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CLASSICAL SOLUTIONS OF MULTIDIMENSIONAL HELE–SHAW MODELS *

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Abstract. Existence and uniqueness of classical solutions for the multidimensional expanding Hele–Shaw problem are proved.

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1. The problem.

We are concerned with a class of moving boundary problems for bounded domains in $\mathbb{R}^n$, which comprise in particular the so-called single phase Hele–Shaw problem. In order to describe precisely the involved geometry, let Ω be a bounded domain in $\mathbb{R}^n$ and assume that its boundary $\partial\Omega$ is of class C^∞. Moreover, assume that $\partial\Omega$ consists of two disjoint nonempty components J and Γ. Later on, we will model over the exterior component Γ a moving interface, whereas the interior component J describes a fixed portion of the boundary. Let ν denote the outer unit normal field over Γ and fix $\alpha \in (0,1)$. Given $a > 0$, set

$$\mathcal{U} := \{\rho \in C^{2+\alpha}(\Gamma)\,;\, \|\rho\|_{C^1(\Gamma)} < a\}.$$

For each $\rho \in \mathcal{U}$ define the map

$$\theta_\rho := id_\Gamma + \rho\nu$$

and let $\Gamma_\rho := \mathrm{im}(\theta_\rho)$ denote its image. Obviously, θ_ρ is a $C^{2+\alpha}$ diffeomorphism mapping Γ onto Γ_ρ, provided $a > 0$ is chosen sufficiently small. In addition, we assume that $a > 0$ is small enough such that Γ_ρ and J are disjoint for each $\rho \in \mathcal{U}$. Let Ω_ρ denote the domain in $\mathbb{R}^n$ being diffeomorphic to Ω and whose boundary is given by J and Γ_ρ. To describe the evolution of the hypersurface Γ_ρ, fix $T > 0$ and set $I := [0,T]$. Then each map $\rho : I \to \mathcal{U}$ defines a collection of domains $\Omega_{\rho(t)},\ t \in I$. For later purposes it is convenient to introduce the following generalized parabolic cylinder:

$$\Omega_{\rho,T} := \big\{(x,t) \in \mathbb{R}^n \times [0,T]\,;\, x \in \Omega_{\rho(t)}\big\} = \bigcup_{t\in I}\big(\Omega_{\rho(t)} \times \{t\}\big)$$

and, correspondingly,

$$\Gamma_{\rho,T} := \big\{(x,t) \in \mathbb{R}^n \times [0,T]\,;\, x \in \Gamma_{\rho(t)}\big\} = \bigcup_{t\in I}\big(\Gamma_{\rho(t)} \times \{t\}\big).$$

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Observe that $\Omega_{0,T}$ is just the standard parabolic cylinder $\Omega \times [0,T]$. Similarly, $\Gamma_{0,T} = \Gamma \times [0,T]$. For the sake of completeness, we write $J_T := J \times [0,T]$.

Now let $\rho_0 \in \mathcal{U}$ be given. Moreover, pick $b \in C(J)$ and $\delta \in \{0,1\}$. Then we consider the *moving boundary problem* of determining a pair (u,ρ) satisfying the following set of equations:

$$(1.1)_{\rho_0} \qquad \begin{aligned} \Delta u &= 0 &&\text{in} && \Omega_{\rho,T}, \\ u &= 0 &&\text{on} && \Gamma_{\rho,T}, \\ (1-\delta)u + \delta(\nabla u|\nu_J) &= b &&\text{on} && J_T, \\ \partial_t N_\rho - (\nabla u|\nabla N_\rho) &= 0 &&\text{on} && \Gamma_{\rho,T}, \\ \rho(0,\cdot) &= \rho_0 &&\text{on} && \Gamma. \end{aligned}$$

Here, Δ and ∇ stand for the Laplacian and the gradient, respectively, in the Euclidean metric. The outer unit normal field over J is denoted by ν_J. The parameter δ is introduced to label the boundary condition on the fixed boundary J (where $\delta = 0$ corresponds to a Dirichlet boundary condition and $\delta = 1$ corresponds to a Neumann condition). Moreover, N_ρ is a defining function for Γ_ρ, i.e., $\Gamma_\rho = N_\rho^{-1}(0)$, $\rho \in \mathcal{U}$. A precise definition of N_ρ is given in section 2.

The set of equations in (1.1) express that the free boundary moves with normal velocity given by the normal derivative of a harmonic function which vanishes on the boundary. More precisely, the motion of the free boundary is governed by $V = -\frac{\partial u}{\partial \nu}$, where the function u satisfies the first three equations in (1.1). Here, V is the normal velocity taken to be positive for expanding hypersurfaces and ν is the outer unit normal field on the moving boundary.

Assume now that $n = 2$, $\delta = 1$, and $b > 0$. Then problem $(1.1)_{\rho_0}$ represents the classical formulation of the expanding two-dimensional Hele–Shaw flow; see Crank [5], Elliott and Ockendon [10], Elliott and Janovsky [9], DiBenedetto and Friedman [7], and Richardson [21]. In this model, u has the meaning of the pressure in an incompressible viscous fluid blob Ω_ρ. Since b is positive, further fluid is injected through the fixed boundary J at the rate b. Hence, the blob is advancing in time, modelled by the moving boundary Γ_ρ. Some authors (see Fasano and Primicerio [15] or Steinbach and Weinelt [22]) consider the above model in the case of prescribed pressure on the fixed boundary, i.e., with the inhomogeneous Dirichlet boundary condition $u = b$ on J. This boundary condition corresponds to the case $\delta = 0$ in $(1.1)_{\rho_0}$. In our model, we cover both cases and we prove the existence of a unique *classical* solution (u,ρ) for the general problem $(1.1)_{\rho_0}$; see the main result below. As pointed out in [5], [8], [9], [10], [16], and [22], there are further applications of $(1.1)_{\rho_0}$ to different multi-dimensional moving boundary problems. We mention the electrochemical machining problem, the one-phase Stefan problem with zero specific heat, the flow of viscous fluid through porous media, and the injection moulding process. These models make sense in higher space dimensions and under general boundary conditions on the fixed boundary J.

To clearly state our result, we need some definitions. Given an open subset U of $\mathbb{R}^m$, let $h^s(U)$ denote the little Hölder space of order $s > 0$, a closed subspace of the usual Hölder space $BUC^s(U)$; see section 2 for a precise definition. Throughout this paper we fix $\alpha \in (0,1)$ and we define

$$\mathcal{V} := h^{2+\alpha}(\Gamma) \cap \mathcal{U}.$$

Moreover, we need the anisotropic function spaces $Ch^{0,s}(\Omega_{\rho,T})$ consisting of all $u : \Omega_{\rho,T} \to \mathbb{R}$ such that, given $(x,t) \in \Omega_{\rho,T}$, the function $u(\cdot,t)$ belongs to $h^s(\Omega_{\rho(t)})$ and the function $u(x,\cdot)$ belongs to $C([0,T])$. A pair (u,ρ) is called a *classical Hölder solution* of (1.1) if

$$(u,\rho) \in Ch^{0,2+\alpha}(\Omega_{\rho,T}) \times \big(C([0,T),\mathcal{V}) \cap C^1([0,T),h^{1+\alpha}(\Gamma))\big)$$

and if (u,ρ) satisfies the equations in (1.1) pointwise. Our main result now reads as follows.

THEOREM 1.1. *Assume that* $b \in h^{2+\alpha-\delta}(J)$ *is nonnegative and not identically equal to zero. Then, given any initial value* $\rho_0 \in \mathcal{V}$*, there exist* $T > 0$ *and a unique classical solution* (u,ρ) *of* $(1.1)_{\rho_0}$ *on* $[0,T]$*. Moreover, the moving boundary* $\rho : (0,T) \to \mathcal{V}$ *is analytic in the time variable.*

It should be emphasized that Theorem 1.1 guarantees a unique classical solution to problem (1.1) for each $C^{2+\alpha}$ initial hypersurface Γ_{ρ_0} which is close to Γ in the sense that ρ_0 belongs to $\mathcal{V}$.

In Elliott [8] and Elliott and Janovsky [9], a variational inequality approach for problem $(1.1)_{\rho_0}$ is developed, and the existence and uniqueness of global weak solutions are proved. However, as stated in the Conclusion of [9] (see p. 106), the existence of classical solutions left an open problem.

Our approach to problem $(1.1)_{\rho_0}$ proposed in this paper is of a different nature. Indeed, transforming the original problem on a fixed domain, we are looking for classical solutions from the very beginning. After a natural reduction of the transformed equations, we are led to an evolution equation for the moving boundary involving a nonlinear and nonlocal pseudodifferential operator of first order. The main result for this pseudodifferential operator can be summarized by the fact that it depends smoothly on the unknown and that the corresponding linearized operator is a nicely behaving operator; i.e., it generates a strongly continuous analytic semigroup on an appropriate subspace of Hölder continuous functions, provided $b \geq 0$ and $b \neq 0$. This generation property of the linearization makes it possible to use the general results of the theory of maximal regularity, due to Da Prato and Grisvard [6], and to construct a unique classical solution of the nonlinear problem. The same technique has been applied to moving boundary problems arising in gravity flows of incompressible fluids through porous media; see [12] and [13].

There is a one-dimensional version of problem $(1.1)_{\rho_0}$; see the work of Fasano and Primicerio [14], [15]. Since the geometry of one-dimensional moving boundary problems is considerably easier to handle, classical solutions are well known to exist in this case.

For two-dimensional simply connected domains and for initial data belonging to an appropriate Gevrey class, Reissig [20] recently proved the existence of analytic solutions to a Hele–Shaw model with a point source.

Let $\rho_0 \in \mathcal{V}$ be given and assume that $b \in h^{2+\alpha-\delta}(J) \setminus \{0\}$ is nonnegative. Moreover, let (u,ρ) denote the classical solution of $(1.1)_{\rho_0}$ constructed in Theorem 1. Then, given $t \in [0,T]$, the pressure $u(\cdot,t) \in h^{2+\alpha}(\Omega_{\rho(t)})$ is the unique solution in $h^{2+\alpha}(\Omega_{\rho(t)})$ of the following elliptic boundary value problem:

$$\Delta u = 0 \quad \text{in} \quad \Omega_{\rho(t)}, \quad u = 0 \quad \text{on} \quad \Gamma_{\rho(t)}, \quad (1-\delta)u + \delta(\nabla u | \nu_J) = b \quad \text{on} \quad J.$$

Hence the strong maximum principle implies that the pressure $u(\cdot,t)$ is strictly positive in $\Omega_{\rho(t)}$. This property is crucial for our approach; see step (b) in the proof of Theorem 4.2.

From a mathematical and a physical point of view, problem $(1.1)_{\rho_0}$ also makes sense for negative b. However, in this so-called ill-posed case, the problem has a completely different feature, as pointed out by Elliott and Ockendon [10] based on numerical investigations, by DiBenedetto and Friedman [7] proving so-called fingering, and by Fasano and Primicerio [15] establishing blow-up and nonexistence results for one-dimensional problems. Our results are also optimal in this sense, since we guarantee the existence of classical solutions in the well-posed case $b \geq 0$, $b \neq 0$, and we *prove* that the linearized reduced problem for the moving boundary is ill-posed in the sense of Hadamard for $b \leq 0$, $b \neq 0$; see Remark 5.3.

2. The transformed problem. In this section we transform the original problem into a problem on a fixed domain, and we introduce a nonlinear, nonlocal pseudo-differential operator Φ of an appropriate reduced problem for the moving boundary Γ_ρ. In addition, we provide a useful representation of the Fréchet derivative of Φ.

Let us first introduce some function spaces which we will need in what follows. Assume that U is an open subset of $\mathbb{R}^m$. Given $k \in \mathbb{N} \cup \{\infty\}$, let $C^k(U)$ denote the space of all $f : U \to \mathbb{R}$ having continuous derivatives up to order k. The closed subspace of $C^k(U)$ consisting of all maps from U into $\mathbb{R}$ which have bounded and uniformly continuous derivatives up to order k is denoted by $BUC^k(U)$. Given $\alpha \in (0,1)$, the space $BUC^{k+\alpha}(U)$ stands for all $f \in BUC^k(U)$ having uniformly α-Hölder continuous derivatives of order k. In addition, $C^\omega(U)$ denotes the subspace of all real analytic functions on U.

Furthermore, we write $\mathcal{S}(\mathbb{R}^m)$ for the Schwartz space, i.e., the Fréchet space of all rapidly decreasing smooth functions on $\mathbb{R}^m$.

Next let r_U denote the restriction operator with respect to U, i.e., $r_U u := u|U$ for $u \in BUC(U)$. Then the *little Hölder spaces* $h^s(U),\ s \geq 0$, are defined as

$$h^s(U) := \text{closure of } r_U\big(\mathcal{S}(\mathbb{R}^m)\big) \text{ in } BUC^s(U).$$

Finally, assume that M is an m-dimensional (sufficiently) smooth submanifold of $\mathbb{R}^n$. Then the spaces $BUC^s(M)$ and $h^s(M),\ s \geq 0$, are defined as usual by means of a smooth atlas for M; see [24].

It is useful to write Γ_ρ as a 0-level set of an appropriate function. For this, pick $a_0 \in (0, \mathrm{dist}(\Gamma, J))$ and let

$$\mathcal{N} : \Gamma \times (-a_0, a_0) \to \mathbb{R}^n, \qquad \mathcal{N}(x,\lambda) := x + \lambda\nu(x).$$

If $a_0 > 0$ is small enough, we have that

$$\mathcal{N} \in \mathrm{Diff}^\infty(\Gamma \times (-a_0, a_0), \mathcal{R}),$$

where $\mathcal{R} := \mathrm{im}(\mathcal{N})$. It is convenient to decompose the inverse of $\mathcal{N}$ into $\mathcal{N}^{-1} = (X, \Lambda)$, where

$$X \in BUC^\infty(\mathcal{R}, \Gamma) \qquad \text{and} \qquad \Lambda \in BUC^\infty(\mathcal{R}, (-a_0, a_0)).$$

Note that $X(y)$ is the nearest point on Γ to y and that $\Lambda(y)$ is the signed distance from y to Γ (that is, to $X(y)$). The neighborhood $\mathcal{R}$ consists of those points with distance less than a_0 to Γ. Given $\rho \in \mathcal{V}$, now define

$$N_\rho : \mathcal{R} \to \mathbb{R}, \qquad N_\rho(y) := \Lambda(y) - \rho(X(y)).$$

Then it is not difficult to verify that $\Gamma_\rho = N_\rho^{-1}(0)$. Therefore, the gradient ∇N_ρ is perpendicular to Γ_ρ, and ∇N_ρ points outward since $N_\rho(y) < 0$ if $y \in \Omega_\rho$. So it follows that the outer unit normal field ν on Γ_ρ is given by $\nu = \frac{\nabla N_\rho}{|\nabla N_\rho|}$. Let $\rho \in C^1([0,T], h^{1+\alpha}(\Gamma))$ be given and set

$$N_\rho(y,t) := \Lambda(y) - \rho(X(y),t), \qquad y \in \mathcal{R}, \quad t \in [0,T].$$

Then

$$V(y,t) := -\frac{\partial_t N_\rho(y,t)}{|\nabla N_\rho(y,t)|} = \frac{\partial_t \rho(X(y),t)}{|\nabla N_\rho(y,t)|}, \qquad y \in \Gamma_{\rho(t)}, \quad t \in [0,T],$$

is the normal velocity of the moving hypersurfaces $\Gamma_{\rho(t)}$ in the direction of the outer normal field. Hence the fourth equation in (1.1) can be rewritten as $-\frac{\partial_t N_\rho}{|\nabla N_\rho|} = -(\nabla u|\nu)$, which shows that the motion of the hypersurfaces $\Gamma_{\rho(t)}$ is governed by $V = -\frac{\partial u}{\partial \nu}$.

Next we introduce an appropriate extension of θ_ρ to $\mathbb{R}^n$. For this we assume that $a \in (0, a_0/4)$, and we fix a $\varphi \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\varphi(\lambda) = \begin{cases} 1 & \text{if} \quad |\lambda| \le a, \\ 0 & \text{if} \quad |\lambda| \ge 3a \end{cases}$$

and such that $\sup |\partial\varphi(\lambda)| < 1/a$. Then we define for each $\rho \in \mathcal{V}$ the map

$$\Theta_\rho(y) := \begin{cases} \mathcal{N}\big(X(y), \Lambda(y) + \varphi(\Lambda(y))\rho(X(y))\big) & \text{if} \quad y \in \mathcal{R}, \\ y & \text{if} \quad y \notin \mathcal{R}. \end{cases}$$

Note that $[\lambda \mapsto \lambda + \varphi(\lambda)\rho]$ is strictly increasing since $|\partial\varphi(\lambda)\rho| < 1$. Then it is not difficult to verify that

$$\Theta_\rho \in \mathrm{Diff}^{\,2+\alpha}(\mathbb{R}^n, \mathbb{R}^n) \cap \mathrm{Diff}^{\,2+\alpha}(\Omega, \Omega_\rho) \quad \text{and} \quad \Theta_\rho|\Gamma = \theta_\rho.$$

Moreover, we observe that there exists an open neighborhood U of J such that

$$\Theta_\rho|U = id_U. \tag{2.1}$$

It should be mentioned that the above diffeomorphism was first introduced by Hanzawa [18] to transform multidimensional Stefan problems to fixed domains. In the following we use the same symbol θ_ρ for both diffeomorphisms θ_ρ and Θ_ρ. The pull-back operator induced by θ_ρ is given as

$$\theta^* u := \theta_\rho^* u := u \circ \theta_\rho \quad \text{for} \quad u \in BUC(\Omega_\rho).$$

Similarly, the corresponding push-forward operator is defined as

$$\theta_* v := \theta_*^\rho v := v \circ \theta_\rho^{-1} \quad \text{for} \quad v \in BUC(\Omega).$$

LEMMA 2.1. *Given $\rho \in \mathcal{V}$ and $k \in \{1, 2\}$, we have*

$$\theta_\rho^* \in \mathrm{Isom}(h^{k+\alpha}(\Omega_\rho), h^{k+\alpha}(\Omega)) \cap \mathrm{Isom}(h^{k+\alpha}(\Gamma_\rho), h^{k+\alpha}(\Gamma))$$

with $[\theta_\rho^*]^{-1} = \theta_*^\rho$.

Proof. Let $\rho \in \mathcal{V}$ and $k \in \{1, 2\}$ be given. It follows from the mean value theorem that

$$\theta_\rho^* \in \mathrm{Isom}(BUC^{k+\alpha}(\Omega_\rho), BUC^{k+\alpha}(\Omega)).$$

Hence, to prove the first assertion, it suffices to show that $\theta_\rho^* u$ belongs to the space $h^{k+\alpha}(\Omega)$, whenever u belongs to $h^{k+\alpha}(\Omega_\rho)$. But this is an easy consequence of the following known characterization of little Hölder spaces: a function $u \in BUC^{k+\alpha}(\Omega)$ belongs to $h^{k+\alpha}(\Omega)$ iff

$$\lim_{\tau \to 0^+} \sup_{0<|x-y|\le\tau} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{\tau^\alpha} = 0, \qquad \beta \in \mathbb{N}^n,\ |\beta| = k.$$

This can be seen by means of local coordinate charts along the lines of Lemma 2.7 and Remark 2.8 in [19]; see also [3]. The second assertion follows analogously. □

Given $\rho \in \mathcal{V}$, we now introduce the following transformed differential operators, acting linearly on $BUC^2(\Omega)$:

$$\begin{aligned} &A(\rho)v := -\theta_\rho^*\big(\Delta(\theta_*^\rho v)\big), \qquad B(\rho)v := \gamma\theta_\rho^*(\nabla(\theta_*^\rho v)|\nabla N_\rho),\\ &Cv := (1-\delta)\gamma_J v + \delta(\gamma_J \nabla v|\nu_J), \end{aligned}$$

where γ and γ_J denote the trace operators with respect to Γ and J, respectively. Assume now that (u, ρ) is a classical Hölder solution of $(1.1)_{\rho_0}$. Then it is not difficult to see that $v := [t \mapsto \theta_{\rho(t)}^* u(t, \cdot)]$ belongs to $C([0, T], h^{2+\alpha}(\Omega))$ and that the pair (v, ρ) satisfies the following equations:

$$(2.2)_{\rho_0} \qquad \begin{aligned} A(\rho)v &= 0 &&\text{in} \quad \Omega_{0,T},\\ v &= 0 &&\text{on} \quad \Gamma_{0,T},\\ Cv &= b &&\text{on} \quad J_T,\\ \partial_t\rho + B(\rho)v &= 0 &&\text{on} \quad \Gamma_{0,T},\\ \rho(0,\cdot) &= \rho_0 &&\text{on} \quad \Gamma. \end{aligned}$$

A pair (v, ρ) is called a *classical Hölder solution* of $(2.2)_{\rho_0}$ if

$$\begin{aligned} &v \in C([0,T], h^{2+\alpha}(\Omega)),\\ &\rho \in C([0,T], \mathcal{V}) \cap C^1([0,T], h^{1+\alpha}(\Gamma)) \end{aligned}$$

and if (v, ρ) satisfies the equations in $(2.2)_{\rho_0}$ pointwise. The following lemma is an obvious consequence of Lemma 2.1 and (2.1).

LEMMA 2.2. *Let* $\rho_0 \in \mathcal{V}$ *be given.*

(a) *If* (u, ρ) *is a classical Hölder solution of* $(1.1)_{\rho_0}$*, then* $(\theta_\rho^* u, \rho)$ *is a classical Hölder solution of* $(2.2)_{\rho_0}$*.*

(b) *If* (v, ρ) *is a classical Hölder solution of* $(2.2)_{\rho_0}$*, then* $(\theta_*^\rho v, \rho)$ *is a classical Hölder solution of* $(1.1)_{\rho_0}$*.*

In the next two lemmas we collect some results for elliptic boundary value problems in little Hölder spaces. We shall use these results in sections 3 and 4.

Lemma 2.3.

$$(A, B) \in C^\omega(\mathcal{V}, \mathcal{L}(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{1+\alpha}(\Gamma))).$$

Proof. Let η denote the standard Euclidean metric on $\mathbb{R}^m$ and let $\theta^*\eta$ be the Riemannian metric on $\overline{\Omega}$ induced by the diffeomorphism θ_ρ, i.e.,

$$\theta_\rho^*\eta|_x(\xi, \zeta) := \eta|_{\theta_\rho(x)}(T_x\theta_\rho\xi, T_x\theta_\rho\zeta)$$

for $x \in \overline{\Omega}$ and $\xi,\ \zeta \in T_x(\overline{\Omega})$. Then $A(\rho)$ and $B(\rho)$ are just the Laplace–Beltrami operator and the outer normal derivative of $(\Omega, \theta_\rho^*\eta)$. Since the metric $\theta_\rho^*\eta$ depends analytically on $\rho \in \mathcal{V}$, the assertion follows easily. □

Lemma 2.4. *Let $\rho \in \mathcal{V}$ be given. Then for each*

$$(f, g, h) \in h^\alpha(\Omega) \times h^{2+\alpha}(\Gamma) \times h^{2+\alpha-\delta}(J)$$

there exists a unique classical solution $v := V(\rho)(f, g, h)$ in $h^{2+\alpha}(\Omega)$ of

$$A(\rho)v = f \quad \text{in} \quad \Omega, \qquad v = g \quad \text{on} \quad \Gamma, \qquad Cv = h \quad \text{on} \quad J.$$

Moreover, there exists a positive constant $C := C(\rho)$ such that

$$\|V(\rho)(f, g, h)\|_{2+\alpha,\Omega} \le C\big(\|f\|_{\alpha,\Omega} + \|g\|_{2+\alpha,\Gamma} + \|h\|_{2+\alpha-\delta,J}\big).$$

Proof. (a) It follows from the proof of Lemma 2.3 and by construction that A is a uniformly elliptic operator having α-Hölder continuous coefficients and that C is a normal boundary operator with regular coefficients too. Hence we conclude from Theorem 7.3 and Remark 2 on p. 669 in [1] that, given any compact subset K of $\mathcal{V}$, there exists a positive constant $C := C(K)$ such that

$$\|v\|_{2+\alpha,\Omega} \le C\big(\|A(\rho)v\|_{\alpha,\Omega} + \|\gamma v\|_{2+\alpha,\Gamma} + \|Cv\|_{2+\alpha-\delta,J}\big)$$

for all $v \in h^{2+\alpha}(\Omega)$ and all $\rho \in K$.

(b) Observe that $\big(A(0), \gamma, C\big)$ is a regular elliptic boundary value problem with constant coefficients on a smooth domain. Hence it follows from formula (3) on p. 236 in [24] that

$$\big(A(0), \gamma, C\big) \in \mathrm{Isom}(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha}(\Gamma) \times h^{2+\alpha-\delta}(J)).$$

Now let $\rho \in \mathcal{V}$ be given and set $K := \{t\rho\,;\ t \in [0, 1]\}$. Then K is a compact subset of $\mathcal{V}$, and therefore it follows from (a) and the continuity method (see Theorem 5.2 in [17]) that

$$\big(A(\rho), \gamma, C\big) \in \mathrm{Isom}(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha}(\Gamma) \times h^{2+\alpha-\delta}(J)).$$

This completes our argumentation. □

Let us now introduce the natural decomposition $V = S \oplus T \oplus R$ of the above solution operator by setting

$$\begin{aligned} S(\rho) &:= V(\rho)(\cdot, 0, 0) \in \mathcal{L}(h^\alpha(\Omega), h^{2+\alpha}(\Omega)),\\ T(\rho) &:= V(\rho)(0, \cdot, 0) \in \mathcal{L}(h^{2+\alpha}(\Gamma), h^{2+\alpha}(\Omega)),\\ R(\rho) &:= V(\rho)(0, 0, \cdot) \in \mathcal{L}(h^{2+\alpha-\delta}(J), h^{2+\alpha}(\Omega)). \end{aligned}$$

Given $v \in BUC^1(\Omega)$, let $\partial_\nu v$ denote the directional derivative with respect to the outer unit normal on Γ, i.e., $\partial_\nu v := \gamma(\nabla v|\nu)$. Using this notation it follows from the strong maximum principle that

$$\partial_\nu(R(\rho)b) < 0, \tag{2.3}$$

provided $b \in h^{2+\alpha-\delta}(J) \setminus \{0\}$ with $b \geq 0$.

Throughout the remainder of this paper we fix

$$b \in h^{2+\alpha-\delta}(J) \setminus \{0\} \quad \text{with} \quad b \geq 0 \tag{2.4}$$

and we set

$$\Phi(\rho) := B(\rho)R(\rho)b \quad \text{for} \quad \rho \in \mathcal{V}.$$

It follows from Lemma 2.3 and the definition of R that Φ maps $\mathcal{V}$ into $h^{1+\alpha}(\Gamma)$. Given $\rho_0 \in \mathcal{V}$, we now consider the nonlinear evolution equation in $h^{1+\alpha}(\Gamma)$ for the operator Φ:

$$\partial_t \rho + \Phi(\rho) = 0, \qquad \rho(0) = \rho_0. \tag{2.5}$$

A function $\rho : I = [0,T] \to h^{1+\alpha}(\Gamma)$ is called a *classical Hölder solution* of (2.5) if

$$\rho \in C(I, \mathcal{V}) \cap C^1(I, h^{1+\alpha}(\Gamma))$$

and if ρ satisfies (2.5) pointwise on I. Using this notation it is now easy to state the following *reduction* of the transformed problem (2.2).

LEMMA 2.5. *Let $\rho_0 \in \mathcal{V}$ be given.*

(a) *If ρ is a classical Hölder solution of* (2.5)*, then the pair* $(R(\rho)b, \rho)$ *is a classical Hölder solution of* (2.2).

(b) *Suppose that* (v, ρ) *is a classical Hölder solution of* (2.2). *Then ρ is a classical Hölder solution of* (2.5).

Proof. This follows immediately from the definition of $R(\rho)$. ∎

In order to treat the nonlinear evolution equation (2.5), we first show that $\Phi(\rho)$ depends smoothly on $\rho \in \mathcal{V}$ and we provide an appropriate representation of the Fréchet derivative $\partial\Phi(\rho)$ of Φ at $\rho \in \mathcal{V}$. For this we introduce for each $\rho \in \mathcal{V}$ the following linear operators:

$$K := K(\rho) := -\partial A(\rho)[\cdot, R(\rho)b] \in \mathcal{L}(h^{2+\alpha}(\Gamma), h^\alpha(\Omega)),$$
$$M := M(\rho) := \partial B(\rho)[\cdot, R(\rho)b] \in \mathcal{L}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)).$$

Here, the notation $\partial A(\rho)[h, v]$ stands for

$$\partial A(\rho)[h,v] = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} A(\rho + \varepsilon h)v, \qquad h \in h^{2+\alpha}(\Gamma),\ v \in h^{2+\alpha}(\Omega).$$

LEMMA 2.6. $\Phi \in C^\omega(\mathcal{V}, h^{1+\alpha}(\Gamma))$ *with*

$$\partial\Phi(\rho) = B(\rho)S(\rho)K(\rho) + M(\rho)$$

for each $\rho \in \mathcal{V}$.

Proof. (a) Due to Lemma 2.3, it suffices to show that

$$[\rho \mapsto R(\rho)b] \in C^\omega(\mathcal{V}, h^{2+\alpha}(\Omega)) \quad \text{with} \quad \partial(R(\rho)b) = S(\rho)K(\rho).$$

(b) Recall that $\mathcal{V}$ is an open subset of $h^{2+\alpha}$. Let γ denote the trace operator with respect to Γ and let

$$F(\rho, v) := (A(\rho)v, \gamma v, Cv - b), \qquad (\rho, v) \in \mathcal{V} \times h^{2+\alpha}(\Omega).$$

Then it follows from Lemma 2.3 that

$$F \in C^\omega\big(\mathcal{V} \times h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha}(\Gamma) \times h^{2+\alpha-\delta}(J)\big).$$

Moreover, given $(\rho, v) \in \mathcal{V} \times h^{2+\alpha}(\Omega)$, we have that

$$\partial_2 F(\rho, v)w = (A(\rho)w, \gamma w, Cw) \quad \text{and} \quad \partial_1 F(\rho, v)h = (\partial A(\rho)[h, v], 0, 0)$$

for $w \in h^{2+\alpha}(\Omega)$ and $h \in h^{2+\alpha}(\Gamma)$. Now the assertion follows from Lemma 2.4 and the implicit function theorem. ∎

The next two sections are devoted to the study of the linearization $\partial\Phi(\rho)$ of Φ. We will see that it is a nicely behaving operator; i.e., we will prove that $-\partial\Phi(\rho)$ generates a strongly continuous analytic semigroup on $h^{1+\alpha}(\Gamma)$.

3. Localizations. Given $\kappa \in (0, a]$, let $\mathcal{R}_\kappa := \mathcal{N}(\Gamma \times (-\kappa, 0])$. Then there exists $m := m_\kappa \in \mathbb{N}$ and an atlas $\{(U_l, \varphi_l)\,;\, 1 \le l \le m\}$ of $\mathcal{R}_\kappa$ such that $\operatorname{diam}(U_l) < 2\kappa$ for all $l \in \{1, \dots, m\}$. Let

$$s_l \in C^\infty((-\delta, \delta)^{n-1}, U_l), \quad l \in \{1, \dots, m\},$$

be a parameterization of $U_l \cap \Gamma$. Furthermore, let $P := (-\delta, \delta)^{n-1}$ and $Q := P \times [0, \delta)$ and define

$$\mu_l : Q \to U_l, \qquad (\omega, r) \mapsto s_l(\omega) - r\nu(s_l(\omega)).$$

Without loss of generality, we may assume that $\delta = \kappa$ and that $\mu_l := \varphi_l^{-1}$ for $1 \le l \le m$. The additional parameter κ is introduced to control the size of the chart domain U_l. This fact will be used in section 5 to prove a perturbation result; cf. Lemma 5.1. Finally, to further economize our notation, we set $\mu := \mu_l$, $U := U_l$ and we let

$$\mu^* u := u \circ \mu, \quad u \in C(U_l) \quad \text{and} \quad \mu_* v := v \circ \mu^{-1}, \quad v \in C(Q)$$

denote the pull-back and push-forward operators, respectively, induced by μ. Given $l \in \{1, \dots, m\}$, we define *local representations* $\mathcal{A} := \mathcal{A}_l$ and $\mathcal{B} := \mathcal{B}_l$ of A and B with respect to (Q, μ_l) by setting

$$\mathcal{A}(\mu^*\rho)\mu^* = \mu^* A(\rho) \quad \text{and} \quad \mathcal{B}(\mu^*\rho)\mu^* = \mu^* B(\rho), \quad \rho \in \mathcal{V},$$

respectively. To determine the coefficients of $\mathcal{A}$ and $\mathcal{B}$, let

$$\hat\rho := \hat\rho_l := \mu_l^* \rho, \quad \rho \in \mathcal{V}$$

and put $d(\omega, r) := \hat\rho(\omega) - r$ for $(\omega, r) \in Q$. In addition, we use the notation

$$\partial_j := \partial_{\omega_j}, \quad 1 \le j \le n-1, \qquad \partial_n := \partial_r.$$

Given $1 \le j,\ k \le n-1$, define

$$w_{jk} := (\partial_j s|\partial_k s) + d\big((\partial_j \mu^* \nu|\partial_k s) + (\partial_k \mu^* \nu|\partial_j s)\big) + d^2(\partial_j \mu^* \nu|\partial_k \mu^* \nu).$$

Clearly, $[w_{jk}]$ is symmetric. In addition, observe that $[(\partial_j s|\partial_k s)]$ is uniformly positive definite on P and that $\sup|d(\omega,r)| \le 2a$. Hence we may assume also that $[w_{jk}]$ is uniformly positive definite on Q, provided $a>0$ is small enough. Let w denote the inverse of $[w_{jk}]$ and let w^{jk} be the components of w. Finally, set

$$D(\omega,r) := \begin{pmatrix} \nabla\hat\rho \otimes \nabla\hat\rho & \nabla\hat\rho \\ (\nabla\hat\rho)^T & 1 \end{pmatrix}, \quad (\omega,r) \in Q,$$

and let

$$g_{jk} := g^l_{jk}(\rho) := (\partial_j \mu^* \theta_\rho|\partial_k \mu^* \theta_\rho), \quad 1 \le j,\ k \le n,$$

denote the components of the metric tensor with respect to (Q,μ). Note that

$$\mu^*\theta_\rho(\omega,r) = \theta_\rho(\mu(\omega,r)) = s(\omega) + d(\omega,r)\nu(s(\omega))$$

since $\varphi \equiv 1$ on $\mu(Q)$. In addition, observe that $d(w,r) = \hat\rho(\omega) - r$ is the function $-N_\rho$ in local coordinates. Using the orthogonality relations $(\partial_j s|\nu) = 0$ and $(\partial_j \nu|\nu) = 0$, direct calculations yield the formulas

$$[g_{jk}] = \begin{pmatrix} w^{-1} & 0 \\ 0 & 0 \end{pmatrix} + D \tag{3.1}$$

and

$$[g^{jk}] = \begin{pmatrix} w & -w\nabla\hat\rho \\ -(w\nabla\hat\rho)^T & 1 + (w\nabla\hat\rho|\nabla\hat\rho) \end{pmatrix}, \tag{3.2}$$

where $[g^{jk}]$ is the inverse of $[g_{jk}]$. From (3.1), (3.2), and the well-known formula (which essentially is Cramer's rule)

$$g_{nn} = \det[g_{jk}]_{1\le j,k\le n} \cdot \det[g^{jk}]_{1\le j,k\le n-1},$$

one then deduces that

$$G := \sqrt{\det[g_{jk}]_{1\le j,k\le n}} = \sqrt{\det w^{-1}}. \tag{3.3}$$

Finally, let W denote the uniformly elliptic second-order differential operator acting on $C^2(P)$ which is induced by w, i.e.,

$$W\sigma := -\sum_{j,k=1}^{n-1} w^{jk}\partial_j\partial_k\sigma, \qquad \sigma \in C^2(P).$$

In the next lemma, we use the following notation: given $\widetilde a \in C^\infty(Q \times \mathbb{R} \times \mathbb{R}^{n-1}, \mathbb{R})$ and $\sigma \in C^1(P)$, let $a(\sigma,\nabla\sigma)$ denote the Nemitskii operator induced by $\widetilde a$, i.e.,

$$a(\sigma,\nabla\sigma)(\omega,r) := \widetilde a((\omega,r),\sigma(\omega),\nabla\sigma(\omega)), \qquad (\omega,r) \in Q.$$

LEMMA 3.1. *There exist*

$$\widetilde{a}_{jk},\ \widetilde{a}_j,\ \widetilde{b}_j \in C^\infty(Q\times(-a,a)\times\mathbb{R}^{n-1},\mathbb{R}),\qquad 1\le j,\ k\le n,$$

such that

$$\begin{aligned}&[\widetilde{a}_{jk}] \quad \text{is symmetric and uniformly positive definite,}\\ &\widetilde{b}_n \quad \text{is uniformly positive}\end{aligned}\tag{3.4}$$

on compact subsets of $\overline{Q}\times(-a,a)\times\mathbb{R}^{n-1}$ *and such that*

$$\begin{aligned}\mathcal{A}(\hat\rho) &= -\sum_{j,k=1}^{n} a_{jk}(\hat\rho,\nabla\hat\rho)\partial_j\partial_k + \sum_{j=1}^{n} a_j(\hat\rho,\nabla\hat\rho)\partial_j + (W\hat\rho)\partial_n,\\ \mathcal{B}(\hat\rho) &= -\sum_{j=1}^{n} b_j(\hat\rho,\nabla\hat\rho)\partial_j.\end{aligned}\tag{3.5}$$

Proof. Recall that $A(\rho)$ and $B(\rho)$ are just the Laplace–Beltrami operator of $(\Omega,\theta_\rho^*\eta)$ and the outer normal derivative on Γ of $(\overline{\Omega},\theta_\rho^*\eta)$, respectively, where η denotes the standard Euclidean metric on $\mathbb{R}^m$; see the proof of Lemma 2.3. Hence assertion (3.4) is obvious, since $(\mathcal{A},\mathcal{B})$ is a representation of (A,B) in local coordinates. The explicit decomposition of the coefficient of ∂_n of $\mathcal{A}$ follows from (3.2). ∎

We close this section by determining the local representations of $K(\rho)$ and $M(\rho)$ according to the parameterization (Q,μ). In order to do this, we introduce

$$\begin{aligned}\mathcal{K} &:= \mathcal{K}(\rho) := -\partial\mathcal{A}(\hat\rho)[\cdot,\mu^*(R(\rho)b)] \in \mathcal{L}(h^{2+\alpha}(P),h^\alpha(\mathring{Q})),\\ \mathcal{M} &:= \mathcal{M}(\rho) := \partial\mathcal{B}(\hat\rho)[\cdot,\mu^*(R(\rho)b)] \in \mathcal{L}(h^{2+\alpha}(P),h^{1+\alpha}(P))\end{aligned}$$

for each $\rho\in\mathcal{V}$.

LEMMA 3.2. *Given* $\rho\in\mathcal{V}$, *we have*

$$\mu^*K(\rho)=\mathcal{K}(\rho)\mu^* \qquad \text{and} \qquad \mu^*M(\rho)=\mathcal{M}(\rho)\mu^*.$$

Proof. Fix $\rho\in\mathcal{V}$. To shorten our notation, we write $v:=R(\rho)b$ and $\hat h:=\mu^*h$ for $h\in h^{2+\alpha}(\Gamma)$. Then we have

$$\begin{aligned}\mu^*K(\rho)h=\mu^*\partial A(\rho)[h,v] &= \mu^*A(\rho+h)v-\mu^*A(\rho)v+o(h)\\ &= \mathcal{A}(\hat\rho+\hat h)\mu^*v-\mathcal{A}(\hat\rho)\mu^*v+o(\hat h)\\ &= \partial\mathcal{A}(\hat\rho)[\hat h,\mu^*v]\\ &= \mathcal{K}(\rho)\mu^*h\end{aligned}$$

as $h\to 0$ in $h^{2+\alpha}(\Gamma\cap U_l)$. The second assertion can be proved analogously. ∎

LEMMA 3.3. *There exist*

$$\widetilde{k}_j,\ \widetilde{m}_j\in C^\infty(Q\times(-a,a)\times\mathbb{R}^{n-1},\mathbb{R}),\quad j=0,\dots,n-1,$$

such that

$$\begin{aligned}\mathcal{K}h &= -\partial_n[\mu^*(R(\rho)b)]Wh+\sum_{j=1}^{n-1}k_j(\hat\rho,\nabla\hat\rho)\partial_j h+k_0(\hat\rho,\nabla\hat\rho)h,\\ \mathcal{M}h &= \sum_{j=1}^{n-1}m_j(\hat\rho,\nabla\hat\rho)\partial_j h+m_0(\hat\rho,\nabla\hat\rho)h\end{aligned}$$

for each $h \in h^{2+\alpha}(P)$. Here again, k_j and m_j denote the Nemitskii operators induced by $\widetilde{k}_j$ and $\widetilde{m}_j$, respectively.

Proof. The above assertions follow easily from Lemma 3.1. ∎

4. Fourier multiplier operators. In this section we are concerned with linear differential operators having constant coefficients, obtained by freezing the local representation $(\mathcal{A}, \mathcal{B})$ of (A, B) at $\rho \in \mathcal{V}$ and at $0 \in Q$. These operators are used to associate a Fourier multiplier operator $\mathcal{G}_1$ to the Fréchet derivative $\partial\Phi(\rho)$ of Φ at ρ.

Throughout this section we fix $\rho \in \mathcal{V}$ and $l \in \{1, \ldots, m_\kappa\}$. Of course, all operators appearing in this section will depend on the choice (ρ, l). However, we will suppress this dependence throughout this section. Let $\mathrm{H}^n = \mathbb{R}^{n-1} \times (0,1)$ denote the truncated half-space in $\mathbb{R}^n$, and let γ_0 denote the restriction operator from H^n to $\mathbb{R}^{n-1} \times \{0\} \equiv \mathbb{R}^{n-1}$. Moreover, we set

$$a^0_{jk} := a_{jk}(\hat\rho)(0), \qquad b^0_j := b_j(\hat\rho)(0), \qquad 1 \le j,\ k \le n, \tag{4.1}$$

and we define the following linear differential operators with constant coefficients:

$$\mathcal{A}_0 := -\sum_{j,k=1}^{n} a^0_{jk}\partial_j\partial_k, \qquad \mathcal{B}_0 := -\sum_{j=1}^{n} b^0_j \gamma_0 \partial_j.$$

Furthermore, let

$$\vec{a} := (a^0_{1n}, \ldots, a^0_{(n-1)n}), \qquad a_0 := \sum_{j,k=1}^{n-1} a^0_{jk}\xi^j\xi^k, \quad \xi \in \mathbb{R}^{n-1},$$

and define for fixed $\xi \in \mathbb{R}^{n-1}$ the following parameter-dependent quadratic polynomial:

$$q_\xi(z) := 1 + a_0(\xi) + 2i(\vec{a}|\xi)z - a^0_{nn}z^2, \quad z \in \mathbb{C}.$$

Since the matrix $[a^0_{jk}]$ is positive definite, it follows that, given $\xi \in \mathbb{R}^{n-1}$, there exists exactly one root $\lambda(\xi)$ of $q_\xi(\cdot)$ with positive real part, which is given by

$$\lambda(\xi) = \frac{i(\vec{a}|\xi)}{a^0_{nn}} + \frac{1}{a^0_{nn}}\sqrt{a^0_{nn}(1 + a_0(\xi)) - (\vec{a}|\xi)^2}.$$

Finally, we set

$$\vec{b} := (b^0_1, \ldots, b^0_{n-1}), \qquad \vec{m} := (m^0_1, \ldots, m^0_{n-1}).$$

In the following, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively, in $\mathbb{R}^{n-1}$. We are now ready to introduce the following Fourier multiplier operators, acting on functions defined on $\mathbb{R}^{n-1}$.

$$\mathcal{T}_0 g(x,y) := [\mathcal{F}^{-1} e^{-\lambda(\cdot)y}\mathcal{F}g](x), \tag{4.2}$$

where $g \in h^{2+\alpha}(\mathbb{R}^{n-1})$ and $(x,y) \in \mathrm{H}^n$. Moreover,

$$\mathcal{S}_0 h(x,y) := \left[\mathcal{F}^{-1}(1 - e^{-\lambda(\cdot)y})\frac{1}{1 + a_0(\cdot)}\,\mathcal{F}h\right](x), \tag{4.3}$$

for $h \in h^\alpha(\mathbb{R}^{n-1})$ and $(x,y) \in \mathrm{H}^n$. Then it can be shown that

$$
\begin{aligned}
&\mathcal{T}_0 \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{2+\alpha}(\mathrm{H}^n)),\\
&\mathcal{S}_0 \in \mathcal{L}(h^{\alpha}(\mathbb{R}^{n-1}), h^{2+\alpha}(\mathrm{H}^n));
\end{aligned}
\tag{4.4}
$$

see Appendices A and B in [12]. Next note that the function $u = \mathcal{T}_0 g$ solves the elliptic boundary value problem

$$(1+\mathcal{A}_0)u = 0 \quad \text{in} \quad \mathrm{H}^n, \qquad \gamma_0 u = g \quad \text{on} \quad \mathbb{R}^{n-1},$$

whereas $v = \mathcal{S}_0 h$ is a solution of

$$(1+\mathcal{A}_0)v = h \quad \text{in} \quad \mathrm{H}^n, \qquad \gamma_0 v = 0 \quad \text{on} \quad \mathbb{R}^{n-1},$$

where we use the same notation for the extended function $\tilde{h}(x,y) := h(x)$, $(x,y) \in \mathrm{H}^n = \mathbb{R}^{n-1} \times (0,1)$. In addition, we define

$$
\begin{aligned}
&k^0 := \big(\partial_n[\mu^*(R(\rho)b)]\big)(0),\\
&w_0^{jk} := w^{jk}(0), \qquad 1 \le j,k \le n-1.
\end{aligned}
\tag{4.5}
$$

Note that $\big(\partial_n[\mu^*(R(\rho)b)]\big)(0) = -\big(\partial_\nu[R(\rho)b]\big)(\mu(0))$. Hence, it follows from (2.3) that k^0 is positive. Given $h \in h^{2+\alpha}(\mathbb{R}^{n-1})$, let

$$(\mathcal{K}_0 h)(x) := -k^0 \left[1 - \sum_{j,k=1}^{n-1} w_0^{jk} \partial_j \partial_k\right] h(x), \quad x \in \mathbb{R}^{n-1}. \tag{4.6}$$

It is then obvious that

$$\mathcal{K}_0 \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{\alpha}(\mathbb{R}^{n-1})). \tag{4.7}$$

Similarly, we set $m_j^0 := m_j(\hat{\rho})(0)$ and define

$$\mathcal{M}_0 h := \sum_{j=1}^{n-1} m_j^0 \partial_j h, \qquad h \in h^{2+\alpha}(\mathbb{R}^{n-1}).$$

Now let $t \in [0,1]$ be given and set

$$\mathcal{G}_t := t(\mathcal{B}_0 \mathcal{S}_0 \mathcal{K}_0 + \mathcal{M}_0) + (1-t)\mathcal{B}_0 \mathcal{T}_0.$$

Observe that $\mathcal{G}_t \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1}))$ for $t \in [0,1]$, as (4.4) and (4.7) show. Since $\mathcal{K}_0$ and $\mathcal{M}_0$ are the principal parts of $\mathcal{K}$ and $\mathcal{M}$, respectively, with coefficients fixed at $\rho \in \mathcal{V}$ and at $0 \in Q$, the operator $\mathcal{G}_1$ may be considered as the constant coefficient operator of the principal part of $\partial\Phi(\rho)$. The operator BT is called the *Dirichlet–Neumann* operator. Hence $\mathcal{G}_0$ is the constant coefficient version of the localization $\mathcal{BT}$ of BT; see also [11]. We should mention that we slightly modified the concepts and notations as introduced in [11] and [12]. However, an inspection of the proofs given in [12] show that formula (4.4) can be proved in the same way by using Fourier multiplier results in Hölder spaces; see [12, App. A.]. We can now prove the following result.

LEMMA 4.1. *Given $t \in [0,1]$, the operator $\mathcal{G}_t$ is a Fourier multiplier operator with symbol g_t; i.e., $\mathcal{G}_t = \mathcal{F}^{-1} g_t \mathcal{F}$ where*

$$g_t(\xi) := b_n^0 \lambda(\xi) \left\{ (1-t) + tk^0 \, \frac{(1 + w_0^{jk} \xi_j \xi_k)}{1 + a_0(\xi)} \right\} + i\big\{ ((t-1)\vec{b} + t\vec{m} | \xi) \big\}$$

for all $\xi \in \mathbb{R}^{n-1}$.

Proof. (a) In a first step we provide a representation of $\mathcal{S}_0 \mathcal{K}_0$. It is an immediate consequence of (4.6) that the Fourier transform of $\mathcal{K}_0 h$ is given by

$$(\mathcal{F}\mathcal{K}_0 h)(\xi) = -k^0 \, (1 + w_0^{jk} \xi_j \xi_k)(\mathcal{F}h)(\xi)$$

for $h \in h^{2+\alpha}(\mathbb{R}^{n-1})$ and $\xi \in \mathbb{R}^{n-1}$. Now it follows from (4.3) that

$$(\mathcal{F} S_0 \mathcal{K}_0 h)(\xi, y) = -(1 - e^{-\lambda(\xi) y}) k^0 \, \frac{(1 + w_0^{jk} \xi_j \xi_k)}{1 + a_0(\xi)} (\mathcal{F}h)(\xi), \tag{4.8}$$

where $\xi \in \mathbb{R}^{n-1}$ and $y \in (0,1)$.

(b) Observe that $\gamma_0 \partial_j u = \partial_j \gamma_0 u$ for $u \in h^{2+\alpha}(\mathrm{H}^n)$ and $j = 1, \ldots, n-1$. Hence (4.8) yields

$$\mathcal{B}_0 \mathcal{S}_0 \mathcal{K}_0 h = \mathcal{F}^{-1} \left[b_n^0 \lambda(\xi) k^0 \, \frac{(1 + w_0^{jk} \xi_j \xi_k)}{1 + a_0(\xi)} \mathcal{F}h \right]. \tag{4.9}$$

From formula (4.2) we infer that

$$b_j^0 \gamma_0 \partial_j \mathcal{T}_0 = \mathcal{F}^{-1} [\xi \mapsto i b_j^0 \xi_j] \mathcal{F}, \qquad j = 1, \ldots, n-1,$$

and

$$b_n^0 \gamma_0 \partial_n \mathcal{T}_0 = -\mathcal{F}^{-1} b_n^0 \lambda(\cdot) \mathcal{F}.$$

Hence we find that

$$\mathcal{B}_0 \mathcal{T}_0 = \mathcal{F}^{-1} [\xi \mapsto b_n^0 \lambda(\xi) - i(\vec{b} | \xi)] \mathcal{F}. \tag{4.10}$$

Finally, it is clear that

$$\mathcal{M}_0 = \mathcal{F}^{-1} [\xi \mapsto i(\vec{m} | \xi)] \mathcal{F}. \tag{4.11}$$

Combining (4.9)–(4.11), we get the assertion. ∎

As a first consequence of Lemma 4.1, we show that $-\mathcal{G}_t$ generates for each $t \in [0,1]$ a strongly continuous analytic semigroup on $h^{1+\alpha}(\mathbb{R}^{n-1})$. To make this precise we need a few definitions. To begin with, assume that $\alpha_* > 0$, $\sigma > 0$ and let

$$\begin{aligned} \mathcal{E}ll\mathcal{S}_\sigma^\infty(\alpha_*) := \big\{ & a \in C^\infty(\mathbb{R}^{n-1} \times (0,\infty)) \,;\, a \text{ is positively homogeneous} \\ & \text{of degree } \sigma, \text{ all derivatives of } a \text{ are bounded on } |\xi|^2 + \mu^2 = 1, \\ & \text{and } \operatorname{Re} a(\xi,\mu) \geq \alpha_* (|\xi|^2 + \mu^2)^{\sigma/2},\ (\xi,\mu) \in \mathbb{R}^{n-1} \times (0,\infty) \big\}. \end{aligned}$$

Given two Banach spaces E_0 and E_1 such that E_1 is continuously and densely embedded in E_0, let $\mathcal{H}(E_1, E_0)$ denote the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as an unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0. It is known (see Remark I.1.2.1(a) in [2]) that $A \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{H}(E_1, E_0)$ if there exist positive constants C and λ_* such that

$$\begin{aligned} &\lambda_* + A \in \operatorname{Isom}(E_1, E_0), \\ &|\lambda|\, \|x\|_{E_0} + \|x\|_{E_1} \leq C\|(\lambda + A)x\|_{E_0}, \quad x \in E_1,\ \lambda \in [\operatorname{Re} z \geq \lambda_*]. \end{aligned} \tag{4.12}$$

THEOREM 4.2. *Suppose that* (2.3) *holds. Then*

$$\mathcal{G}_t \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1})), \qquad t \in [0,1].$$

Proof. (a) Basically, the idea is to use Lemma 4.1 together with appropriate results on Fourier multipliers to verify the generation property of $\mathcal{G}_t$. Having this intention, it is well known that homogeneous symbols are much easier to handle. Hence, in a first step we introduce a parameter-dependent version of the symbol g_t, which is positively homogeneous of degree 1. Given $(\xi, \mu) \in \mathbb{R}^{n-1} \times (0, \infty)$, let

$$\lambda(\xi, \mu) := \frac{i(\vec{a}|\xi)}{a_{nn}^0} + \frac{1}{a_{nn}^0}\sqrt{a_{nn}^0(\mu^2 + a_0(\xi)) - (\vec{a}|\xi)^2}$$

and $r(\xi, \mu) := \operatorname{Re}(\lambda(\xi, \mu))$. Then we set

$$\widetilde{g}_t(\xi, \mu) := b_n^0 \lambda(\xi, \mu)\left\{(1-t) + tk^0\, \frac{(\mu^2 + w_0^{jk}\xi_j\xi_k)}{\mu^2 + a_0(\xi)}\right\} + i\{((t-1)\vec{b} + t\vec{m}|\xi)\},$$

for $(\xi, \mu) \in \mathbb{R}^{n-1} \times (0, \infty)$ and $t \in [0,1]$. Obviously, $\widetilde{g}_t(\cdot, 1) = g_t$. Moreover, it is clear that $\widetilde{g}_t \in C^\infty(\mathbb{R}^{n-1} \times (0, \infty), \mathbb{C})$ and that each $\widetilde{g}_t$ is positively homogeneous of degree 1. In addition, it is easily verified that all derivatives of a are bounded on $|\xi|^2 + \mu^2 = 1$.

(b) Observe that $k^0 > 0$, thanks to assumption (2.4) and (2.3). In addition, we know from (3.4) and (3.5) that $a_{nn}^0 > 0$ and $b_n^0 > 0$. Furthermore, there exist positive constants K and r_* such that

$$\mu^2 + a_0(\xi) \leq K(\mu^2 + |\xi|^2), \qquad r(\xi, \mu) \geq r_*\sqrt{\mu^2 + |\xi|^2}$$

for all $(\xi, \mu) \in \mathbb{R}^{n-1} \times (0, \infty)$. The first estimate follows immediately from the definition of a_0. The second one is a consequence of the ellipticity of $[a_{jk}]_{1 \leq j,k \leq n}$. Finally, recall that w is uniformly positive definite; see section 3. Hence there is a positive constant $w_* > 0$ such that $(\mu^2 + w_0^{jk}\xi_j\xi_k) \geq w_*(\mu^2 + |\xi|^2)$ for all $(\xi, \mu) \in \mathbb{R}^{n-1} \times (0, \infty)$. This leads to an estimate

$$\begin{aligned} \operatorname{Re} \widetilde{g}_t(\xi, \mu) =&\ b_n^0 r(\xi, \mu)\left\{(1-t) + tk^0\, \frac{(\mu^2 + w_0^{jk}\xi_j\xi_k)}{\mu^2 + a_0(\xi)}\right\} \\ \geq&\ b_n^0 r_*\sqrt{\mu^2 + |\xi|^2}\left\{(1-t) + tk^0\, \frac{w_*(\mu^2 + |\xi|^2)}{K(\mu^2 + |\xi|^2)}\right\} \\ \geq&\ b_n^0 r_*\sqrt{\mu^2 + |\xi|^2}\{(1-t) + tk_*\}, \end{aligned}$$

where $k_* := k^0 K^{-1} w_* > 0$. Now, letting

$$\alpha_* := r_* b_n^0 \min\{1, k_*\} > 0,$$

we find that $\widetilde{g}_t \in \mathcal{E}ll\mathcal{S}_1^\infty(\alpha_*)$ for all $t \in [0,1]$. Now the assertion is implied by a general result due to Amann, which in particular states that given $a \in \mathcal{E}ll\mathcal{S}_1^\infty(\alpha_*)$ and $\mu_0 > 0$; it follows that $a(\cdot, \mu_0) \in \mathcal{H}(h^{2+\alpha}(\mathbb{R}^{n-1}), h^{1+\alpha}(\mathbb{R}^{n-1}))$; see [3]. ∎

5. Perturbations. In this section we prove that, given $\rho \in \mathcal{V}$, the linearization $-\partial\Phi(\rho)$ of $-\Phi$ at ρ generates a strongly continuous analytic semigroup on $h^{1+\alpha}(\Gamma)$. The main technical tool is a perturbation result contained in Lemma 5.1. To state this result we need some preparation. First let

$$\partial\Phi_t(\rho) := t\partial\Phi(\rho) + (1-t)B(\rho)T(\rho)$$

for $\rho \in \mathcal{V}$ and $t \in [0,1]$. Obviously, $\partial\Phi_t(\rho)$ is a convex combination connecting $\partial\Phi(\rho)$ and the Dirichlet–Neumann operator $B(\rho)T(\rho)$; see [11].

Next, given $\kappa \in (0,a]$, choose smooth test functions $\psi_l \in \mathcal{D}(U_l)$ such that $\{(U_l,\psi_l)\ ;\ 1 \le l \le m_\kappa\}$ is a partition of unity on $\mathcal{R}_\kappa$; see section 3 for the definition of $\mathcal{R}_\kappa$. Call such a family $\{(U_l,\psi_l)\ ;\ 1 \le l \le m_\kappa\}$ a (finite) *localization sequence* for $\mathcal{R}_\kappa$. Moreover, we fix $\hat{x}_l \in \Gamma$ such that $\hat{x}_l \in U_l$, $l = 1,\dots,m_\kappa$. We may further assume that $\mu_l(0) = \hat{x}_l$ for $l = 1,\dots,m_\kappa$.

To economize our notation, the symbols $|\cdot|_s$ and $\|\cdot\|_s$ are exclusively used for the norms in $h^s(\mathbb{R}^{n-1})$ and $h^s(\Gamma)$, respectively.

Finally, throughout this section we fix $\rho \in \mathcal{V}$ and $\beta \in (0,\alpha)$.

LEMMA 5.1. *Given $\varepsilon > 0$, there exists $\kappa \in (0,a]$, a localization sequence $\{(U_l,\psi_l);$ $1 \le l \le m_\kappa\}$ for $\mathcal{R}_\kappa$, and a positive constant $C := C(\rho,\varepsilon,\kappa)$ such that*

$$|\mu_l^*(\psi_l\partial\Phi_t(\rho)h) - \mathcal{G}_t(\rho,l)\mu_l^*(\psi_l h)|_{1+\alpha} \le \varepsilon|\mu_l^*(\psi_l h)|_{2+\alpha} + C\|h\|_{2+\beta}$$

for all $h \in h^{2+\alpha}(\Gamma)$, $l \in \{1,\dots,m_\kappa\}$, and $t \in [0,1]$.

Proof. (a) We fix $\rho \in \mathcal{V}$, $l \in \{1,\dots,m_\kappa\}$ and suppress the pair (ρ,l) in our notation. Moreover, given $\varepsilon > 0$ and $\beta \in (0,\alpha)$, we only show explicitly the existence of a positive constant C such that

$$|\mu^*(\psi BSKh) - \mathcal{B}_0\mathcal{S}_0\mathcal{K}_0\mu^*(\psi h)|_{1+\alpha} \le \varepsilon|\mu^*(\psi h)|_{2+\alpha} + C\|h\|_{2+\beta}$$

for all $h \in h^{2+\alpha}(\Gamma)$. The remaining two terms

$$|\mu^*(\psi BTh) - \mathcal{B}_0\mathcal{T}_0\mu^*(\psi h)|_{1+\alpha}, \qquad |\mu^*(\psi Mh) - \mathcal{M}_0\mu^*(\psi h)|_{1+\alpha}$$

can be estimated similarly (and are even easier to handle). Our argumentation follows the proof of Lemma 6.1 in [12] and uses in particular obvious generalizations of Lemmas 6.5, 6.6, and 6.7 in [12] to the n-dimensional case.

(b) Choose a smooth test-function $\chi \in \mathcal{D}(U)$ such that $\chi|\mathrm{supp}(\psi) = 1$. Then we have

$$\mu^*\psi BSK - \mathcal{B}_0\mathcal{S}_0\mathcal{K}_0\mu^*\psi = \mu^*\chi BSK\psi - \mathcal{B}_0\mathcal{S}_0\mathcal{K}_0\mu^*\chi\psi - \mu^*\chi[BSK,\psi],$$

where ψ and χ also denote the linear operators induced by pointwise multiplication by ψ and χ, respectively, and where $[A,B] := AB - BA$ denotes the commutator of A and B. It follows, essentially from Leibniz' rule (see Lemma 6.5(b) in [12]), that there exists a positive constant C such that

$$\|[BSK,\psi]h\|_{1+\alpha} \le C\|h\|_{2+\beta}, \qquad h \in h^{2+\alpha}(\Gamma).$$

Hence, it suffices to estimate the operator

$$\mu^*\chi BSK - \mathcal{B}_0\mathcal{S}_0\mathcal{K}_0\mu^*\chi.$$

In addition, we split that operator in the following way:

$$\mu^*\chi BSK - \mathcal{B}_0\mathcal{S}_0\mathcal{K}_0\mu^*\chi = \mu^*\chi BSK - \mathcal{B}_0\mu^*\chi SK + \mathcal{B}_0\{\mu^*\chi S - \mathcal{S}_0\mu^*\chi\}K + \mathcal{B}_0\mathcal{S}_0\{\mu^*\chi K - \mathcal{K}_0\mu^*\chi\}. \tag{5.1}$$

(c) Let us start with the first term $\mu^*\chi BSK - \mathcal{B}_0\mu^*\chi SK$. Again, by Leibniz' rule, the commutator $[\mu^*\chi, \mathcal{B}_0]$ can be estimated as

$$|[\mu^*\chi, \mathcal{B}_0]u|_{1+\alpha} \le C|u|_{1+\alpha, H^n}, \qquad u \in h^{2+\alpha}(\mathrm{H}^n). \tag{5.2}$$

Thus we are left to control the operator $\mu^*\chi B - (\mu^*\chi)\mathcal{B}_0\mu^*$. By the definition of $\mathcal{B}$ we get the formula

$$\mu^*\chi B - (\mu^*\chi)\mathcal{B}_0\mu^* = (\mu^*\chi)\{\mathcal{B} - \mathcal{B}_0\}\mu^*. \tag{5.3}$$

But, as in [12, Lemma 6.7(a)], we find positive constants C and C_κ such that

$$|(\mu^*\chi)\{1 + \mathcal{A}_0 - \mathcal{A}\}(\mu^*v)|_{\alpha, H^n} + |(\mu^*\chi)\{\mathcal{B} - \mathcal{B}_0\}(\mu^*v)|_{1+\alpha} \le C\kappa^{1-\alpha}\|v\|_{2+\alpha,\Omega} + C_\kappa\|v\|_{1+\alpha,\Omega} \tag{5.4}$$

for all $v \in h^{2+\alpha}(\Omega)$. Finally, observe that

$$S \in \mathcal{L}(h^\gamma(\Omega), h^{2+\gamma}(\Omega)), \qquad K \in \mathcal{L}(h^{2+\gamma}(\Gamma), h^\gamma(\Omega)) \tag{5.5}$$

for $\gamma \in [\beta, \alpha]$ and that

$$\mu^* \in \mathrm{Diff}^\infty(h^{2+\alpha}(\Gamma \cap U), h^{2+\alpha}(P)). \tag{5.6}$$

Combining (5.2)–(5.6), we can find a $\kappa_1 \in (0, a]$ and a positive constant C such that

$$|\mu^*(\chi BSKg) - \mathcal{B}_0\mu^*(\chi SKg)|_{1+\alpha} \le \frac{\varepsilon}{3}|\mu^*g|_{2+\alpha} + C\|g\|_{2+\beta} \tag{5.7}$$

for all $g \in h^{2+\alpha}(\Gamma \cap U)$.

(d) In a next step we estimate the operator $\mu^*\chi S - \mathcal{S}_0\mu^*\chi$. To achieve this, we use the representation

$$\mu^*\chi S - \mathcal{S}_0\mu^*\chi = \mathcal{S}_0\big\{[\mathcal{A}_0, \mu^*\chi]\mu^*S + (\mu^*\chi)\{1 + \mathcal{A}_0 - \mathcal{A}\}\mu^*S\big\}, \tag{5.8}$$

which follows from Lemma 6.6 in [12]. Again, the operator $[\mathcal{A}_0, \mu^*\chi]$ is of lower order in the sense that there exists a positive constant C such that

$$|[\mathcal{A}_0, \mu^*\chi]u|_{\alpha, \mathring{Q}} \le C|u|_{1+\alpha, \mathring{Q}}, \qquad u \in h^{1+\alpha}(\mathring{Q}). \tag{5.9}$$

Hence, it follows from (5.4), (5.5), (5.6), (5.8), and (5.9) that there is a $\kappa_2 \in (0, a]$ such that

$$|\mathcal{B}_0\big\{\mu^*(\chi SKg) - \mathcal{S}_0\mu^*(\chi Kg)\big\}|_{1+\alpha} \le \frac{\varepsilon}{3}|\mu^*g|_{2+\alpha} + C\|g\|_{2+\beta} \tag{5.10}$$

for all $g \in h^{2+\alpha}(\Gamma \cap U)$.

(e) From Lemma 3.2 we know that

$$\mu^*\chi K - \mathcal{K}_0\mu^*\chi = (\mu^*\chi)\{\mathcal{K} - \mathcal{K}_0\}\mu^* + [\mu^*\chi, \mathcal{K}_0]\mu^*.$$

But here again, it follows from Leibniz' rule that there is a $C > 0$ such that

$$|[\mu^*\chi, \mathcal{K}_0]\mu^* g|_{\alpha,H^n} \le C\|g\|_{1+\alpha}, \qquad g \in h^{2+\alpha}(\Gamma \cap U). \tag{5.11}$$

Finally, we infer from Lemma 6.7(b) in [12] that there are positive constants C and C_κ such that

$$|(\mu^*\chi)\{\mathcal{K} - \mathcal{K}_0\}g|_{\alpha,H^n} \le \kappa^{1-\alpha}C|\mu^* g|_{2+\alpha} + C_\kappa\|g\|_{1+\alpha} \tag{5.12}$$

for all $g \in h^{2+\alpha}(\Gamma \cap U)$. Since $\mathcal{B}_0\mathcal{S}_0 \in \mathcal{L}(h^\alpha(\mathrm{H}^n), h^{1+\alpha}(\Gamma))$, we conclude from (5.11) and (5.12) that there is a $\kappa_3 \in (0, a]$ and a $C > 0$ such that

$$|\mathcal{B}_0\mathcal{S}_0\big\{\mu^*\chi K - \mathcal{K}_0\mu^*\chi\big\}g|_{1+\alpha} \le \frac{\varepsilon}{3}|\mu^* g|_{2+\alpha} + C\|g\|_{2+\beta} \tag{5.13}$$

for all $g \in h^{2+\alpha}(\Gamma \cap U)$. Now, letting $\kappa := \min\{\kappa_1, \kappa_2, \kappa_3\}$, the assertion follows from (5.7), (5.10), and (5.13). □

THEOREM 5.2. *We have*

$$\partial\Phi_t(\rho) \in \mathcal{H}(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)), \qquad \rho \in \mathcal{V}, \quad t \in [0, 1].$$

Proof. (a) In a first step we provide a parameter-dependent a priori estimate for $\partial\Phi_t(\rho)$. To begin with, we know from Theorem 4.2 that there are positive constants λ_1 and C_1, independent of $\kappa \in (0, a]$ and $l \in \{1, \ldots, m_\kappa\}$, such that

$$|g|_{2+\alpha} + |\lambda||g|_{1+\alpha} \le C_1|(\lambda + \mathcal{G}_t(\rho, l))g|_{1+\alpha} \tag{5.14}$$

for all $g \in h^{2+\alpha}(\mathbb{R}^{n-1}),\ \lambda \in [\mathrm{Re}\, z \ge \lambda_1]$, and $l \in \{1, \ldots, m_\kappa\}$. Furthermore, Lemma 5.1 guarantees the existence of positive constants $\kappa,\ C_2$, and a localization sequence $\{(U_l, \psi_l)\,;\, 1 \le l \le m_\kappa\}$ such that

$$|\mu_l^*\big(\psi_l\partial\Phi_t(\rho)h\big) - \mathcal{G}_t(\rho, l)\mu_l^*(\psi_l h)|_{1+\alpha} \le \frac{1}{2C_1}|\mu_l^*(\psi_l h)|_{2+\alpha} + C_2\|h\|_{2+\beta}$$

for all $h \in h^{2+\alpha}(\Gamma)$, $l \in \{1, \ldots, m_\kappa\}$, and $t \in [0, 1]$. Consequently, it follows from (5.14) that

$$\begin{aligned} |\mu_l^*(\psi_l h)|_{2+\alpha} &+ |\lambda||\mu_l^*(\psi_l h)|_{1+\alpha} \\ &\le 2C_1\big\{|\mu_l^*\big(\psi_l(\lambda + \partial\Phi_t(\rho))h\big)|_{1+\alpha} + C_2\|h\|_{2+\beta}\big\} \end{aligned} \tag{5.15}$$

for all $h \in h^{2+\alpha}(\Gamma),\ \lambda \in [\mathrm{Re}\, z \ge \lambda_1]$, $l \in \{1, \ldots, m_\kappa\}$, and $t \in [0, 1]$. Next observe that

$$\big[h \mapsto \max_{1 \le l \le m_\kappa} |\mu_l^*(\psi_l h)|_{k+\alpha}\big]$$

defines an equivalent norm on $h^{k+\alpha}(\Gamma),\ k = 1, 2$, due to the fact that the family $\{(U_l, \psi_l)\,;\, 1 \le l \le m_\kappa\}$ is a localization sequence for $\mathcal{R}_\kappa$; see [24]. Hence (5.15) implies the existence of a positive constant C such that

$$\|h\|_{2+\alpha} + |\lambda|\|h\|_{1+\alpha} \le \frac{C}{2}\|(\lambda + \partial\Phi_t(\rho))h\|_{1+\alpha} + C\|h\|_{2+\beta} \tag{5.16}$$

for all $h \in h^{2+\alpha}(\Gamma),\ \lambda \in [\mathrm{Re}\, z \ge \lambda_1]$, and $t \in [0, 1]$.

Finally, let $(\cdot,\cdot)^0_{\theta,\infty}$ denote the continuous interpolation functor of Da Prato and Grisvard; see [6]. It is known that

$$h^{2+\beta}(\Gamma) = \big(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\Gamma)\big)^0_{1-\alpha+\beta,\infty}. \tag{5.17}$$

Hence there exists a positive constant C_3 such that

$$\|h\|_{2+\beta} \le \frac{1}{2C}\|h\|_{2+\alpha} + C_3\|h\|_{1+\alpha}, \qquad h \in h^{2+\alpha}(\Gamma).$$

Now we conclude from (5.17) that

$$\|h\|_{2+\alpha} + |\lambda|\|h\|_{1+\alpha} \le C\|(\lambda + \partial\Phi_t(\rho))h\|_{1+\alpha} \tag{5.18}$$

for all $h \in h^{2+\alpha}(\Gamma)$, $\lambda \in [\operatorname{Re} z \ge \lambda_*]$, and $t \in [0,1]$, where we have set $\lambda_* := 2\max\{\lambda_1, CC_3\}$.

(b) In view of (4.12) and (5.18), it remains to prove that $\partial\Phi_t(\rho)$ is surjective for each $t \in [0,1]$. Moreover, since the estimate (5.18) is uniform in $t \in [0,1]$, a well-known homotopy argument (see Theorem 5.2 in [17]) implies that it is sufficient to prove that $\partial\Phi_0(\rho)$ is onto. Thus, let $g \in h^{1+\alpha}(\Gamma)$ be given. Then we find, as in the proof of Lemma 2.4, a unique $v \in h^{2+\alpha}(\Omega)$ such that

$$\big(A(\rho), \lambda_*\gamma + B(\rho), C\big)v = (0, g, 0). \tag{5.19}$$

The first and the third components of this identity imply that

$$T(\rho)\gamma v = \big(A(\rho), \gamma, B(\rho)\big)^{-1}(0, \gamma v, 0) = v;$$

see section 2 for the definition of the operator $T(\rho)$. Now, putting $h := \gamma v \in h^{2+\alpha}(\Gamma)$, the second component of (5.19) gives

$$\big(\lambda_* + B(\rho)T(\rho)\big)h = \big(\lambda_*\gamma + B(\rho)\big)v = g,$$

which completes our argumentation. □

Remark 5.3. Let $\rho \in \mathcal{V}$ be given. Then the proofs of Theorems 4.2 and 5.2 show that $-\partial\Phi(\rho)$ does not generate a strongly continuous semigroup on $h^{1+\alpha}(\Gamma)$ if $b \in h^{2+\alpha-\delta}(J)\backslash\{0\}$ is nonpositive. Hence, for such b, the linearized evolution equation for the moving boundary

$$\partial_t\sigma + \partial\Phi(\rho)\sigma = 0, \qquad \sigma(0) = \sigma_0$$

is not well posed in $h^{1+\alpha}(\Gamma)$ in the sense of Hadamard.

Proof of Theorem 1. Let $\rho_0 \in \mathcal{V}$ be given. Thanks to Lemmas 2.2 and 2.5 we only have to prove the existence and uniqueness of a classical Hölder solution of (2.5). To show this, fix $\beta \in (0,\alpha)$. Then it follows from Theorem 5.2 that

$$\partial\Phi(\rho) \in \mathcal{H}(h^{2+\gamma}(\Gamma), h^{1+\gamma}(\Gamma)), \quad \rho \in \mathcal{V},\ \gamma \in [\beta,\alpha].$$

From this and the known fact that little Hölder spaces are stable under continuous interpolation one finds that

$$\partial\Phi(\rho) \in \mathcal{M}_1(h^{2+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)), \qquad \rho \in \mathcal{V}, \tag{5.20}$$

where $\mathcal{M}_1(E_1, E_0)$ denotes the class of all operators in $\mathcal{L}(E_1, E_0)$, having the property of maximal regularity in the sense of Da Prato and Grisvard [6]; see also [4] and [23]. The assertions now follow from Theorem 2.7 in [4].

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