}/(1+\lambda)$ and $\rho = \rho_0 + \bar\rho$ with $(\rho_0|1)_\Sigma = 0$. Since $(A_\Sigma\rho_0|\rho_0)_\Sigma \geq 0$ we see that all of the terms in the previous line are nonnegative, provided $\lambda(\delta+\gamma l) \geq s\sigma/R^2$, i.e., for small $s$. Hence, for small $s > 0$, the operator $B_\lambda(s)$ is positive definite, which means that $\lambda$ cannot be an eigenvalue of (1.9), where in the third line of (1.9) $\sigma$ is replaced by $s\sigma$. On the other hand, choosing $\rho = 1$ we have

$$(B_\lambda(s)1|1)_\Sigma = \lambda\delta|\Sigma| + \lambda l(T_\lambda 1|1)_\Sigma - s\sigma|\Sigma|/R^2 < 0$$

if $s$ becomes large. Now we set

$$s_* := s_*(\lambda) := \sup\{s > 0 :\ B_\lambda(s) \text{ is positive definite}\}.$$

Then $B_\lambda(s_*)$ is still semi-definite, but not definite, and hence, by compactness of the resolvent, has a nontrivial kernel. Therefore, for a given $\lambda > 0$ there is an $s_* = s_*(\lambda)$ such that $\lambda$ is an eigenvalue of (1.9), where $\sigma A_\Sigma$ is replaced by $s_*\sigma A_\Sigma$ in the third line.

Next, we show that positive eigenvalues are simple. Rewrite (5.5) as

$$\lambda\delta\rho_0 + \lambda l T_\lambda\rho_0 + \sigma A_\Sigma\rho_0 = -\big\{\lambda\delta + \lambda l T_\lambda 1 - \sigma/R^2\big\}\bar\rho.$$

Since $B_\lambda$ is positive definite on $L_{2,0}(\Sigma)$, this equation has precisely one solution for given $\bar\rho$, which shows that the eigenspace $N(\lambda - L_\delta)$ is at most one-dimensional for any given $\lambda > 0$.

To show that nontrivial eigenvalues are semi-simple, suppose that

$$(\lambda - L_\delta)(v,\rho) = (v_1,\rho_1), \quad (\lambda - L_\delta)(v_1,\rho_1) = 0.$$

Then by Green's formula

$$\begin{aligned}\|\sqrt{\kappa}v_1\|_\Omega^2 &= (\lambda\kappa v - d\Delta v|v_1)_\Omega \\ &= (v|\lambda\kappa v_1 - d\Delta v_1)_\Omega + ([d\partial_\nu v]|v_1)_\Sigma - (v|[d\partial_\nu v_1])_\Sigma \\ &= (\delta/l)([d\partial_\nu v]|[d\partial_\nu v_1])_\Sigma + l\sigma(\lambda\rho - \rho_1|A_\Sigma\rho_1)_\Sigma \\ &\quad - (\delta/l)([d\partial_\nu v]|[d\partial_\nu v_1])_\Sigma - \lambda l\sigma(A_\Sigma\rho|\rho_1)_\Sigma \\ &= -l\sigma(\rho_1|A_\Sigma\rho_1),\end{aligned}$$

which yields

$$\|\sqrt{\kappa}v_1\|_\Omega^2 + l\sigma(\rho_1|A_\Sigma\rho_1).$$

It follows now from (4.1) that $v_1$ is constant on $\Omega\setminus\Sigma$. Since $\lambda \neq 0$, we then obtain from (1.9) that $(v_1,\rho_1) = (0,0)$. Thus any nontrivial eigenvalue is semi-simple, and, in particular, positive eigenvalues are algebraically simple.

We want to show that for $\zeta > 1$ there is precisely one positive eigenvalue $\lambda_* > 0$. For this purpose we fix the parameters $d$, $l$, $\sigma$, $\delta$ as well as $R$, but replace $\kappa$ by $s\kappa$

in the first line of (1.9). Fixing $\mu = \lambda s$ and scaling $\rho \mapsto \rho/s$, we obtain the scaling $\sigma \mapsto s\sigma$. The argument given previously then shows that there is $s_* > 0$ such that $\mu$ is a simple eigenvalue of the scaled problem; hence, $\lambda_* = \mu/s_* > 0$ is a simple positive eigenvalue for (1.9) with $\kappa$ replaced by $s_*\kappa$ in the first line. Since $\lambda_* = \lambda_*(s_*)$ is simple, the eigenvalue problem

$$(5.6)\qquad \begin{cases} \lambda_* s\kappa v - d\Delta v = 0 & \text{in} \quad \Omega\setminus\Sigma,\\ \partial_\nu v = 0 & \text{on} \quad \partial\Omega,\\ v = \sigma\mathcal{A}_\Sigma\rho + \lambda_*\delta\rho & \text{on} \quad \Sigma,\\ \lambda_* l\rho - [d\partial_\nu v] = 0 & \text{on} \quad \Sigma \end{cases}$$

has a smooth (analytic) family $[s \mapsto (\lambda_*(s), v(s), \rho(s))]$ of solutions, which exists as long as $\lambda_*(s)$ remains a simple eigenvalue. As $\zeta(s) := s\sigma(\kappa|1)_\Omega/l|\Sigma|R^2$ approaches the value $\zeta = 1$ from above, we must have $\lambda_*(s) \to 0$ from the right. This means that at the value

$$(5.7)\qquad s = s_0 := \frac{l|\Sigma|R^2}{\sigma(\kappa|1)_\Omega}$$

the eigenvalue $\lambda_*(s)$ passes through the origin, in accordance with the jump of the algebraic multiplicity by 1 of the eigenvalue 0 for $L_\delta$ at $\zeta = 1 = \zeta(s_0)$. This shows that there can be only one positive eigenvalue for (5.6), independently of the values of the parameters, and there is precisely one iff $\zeta > 1$.

If $\zeta = \sigma(\kappa|1)\Omega/l|\Sigma|R^2 > 1$, then we have that $s_0 < 1$. The argument given above shows that the modified eigenvalue problem (5.6) has for each $s > s_0$ exactly one simple eigenvalue. This is, in particular, true for $s = 1$, thus establishing Theorem 2.1(f).

Now we turn our attention to the case $l < 0$. As before, we conclude that the operator $L_\delta$ has countably many eigenvalues. We note that the argument given in section 4 also applies to the case $l < 0$ and $\delta = 0$, showing that all eigenvalues of (1.9) are real in this case.

In the following, we assume that $l < 0$ and $\delta > 0$. In order to show Theorem 1.1(f), we consider the operators $B_\lambda := \lambda\delta I + \lambda l T_\lambda + \sigma A_\Sigma$ for $\lambda > 0$. By Proposition 5.1 we have

$$(B_\lambda\rho|\rho)_\Sigma \geq \big(\delta\lambda - |l|M_0\lambda^{1/2} - \sigma/R^2\big)\|\rho\|_\Sigma^2 \geq \|\rho\|_\Sigma^2$$

provided that $\lambda \geq \mu_0$, for some $\mu_0 \geq \lambda_0$. Hence, $B_\lambda$ is positive definite for large $\lambda > 0$. On the other hand we have

$$(B_\lambda 1|1)_\Sigma = \lambda\delta|\Sigma| - \lambda|l|(T_\lambda 1|1)_\Sigma - \sigma|\Sigma|/R^2 \leq \lambda\delta|\Sigma| - \sigma|\Sigma|/R^2.$$

Thus for $\lambda$ small we see that $B_\lambda$ is not positive. Let

$$\lambda_* := \inf\{\lambda > 0 : B_\mu \text{ is positive definite for all } \mu \geq \lambda\}.$$

Then $B_{\lambda_*}$ is still semi-definite, but not definite, and hence, by compactness of the resolvent, has a nontrivial kernel. This shows that $\lambda_*$ is an eigenvalue of (1.9), proving Theorem 1.1(f).

*Remark* 5.2. Suppose $l < 0$ and $\delta > 0$.

(a) While it is still true that all nontrivial eigenvalues of (1.9) are semi-simple, we cannot conclude that positive eigenvalues are simple.

(b) We do not know whether all eigenvalues of (1.9) are real if $l < 0$ and $\delta > 0$. We can, however, prove that every sector $[|\arg\lambda| \le \theta]$ can only contain finitely many eigenvalues for a fixed $\theta \in (0, \pi)$. This can be shown as follows. Let $\theta \in (\pi/2, \pi)$ be fixed, and suppose that $|\arg\lambda| \le \theta$. Moreover, let $\alpha \in (0, \pi/2)$ be an arbitrary fixed number. Then one verifies that

$$|\lambda + \mu| \ge \min\{\sin\alpha, \sin(\pi - \theta)\}|\lambda|, \quad \text{whenever } \mu \in \mathbb{R},\ \alpha \le |\arg\lambda| \le \theta,$$

and this shows that there exists a constant $c > 0$ such that

$$|\lambda\delta + \sigma(A_\Sigma\rho|\rho)_\Sigma| \ge c|\lambda|, \quad \alpha \le |\arg\lambda| \le \theta. \tag{5.8}$$

Using that $\sigma(A_\Sigma\rho|\rho) \ge -\sigma/R^2\|\rho\|_\Sigma^2$, we see that

$$|\lambda\delta + \sigma(A_\Sigma\rho|\rho)_\Sigma| \ge (\delta/2)|\lambda|, \quad \|\rho\|_\Sigma = 1, \quad \operatorname{Re}\lambda \ge 2\sigma/\delta R^2. \tag{5.9}$$

Combining (5.8)–(5.9) yields

$$|\lambda\delta(\rho|\rho)_\Sigma + \sigma(A_\Sigma\rho|\rho)_\Sigma| \ge k|\lambda|, \quad \|\rho\|_\Sigma = 1, \quad |\arg\lambda| \le \theta, \quad |\lambda| \ge \eta, \tag{5.10}$$

where $k = \min\{c, \delta/2\}$ and $\eta = (2\sigma/\delta R^2)(1/\cos\alpha)$. Let $\lambda_0$ and $M_0$ be as in Proposition 5.1. Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg\lambda| \le \theta$ is an eigenvalue of (1.9) with eigenfunction $(v, \rho)$. Then we have

$$\lambda\delta\rho + \sigma A_\Sigma\rho = \lambda|l|T_\lambda\rho \tag{5.11}$$

and we can assume, without loss of generality, that $\|\rho\|_\Sigma = 1$. If $|\lambda| \ge \max\{\lambda_0, \eta\}$, then we conclude from (5.10)–(5.11) and Proposition 5.1 that

$$k|\lambda| \le M_0|l||\lambda|^{1/2}$$

and so $|\lambda|$ is bounded by $(M_0 l/k)^2$. Clearly, if $|\lambda| \le \max\{\lambda_0, \eta\}$, we have a trivial bound. This shows that all possible eigenvalues in the sector $[|\arg\lambda| \le \theta]$ are bounded. Since eigenvalues cannot accumulate in a bounded set, we see that (1.9) can only have finitely many eigenvalues in the sector $[|\arg\lambda| \le \theta]$.

**6. Analysis of the unstable eigenvalue.** In this section we analyze the properties of the unstable eigenvalue $\lambda_*$ of problem (5.6) in case $l > 0$ in more detail. In particular, we study the behavior of $\lambda_*(s)$ and the corresponding eigenfunctions near the critical value 1 of $\zeta(s)$, i.e., for $s$ near $s_0$; see (5.7).

*Proof Theorem* 1.3. (a) We will first analyze the behavior of $(\lambda_*(s), v(s), \rho(s))$ for $s$ near $s_0$. In order to do so we use the following ansatz:

$$\begin{aligned} \lambda_*(s) &= (s - s_0)\lambda_1(s), \\ v(s) &= -1 + (s - s_0)\lambda_1(s)v_1(s), \\ \rho(s) &= \rho_0 + (s - s_0)\eta + (s - s_0)\lambda_1(s)\big(\rho_1(s) + \vec\beta(s)\cdot\vec y\,\big), \\ &(v_1(s)|\kappa)_\Omega = 0, \quad (\rho_1(s)|1)_\Sigma = (\rho_1(s)|Y_j) = 0,\ 1 \le j \le n, \end{aligned} \tag{6.1}$$

with

$$\rho_0 := R^2/\sigma + \vec\alpha\cdot\vec y\,, \qquad \eta := (\kappa|1)_\Omega/l\Sigma\,, \tag{6.2}$$

where $\vec{\alpha}$, $\vec{\beta}(s) \in \mathbb{R}^n$ and $\vec{y} = (Y_1, \dots, Y_n)$. Setting $r = s - s_0$ and inserting this ansatz into the eigenvalue problem (5.6), we obtain the following system of equations:

$$\text{(6.3)} \qquad \begin{cases} -d\Delta v_1 = s_0\kappa + r\kappa(1 - s\lambda_1 v_1) & \text{in } \Omega \setminus \Sigma, \\ \partial_\nu v_1 = 0 & \text{on } \partial\Omega, \\ [v_1] = 0 & \text{on } \Sigma, \\ -[d\partial_\nu v_1] = -l\rho_0 - rl\eta - r\lambda_1 l(\rho_1 + \vec{\beta} \cdot \vec{y}\,) & \text{on } \Sigma, \\ \sigma A_\Sigma \rho_1 = \dfrac{\sigma\eta}{R^2\lambda_1} - \delta\rho_0 + v_1 - r\delta\eta - r\lambda_1\delta(\rho_1 + \vec{\beta} \cdot \vec{y}\,) & \text{on } \Sigma. \end{cases}$$

We first observe that due to (5.7), $(6.1)_4$, (6.2), and the fact that $(Y_i|1)_\Sigma = 0$, the compatibility condition

$$\text{(6.4)} \qquad \big(1|s_0\kappa + r\kappa(1 - s\lambda_1 v_1)\big)_\Omega - l\big(1|\rho_0 + r\eta + r\lambda_1(\rho_1 + \vec{\beta} \cdot \vec{y}\,)\big)_\Sigma = 0$$

holds. It is our intention to apply the implicit function theorem to find a smooth (analytic) curve of solutions

$$[s \mapsto (\lambda_1(s), v_1(s), \rho_1(s), \vec{\beta}(s))]$$

of (6.3) for $s$ near $s_0$. The idea is to use the $(n+1)$ orthogonality conditions

$$(A_\Sigma\rho_1|1) = (A_\Sigma\rho_1|Y_j) = 0, \quad 1 \le j \le n,$$

to determine the $(n+1)$ scalar functions $\lambda_1$ and $\beta_j$. In order to do so, we will first derive an expression for $\vec{\alpha}$ and $\vec{\beta}$. Taking the inner product of $(6.3)_5$ with $Y_j$ yields

$$\text{(6.5)} \qquad 0 = -\delta\,\vec{\alpha} + (v_1|\,\vec{y}\,)_\Sigma - r\delta\lambda_1\,\vec{\beta},$$

where $(v_1|\,\vec{y}\,)_\Sigma$ denotes the vector in $\mathbb{R}^n$ with components $(v_1|Y_j)_\Sigma$, $1 \le j \le n$. Due to Proposition 3.2 we have

$$\begin{aligned} (v_1|Y_j)_\Sigma &= (v_1| - [d\partial_\nu T_0 Y_j])_\Sigma = \int_\Omega d\,\mathrm{div}(v_1 T_0 Y_j)\,dx \\ &= (d\,\nabla v_1|\nabla T_0 Y_j)_\Omega = (-d\Delta v_1|T_0Y_j)_\Omega + (-[d\partial_\nu v_1]|T_0Y_j)_\Sigma \\ &= (s_0\kappa + r\kappa(1 - s\lambda_1 v_1)|T_0Y_j)_\Omega - l(\rho_0 + r\eta + r\lambda_1(\rho_1 + \vec{\beta} \cdot \vec{y}\,)|T_0Y_j)_\Sigma. \\ &= -rs\lambda_1(\kappa v_1|T_0Y_j)_\Omega - l(\rho_0 + r\eta + r\lambda_1(\rho_1 + \vec{\beta} \cdot \vec{y}\,)|T_0Y_j)_\Sigma \end{aligned}$$

for $j = 1, \dots, n$. Setting first $r = 0$ yields an equation for $\vec{\alpha}$, namely,

$$0 = -\delta\,\vec{\alpha} - (lR^2/\sigma)(1|T_0\,\vec{y}\,)_\Sigma - l(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma\,\vec{\alpha},$$

i.e.,

$$\text{(6.6)} \qquad \vec{\alpha} = -\big(\delta I + l(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma\big)^{-1}(lR^2/\sigma)(1|T_0\,\vec{y}\,)_\Sigma,$$

where $(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma$ denotes the symmetric matrix with entries $[(T_0Y_i|Y_j)_\Sigma]_{1\le i,j\le n}$. Here we remind the reader that

$$\text{(6.7)} \qquad \langle (T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma\xi|\xi\rangle = \sum_{i,j=1}^n \xi_i\xi_j(T_0Y_i|Y_j)_\Sigma = (T_0(\xi \cdot \vec{y}\,)|(\xi \cdot \vec{y}\,))_\Sigma \ge c\|\xi\|^2$$

for $\xi \in \mathbb{R}^n$, where $\langle\cdot|\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$. This shows that the matrix $(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma$ is positive definite, and hence $\delta I + l(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma$ is invertible for any $\delta \ge 0$. Next, we obtain for $\vec{\beta}$ that

$$\vec{\beta} = -\big(\delta I + l(T_0\,\vec{y}\,|\,\vec{y}\,)_\Sigma\big)^{-1}\big\{s\big(\kappa v_1|T_0\,\vec{y}\,\big)_\Omega + l\big(\eta/\lambda_1 + \rho_1|T_0\,\vec{y}\,\big)_\Sigma\big\}. \tag{6.8}$$

Thus we have a function $\vec{\beta} = \vec{\beta}(\lambda_1, v_1, \rho_1, s)$. Finally, we obtain an equation for $\lambda_1$ by taking the inner product of $(6.3)_5$ with 1:

$$0 = \sigma\eta|\Sigma|/R^2\lambda_1 - \delta R^2|\Sigma|/\sigma + (v_1|1)_\Sigma - r\delta|\Sigma|\eta. \tag{6.9}$$

Employing the relation $(\kappa|v_1)_\Omega = 0$ and $(6.3)_{1,4}$ as well as (6.5) yields

$$\begin{aligned}
-rs\lambda_1(\kappa v_1|v_1)_\Omega &= (s_0\kappa + r\kappa(1 - s\lambda_1 v_1)|v_1)_\Omega\\
&= (-d\Delta v_1|v_1)_\Omega = \|\sqrt{d}\,\nabla v_1\|_\Omega^2 + ([d\partial_\nu v_1]|v_1)_\Sigma\\
&= \|\sqrt{d}\,\nabla v_1\|_\Omega^2 + (l\rho_0 + rl\eta + r\lambda_1 l(\rho_1 + \vec{\beta}\cdot\vec{y}\,)|v_1)_\Sigma\\
&= \|\sqrt{d}\,\nabla v_1\|_\Omega^2 + \big\{lR^2/\sigma + rl\eta\big\}(v_1|1)_\Sigma + l\delta\big|\,\vec{\alpha} + r\lambda_1\,\vec{\beta}\big|^2 + r\lambda_1 l(\rho_1|v_1)_\Sigma.
\end{aligned}$$

This leads to

$$\begin{aligned}
0 = \|\sqrt{d}\,\nabla v_1\|_\Omega^2 &- \big\{lR^2/\sigma + rl\eta\big\}\big\{(\sigma|\Sigma|\eta/R^2\lambda_1) - \delta R^2|\Sigma|/\sigma - r\delta|\Sigma|\eta\big\}\\
&+ l\delta\big|\,\vec{\alpha} + r\lambda_1\,\vec{\beta}\big|^2 + r\lambda_1\big\{l(\rho_1|v_1)_\Sigma + s(\kappa v_1|v_1)_\Omega\big\},
\end{aligned} \tag{6.10}$$

where we used (6.9).

Suppose now that $v_1$ solves the first four equations of (6.3). Then one easily verifies that (6.5) is equivalent to (6.6) and (6.8). Moreover, assuming once again that $v_1$ satisfies the first four equations of (6.3), and that $\vec{\alpha}$ and $\vec{\beta}$ satisfy (6.6) and (6.8), one verifies that

$$(6.9) \iff (6.10). \tag{6.11}$$

For $r = 0$, that is, for $s = s_0$, we obtain from (6.10)

$$\lambda_1(s_0) = l\Sigma\eta/\big\{\|\sqrt{d}\,\nabla v_1(s_0)\|_\Omega^2 + l\delta\big(|\,\vec{\alpha}|^2 + R^4|\Sigma|/\sigma^2\big)\big\}, \tag{6.12}$$

where $v_1(s_0)$ is the unique solution of problem (3.4) with $(f,g) = (s_0\kappa, -l\rho_0)$ and $(\kappa|v_1(s_0))_\Omega = 0$; see Proposition 3.2. This shows that $\lambda_1(s_0)$ is uniquely defined and strictly positive. Moreover, we also know from (6.11) that

$$0 = (\sigma\eta|\Sigma|/R^2\lambda_1(s_0)) - \delta R^2|\Sigma|/\sigma + (v_1(s_0)|1)_\Sigma - r\delta|\Sigma|\eta.$$

We obtain $\rho_1(s_0)$ by solving

$$\sigma A_\Sigma\rho_1 = \sigma\eta/R^2\lambda_1(s_0) - \delta\rho_0 + v_1(s_0)$$

for $\rho_1$, which is possible since we chose $\lambda_1(s_0)$ and $\vec{\alpha}$ in such a way that the necessary orthogonality conditions of Proposition 3.1(d) hold. Equation (6.8) shows that the mapping

$$\begin{aligned}
&I \times \Big\{v \in W_2^2(\Omega\setminus\Sigma) : [v] = 0 \text{ on } \Sigma\Big\} \times W_2^2(\Sigma) \times \mathbb{R} \to \mathbb{R}^n,\\
&[(\lambda_1, v_1, \rho_1, s) \mapsto \vec{\beta}(\lambda_1, v_1, \rho_1, s)]
\end{aligned} \tag{6.13}$$

is analytic where $I \subset \mathbb{R}$ is an open interval that contains $\lambda_1(s_0)$ but does not contain 0.

We are now in a position to apply the implicit function theorem at the point $(v_1(s_0), \rho_1(s_0), s_0)$ to solve the first four equations in (6.3) and (6.9) for $(\lambda_1, v_1)$ in terms of $(\rho_1, s)$. We choose the functional analytic setting

$$\begin{aligned} X_1 &:= \{v \in W_2^2(\Omega \setminus \Sigma) : \partial_\nu v = 0 \text{ on } \partial\Omega,\ [v] = 0 \text{ on } \Sigma,\ (\kappa|v)_\Omega = 0\}, \\ X_2 &:= \{\rho \in W_2^2(\Sigma) : (\rho|1)_\Sigma = (\rho|Y_j)_\Sigma = 0,\ 1 \le j \le n\}, \\ X &:= \mathbb{R} \times X_1 \times X_2 \times \mathbb{R}, \\ Y &:= \mathbb{R} \times \{(f, g) \in L_2(\Omega) \times W_2^{1/2}(\Sigma) : (f|1)_\Omega + (g|1)_\Sigma = 0\} \end{aligned}$$

and we define $F : V \subset X \to Y$ by means of

$$F(\lambda_1, v_1, \rho_1, s) := \begin{pmatrix} \sigma\eta|\Sigma|/R^2\lambda_1 - \delta R^2|\Sigma|/\sigma + (v_1|1)_\Sigma - r\delta|\Sigma|\eta \\ - d\Delta v_1 - s_0\kappa - r\kappa(1 - s\lambda_1 v_1) \\ - [d\partial_\nu v_1] + l(\rho_0 + r\eta + r\lambda_1(\rho_1 + \vec{\beta} \cdot \vec{y}\,)) \end{pmatrix},$$

where $V := I \times X_1 \times X_2 \times \mathbb{R}$.

Equation (6.4) implies that $F$ maps $V$ into $Y$, and (6.13) and the definition of $F$ show that

$$[(\lambda_1, v_1, \rho_1, s) \mapsto F(\lambda_1, v_1, \rho_1, s)] \in C^\omega(V, Y).$$

Clearly, the first four equations of (6.3) together with (6.9) are equivalent to $F(\lambda_1, v_1, \rho_1, s) = (0, 0, 0)$. Since we already know that

$$F(\lambda_1(s_0), v_1(s_0), \rho_1(s_0), s_0) = (0, 0, 0),$$

we are left with verifying that the derivative of $F$ at the point $(v_1(s_0), \rho_1(s_0), s_0)$ w.r.t. $(\rho_1, v_1)$ is an isomorphism, i.e.,

$$\mathbb{D}_1 F(\lambda_1(s_0), v_1(s_0), \rho_1(s_0), s_0) \in \mathrm{Isom}(\mathbb{R} \times X_1, Y).$$

It follows from Proposition 3.2(a) that the problem

$$\mathbb{D}_1 F(\lambda_1(s_0), v_1(s_0), \rho_1(s_0), s_0)(\lambda, w) = (\mu, f, g)$$

has for each $(\mu, f, g) \in Y$ a unique solution $(\lambda, w) \in \mathbb{R} \times X_1$, namely,

$$w = R_0(f, g), \quad \lambda = \frac{R^2\lambda_1^2(s_0)}{\sigma\eta|\Sigma|}\big((R_0(f, g)|1)_\Sigma - \mu\big),$$

where $w = R_0(f, g)$ is the unique solution of (3.4) with $(\kappa|w)_\Omega = 0$.

The implicit function theorem then yields a neighborhood $U$ of $(\rho_1(s_0), s_0)$ in $X_2 \times \mathbb{R}$ such that

$$\begin{aligned} &[(\rho_1, s) \mapsto (\lambda_1(\rho_1, s), v_1(\rho_1, s))] \in C^\omega(U, \mathbb{R} \times X_1) \\ &F(\lambda_1(\rho_1, s)v_1(\rho_1, s), \rho_1, s) = (0, 0, 0), \quad (\rho_1, s) \in U. \end{aligned} \tag{6.14}$$

Combining all of the above results, we conclude that

$$[(\rho_1, s) \mapsto (\lambda_1(\rho_1, s), v_1(\rho_1, s), \vec{\beta}(\rho_1, s))] \in C^\omega(U, \mathbb{R} \times X_1 \times \mathbb{R}^n) \tag{6.15}$$

and that the functions $(\lambda_1(\rho_1, s), v_1(\rho_1, s), \vec{\alpha}, \vec{\beta}(\rho_1, s))$ satisfy the first four equations of (6.3) as well as (6.5) and (6.9). We now insert these functions into the equation for $\rho_1$ which gives an equation of the form

$$G(\rho_1, s) := \sigma A_\Sigma \rho_1 - \sigma\eta/R^2 \lambda_1(s_0) + \delta\rho_0 - v_1(s_0) - (s - s_0)R(\rho_1, s) = 0,$$

where $[(\rho_1, s) \mapsto R(\rho_1, s)] \in C^\omega(U, L_2(\Sigma))$. By (6.5) and (6.9) we know that

$$G : X_2 \times \mathbb{R} \to Z := \big\{g \in L_2(\Sigma) :\ (g|1)_\Sigma = (g|Y_j) = 0,\ 1 \le j \le n\big\}.$$

Moreover, we also know that $G(\rho_1(s_0), s_0) = 0$. The derivative of $G$ with respect to $\rho_1$ at $(\rho_1(s_0), s_0)$ is $\sigma A_\Sigma$, and Proposition 3.1(d) and the implicit function theorem then yield an analytic curve

$$[s \mapsto \rho_1(s)] \in C^\omega((s_0 - \varepsilon_0, s_0 + \varepsilon_0), X_2)$$

such that $G(\rho_1(s), s) = 0$.

Combining all of the results, we obtain an analytic curve of solutions

$$[s \mapsto (\lambda_*(s), v(s), \rho(s))]$$

of (5.6) for $s \in (s_0 - \varepsilon_0, s_0 + \varepsilon_0)$. If $s > s_0$, then the statement in (a) follows from the considerations in section 5.

(b) The proof of part (a) shows that the eigenvalue curve $[s \mapsto \lambda_*(s)]$ is analytic near the critical value $s = s_0$ and crosses the imaginary axis at $s = s_0$ with positive speed $\lambda_1(s_0)$; see (6.12).

(c) We show that $\lambda_*(s)$ is strictly increasing. To see this, we differentiate (5.6) w.r.t. $s$ and form the inner product of the resulting equation in $\Omega$ with $v = v(s)$. This yields with Green's formula

$$\begin{aligned}
-(s\lambda_*(s))'\|\sqrt{\kappa}v\|_\Omega^2 &= s\lambda_*(\kappa v'|v)_\Omega - (d\Delta v'|v)_\Omega \\
&= (v'|s\lambda_*\kappa v - d\Delta v)_\Omega + ([d\partial_\nu v']|v)_\Sigma - (v'|[d\partial_\nu v])_\Sigma \\
&= (\delta/l)([d\partial_\nu v']|[d\partial_\nu v])_\Sigma + (\lambda_* l\rho' + \lambda_*' l\rho|\sigma A_\Sigma \rho)_\Sigma \\
&\quad - (\delta/l)([d\partial_\nu v]|[d\partial_\nu v'])_\Sigma - (\lambda_* l\rho|\sigma A_\Sigma \rho')_\Sigma \\
&= \lambda_*' l\sigma(A_\Sigma\rho|\rho)_\Sigma;
\end{aligned}$$

hence

$$\lambda_*\|\sqrt{\kappa}v\|_\Omega^2 + \lambda_*'\big\{\|\sqrt{s\kappa}\,v\|_\Omega^2 + l\sigma(A_\Sigma\rho|\rho)_\Sigma\big\} = 0.$$

Employing (4.1) once more (with $\kappa$ replaced by $s\kappa$), we obtain

$$\lambda_*'(s) = \lambda_*^2(s)\|\sqrt{\kappa}v\|_\Omega^2 / \big\{\|\sqrt{d}\,\nabla v\|_\Omega^2 + (\delta/l)\|[d\partial_\nu v]\|_\Sigma^2\big\},$$

which yields $\lambda_*'(s) > 0$ for $s \neq s_0$. If $s = s_0$, then we have already established in (b) that $\lambda_*'(s_0) > 0$, and this shows the assertion of Theorem 1.3(c).

(d) If the stability condition (5.1) is violated, then we can conclude from the identity (4.4), where $\kappa$ is now replaced with $s\kappa$ and $\lambda$ is replaced with $\lambda_*$, that

$$\Big(\lambda_*(s)l\delta|\Sigma| + \frac{l^2|\Sigma|^2}{s(\kappa|1)_\Omega} - \frac{l\sigma|\Sigma|}{R^2}\Big) \le 0,$$

which shows that $\lambda_*(s) \le \frac{\sigma}{\delta R^2}\big(1 - 1/\zeta(s)\big)$ provided that $\delta > 0$, i.e., if kinetic undercooling is present.

(e) To show that $\lambda_*(s) \to \infty$ as $s \to \infty$ in case $\delta = 0$, we employ the estimate in Proposition 5.1(b). We first observe that due to the fact that $\lambda_*(s)$ is increasing in $s$, there exists a number $s_1 > s_0$ such that $s\lambda_*(s) \ge \lambda_0$ for $s \ge s_1$, where $\lambda_0$ is the number occurring in Proposition 5.1(b). It then follows from the relation $\lambda_* T_{s\lambda_*}\rho + \sigma A_\Sigma \rho = 0$ that

$$\begin{aligned} \sigma^2\|A_\Sigma\rho\|_\Sigma^2 = \sigma^2\|A_\Sigma\rho_0\|_\Sigma^2 + (\sigma^2|\Sigma|/R^2)\bar\rho^2 &= \lambda_*^2\|T_{s\lambda_*}\rho\|_\Sigma^2 \\ &\le M_1\|\rho\|_\Sigma^2\,\lambda_*(s)/s, \quad s \ge s_1, \end{aligned} \tag{6.16}$$

where we write $\rho = \rho_0 + \bar\rho$ with $(\rho_0|1)_\Sigma = 0$. Multiplying the eigenvalue problem (5.6) with $(s\lambda_* v - (d/\kappa)\Delta v)$ and using the divergence theorem and (6.16), we get

$$\begin{aligned} (s\lambda_*)^2\,\|\sqrt{\kappa}v\|_\Omega^2 + 2s\lambda_*\,\|\sqrt{d}\nabla v\|_\Omega^2 + \|(d/\sqrt{\kappa})\Delta v\|_\Omega^2 + 2s\lambda_*^2 l\sigma(\rho_0|A_\Sigma\rho_0)_\Sigma \\ = \big(2s\lambda_*^2 l\sigma|\Sigma|/R^2\big)\bar\rho^2 \le M_2\|\rho\|_\Sigma^2\,\lambda_*^3, \quad s \ge s_1. \end{aligned}$$

The relation $\lambda_* l\rho = [d\partial_\nu v]$ and the inequality above yield

$$\begin{aligned} \lambda_*^2\|\rho\|_\Sigma^2 = (1/l)^2\|[d\partial_\nu v]\|_\Sigma^2 &\le C\|v\|^2_{W_2^{3/2+\varepsilon}(\Omega\setminus\Sigma)} \le C\|v\|_\Omega^{(1/2-\varepsilon)}\|v\|^{3/2+\varepsilon}_{W_2^2(\Omega\setminus\Sigma)} \\ &\le C\Big\{\|v\|_\Omega^2 + \|v\|_\Omega^{(1/2-\varepsilon)}\|\Delta v\|_\Omega^{3/2+\varepsilon}\Big\} \le C\|\rho\|_\Sigma^2\,\lambda_*^3\left\{\frac{1}{(s\lambda_*)^2} + \frac{1}{(s\lambda_*)^{1/2-\varepsilon}}\right\} \\ &\le C\|\rho\|_\Sigma^2\,\lambda_*^3/(s\lambda_*)^{1/2-\varepsilon} = C\|\rho\|_\Sigma^2\,\lambda_*^{5/2+\varepsilon}/s^{1/2-\varepsilon} \end{aligned}$$

for $s \ge s_1$, where $C$ is a generic constant that may change from line to line. Dividing by $\lambda_*^2$ and by $\|\rho\|_\Sigma^2$ implies

$$\lambda_*(s) \ge cs^{(1-2\varepsilon)/(1+2\varepsilon)}, \quad s \ge s_1.$$

Thus we can conclude that $\liminf_{s\to\infty}\lambda_*(s)/s^\theta = \infty$ for each $\theta < 1$. ∎

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