# Center Manifolds for Quasilinear Reaction-Diffusion Systems

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**Abstract:** We consider strongly coupled quasilinear reaction-diffusion systems subject to nonlinear boundary conditions. Our aim is to develop a geometric theory for this type of equations. Such a theory is necessary in order to describe the dynamical behavior of solutions. In our main result we establish the existence and attractivity of center manifolds under suitable technical assumptions. The technical ingredients we need consist of the theory of strongly continuous analytic semigroups, maximal regularity, interpolation theory and evolution equations in extrapolation spaces.

## 1. Introduction

Already for ordinary differential equations, center manifolds form one of the cornerstones in the development of a qualitative theory. For partial differential equations, these are of even greater importance.

In this work we shall show the existence and attractivity of center manifolds for quasilinear parabolic evolution equations in a neighborhood of a non-hyperbolic critical point. We would like to illustrate our intention and some of our results in the important case of quasilinear reaction-diffusion systems.

For this, let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $\nu = (\nu^1, \ldots, \nu^n) \in \mathbb{R}^n$ be the unit outer normal field on $\partial\Omega$. We then consider the following system of **quasilinear reaction-diffusion** equations

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot, u) && \text{in} \quad \Omega \times (0,\infty)\ , \\ \mathcal{B}(u)u &= g(\cdot, u) && \text{on} \quad \partial\Omega \times (0,\infty)\ , \\ u(0) &= u_0 && \text{in} \quad \Omega\ . \end{aligned} \tag{1.1}$$

Here, $u = (u^1, \ldots, u^N)$ is a function with $N$- components and $\mathcal{A}(v)$ denotes a second order differential operator, i.e.

$$\mathcal{A}(v)u := -\partial_j(a_{jk}(\cdot, v)\partial_k u) + a_j(\cdot, v)\partial_j u + a_0(\cdot, v)u \tag{1.2}$$

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for $u \in W_p^2(\Omega, \mathbb{R}^N)$ and $v$ belonging to an appropriate function space. For the coefficients and the functions $f, g$, the dependence on the space variable is indicated by a dot and the summation convention is used.

For $\mathcal{B}(v)$ we take a boundary operator of Neumann type, that is,

$$\mathcal{B}(v)u := a_{jk}(\cdot, v)\nu^j \gamma \partial_k u + b_0(\cdot, v)\gamma u \ , \tag{1.3}$$

where $\gamma$ denotes the trace operator. We may also admit boundary operators which correspond to a Dirichlet condition on one part of the boundary and to a Neumann condition on another part and this could even vary in each component of $u$. This general situation will be explained in section 7.

In order to have simple statements we take the functions to be smooth, i.e

$$\begin{aligned} &a_{jk},\ a_j,\ a_0 \in C^\infty(\bar{\Omega} \times \mathbb{R}^N, \mathcal{L}(\mathbb{R}^N)),\ b_0 \in C^\infty(\partial\Omega \times \mathbb{R}^N, \mathcal{L}(\mathbb{R}^N)) \ , \\ &f \in C^\infty(\bar{\Omega} \times \mathbb{R}^N, \mathbb{R}^N),\ g \in C^\infty(\partial\Omega \times \mathbb{R}^N, \mathbb{R}^N) \ . \end{aligned} \tag{1.4}$$

Moreover, we impose appropriate ellipticity and complementing conditions upon the boundary value problem

$$(\mathcal{A}(v), \mathcal{B}(v)) \ , \tag{1.5}$$

i.e. $(\mathcal{A}(v), \mathcal{B}(v))$ is **normally elliptic** in the sense of Amann [**6**] for each $v$, see section 7.

Note that equation (1.1) is a **strongly coupled system of quasilinear parabolic equations subject to nonlinear boundary conditions**.

Our aim is to develop a qualitative theory (a geometric theory) for abstract quasilinear parabolic equations which covers the reaction-diffusion system (1.1). We pose the following questions:

Does there exist a space $X$ (a phase space), such that

a) given any $u_0 \in X$, the reaction-diffusion equation (1.1) has a unique (classical) solution $u := u(\cdot, u_0)$ on a time interval $(0, t^+(u_0))$?

b) the map $(t, u_0) \mapsto u(t, u_0)$ defines a semiflow on $X$?

The next steps towards a dynamic theory are, for example:

c) The existence of invariant manifolds on $X$, especially of center manifolds.

d) The study of bifurcation problems (e.g. Hopf bifurcation).

e) A stability analysis of bifurcating solutions (e.g. of bifurcating periodic solutions.)

By setting

$$X := W_p^1(\Omega, \mathbb{R}^N) \ , \quad p > n \ , \tag{1.6}$$

we have, due to the results of Amann in [**6**]:

*For any initial value $u_0 \in X$ there exists a unique classical solution*

$$u(\cdot, u_0) \in C([0, t^+(u_0)), X) \cap C^\infty(\bar{\Omega} \times (0, t^+(u_0)), \mathbb{R}^N) \tag{1.7}$$

*of the quasilinear equation (1.1) and the map*

$$(t, u_0) \mapsto u(t, u_0) \tag{1.8}$$

*defines a smooth semiflow on $X$. Moreover, bounded orbits are relatively compact in $X$ and bounded in $W_p^2$ for $t > 0$.*

For more general and additional results we refer to [**6**]. We merely assumed 'Neumann type boundary conditions' in order to have simpler statements. In the presence of Dirichlet conditions on some parts of the boundary, we take $X$ to be the Banach space of all functions in $W_p^1(\Omega, \mathbb{R}^N)$ satisfying the requested Dirichlet conditions. In addition, the coefficients and functions might be defined on $\bar{\Omega} \times G$ only, where $G$ is an open subset of $\mathbb{R}^N$. For example, $G$ can be an open neighborhood of zero in $\mathbb{R}^N$. This will occur when studying small solutions and the existence of invariant manifolds in a small neighborhood of 0. This general situation will be considered in section 8.

Let us remark that $W_p^1(\Omega, \mathbb{R}^N)$ is a very natural choice for a phase space. First, it is a 'simple' space which makes the statements easily accessible to the reader mainly interested in applications. Second, we do not have to assume nonlinear compatibility conditions on the initial values of (1.1), despite our (nonlinear) boundary conditions. Such compatibility conditions would come in by working in spaces with too much regularity, say in $W_p^2(\Omega, \mathbb{R}^N)$ or in Hölder spaces. This would force us to work in nonlinear Banach manifolds, which is considerably more complicated. On the other hand, due to the character of partial differential equations in infinite dimensional function spaces, there is not a distinguished phase space and the Sobolev space $W_p^1$ is far from being the only possible choice. In fact, there are many other function spaces, including some of the Bessel potential spaces, the Besov spaces and the so called little Nikol'skii spaces. Indeed, the techniques used in [**4**,**6**] and [**35**] produce results for a variety of spaces which are intimately connected with various interpolation methods.

In this paper we focus our attention on c). A forthcoming note shall be devoted to d) and e). Suppose

$$(f(\cdot, 0), g(\cdot, 0)) = (0, 0) \ , \tag{1.9}$$

such that 0 is an equilibrium of (1.1).

Let $(\mu_k)_{k \in \mathbb{N}}$ be the sequence of eigenvalues of the linear elliptic eigenvalue problem

$$\begin{aligned} [\,-\mathcal{A}(0) + \partial_2 f(\cdot, 0)\,]\, v &= \mu v \quad \text{in} \quad \Omega \ , \\ [\,-\mathcal{B}(0) + \partial_2 g(\cdot, 0)\,]\, v &= 0 \quad\ \ \text{on} \quad \partial\Omega \end{aligned} \tag{1.10}$$

and assume

$$\{\,\mu_k\,;\,k\in\mathbb{N}\,\}=\sigma_c\cup\sigma_s$$

with

$$\sigma_c\subset i\mathbb{R}\quad\text{and}\quad\sigma_s\subset[\,Re\,z<0\,]\;. \tag{1.11}$$

Observe that the $L_p$ realization of (1.10) has compact resolvent so that the eigenvalue problem is indeed well posed. Then we can state the following theorem.

**Theorem 1** *Existence and attractivity of center manifolds*

*a) For any $k\in\mathbb{N}^*$ there exists a finite dimensional, locally invariant $C^k$-center manifold*

$$\mathcal{M}^c=\mathcal{M}_k^c\subset W_p^1(\Omega,\mathbb{R}^N) \tag{1.12}$$

*(living in a suitable small neighborhood of 0) for the quasilinear reaction-diffusion equation*

$$\begin{aligned}\partial_t u+\mathcal{A}(u)u&=f(\cdot,u)\quad in\quad\ \Omega\times(0,\infty)\;,\\ \mathcal{B}(u)u&=g(\cdot,u)\quad on\quad\partial\Omega\times(0,\infty)\;.\end{aligned} \tag{QRD}$$

*Moreover, $\mathcal{M}_k^c$ is tangential at 0 to the finite dimensional space $X^c$ given by*

$$X^c:=\bigoplus_{\mu_j\in\sigma_c}N(\mu_j)\;,\quad N(\mu_j)\ \textit{ the algebraic eigenspace of }\ \mu_j\;.$$

*b) Each $\mathcal{M}_k^c$ attracts solutions of $(QRD)$ with initial values in a small neighborhood of $X$ at an exponential rate.*

The proof relies on maximal regularity results. (See [**20**] and [**10**] for an improvement which allowed us to find invariant and attractive manifolds in the Sobolev space $W_p^1(\Omega,\mathbb{R}^N)$ and also in several other function spaces). Moreover, we use results of Amann on quasilinear parabolic equations, evolution equations in interpolation and extrapolation spaces. We will rely quite heavily on interpolation theory. For many additional results, see [**35**].

Existence and attractivity results for center manifolds, using maximal regularity, were first shown in [**21**], and, independently, in [**30**]. (The latter paper establishes the existence without considering attractivity). The authors obtained center manifolds even for fully nonlinear parabolic equations. However, in the context of quasilinear equations, our results provide some important improvements. We show existence and attractivity in spaces which do not have the property of maximal regularity, for example in the Sobolev space $W_p^1(\Omega,\mathbb{R}^N)$. Our results on attractivity are optimal and take also the smoothing property of quasilinear equations into consideration. Moreover, we can treat equations subject to **nonlinear** boundary conditions. We are not aware of any other results on existence of invariant manifolds in this context. As it is not simple to verify the conditions involved with maximal regularity, our work can also be considered to give an application of the results in [**21**] to a wide class of nonlinear parabolic equations. (But we mention that we have improved their results in the case of quasilinear equations). Our method gives the possibility of working in spaces which are related in a natural way to the given equations.

Recently, center manifolds have become a subject of great interest. But while there are numerous contributions in the case of semilinear equations - cf. the work of Carr [**14**], Henry [**24**], Chow and Lu [**15**,**16**], Bates and Jones [**12**], Mielke [**29**], Iooss and Vanderbauwhede [**25**] to mention only a few - we know only of the work of Da Prato and Lunardi [**21**], Mielke [**30**], and of the contribution of Mielke [**29**] in the case of quasilinear equations. In the latter paper, the author gets an existence result using $L_p$ maximal regularity. However, he is mainly interested in quasilinear elliptic equations in Hilbert spaces.

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*Notation*: Let $E$ and $F$ be two Banach spaces over the same field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then we denote by $\mathcal{L}(E,F)$ the vector space of all bounded linear operators from $E$ to $F$ and we equip this space with the uniform operator norm. We denote by $Isom\,(E,F)$ the (open) subset of $\mathcal{L}(E,F)$ consisting of all isomorphisms from $E$ onto $F$. If two Banach spaces $E,F$ coincide except for equivalent norms, we express this by writing $E \doteq F$. If $E$ is a subspace of $F$, $E \hookrightarrow F$ means that the natural injection is continuous, that is, $E$ is continuously embedded in $F$. $E \overset{d}{\hookrightarrow} F$ then stands for dense embedding, i.e $E \subset F$ is densely and continuously embedded.

If $E \hookrightarrow F$ and $A : \mathrm{dom}\,(A) \subset F \to F$ is a linear operator, defined on a linear subspace $dom\,(A)$ of $F$, we define the (maximal) $E$-realization of $A$, $A_E$, by

$$\mathrm{dom}\,(A_E) := \{\, x \in E \cap \mathrm{dom}\,(A)\ ;\ Ax \in E\,\}\ , \quad A_E x = Ax\ .$$

For a Banach space $E$, the (continuous) dual is denoted by $E'$ and

$$< \cdot, \cdot >_E\, :\, E' \times E \to \mathbb{K}\ , \quad (e', e) \mapsto e'(e)$$

is the duality pairing. If $X_1$ and $X_0$ are two Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$ we define

$$\begin{aligned} \mathcal{H}(X_1, X_0) := \{&A \in \mathcal{L}(X_1, X_0); \\ &\text{-A generates an analytic } C_0\text{-semigroup on } \mathcal{L}(X_0)\}\ . \end{aligned} \tag{1.13}$$

Note that $\mathcal{H}(X_1, X_0)$ is an open subset of $\mathcal{L}(X_1, X_0)$. Finally, if $U \subset E$ is an open subset and $f : U \to F$ is Fréchet differentiable, we write $\partial f(x)$ for the derivative of $f$ at a point $x \in U$.

## 2. Maximal regularity

Our results on the existence and attractivity of center manifolds for quasilinear equations rely on maximal regularity results. These in turn are intimately connected with the continuous interpolation spaces introduced by Da Prato and Grisvard in [**20**]. In this section we present some results about maximal regularity. We refer to [**20**,**27**,**17**,**10**,**35**].

To describe what maximal regularity is about, we consider the linear Cauchy problem

$$(CP)_{(A,f,x)} \quad \begin{cases} \partial_t u + Au = f(t) \ , \\ \qquad u(0) = x \ . \end{cases} \tag{2.1}$$

on a Banach space $X$. Here, $-A$ is the generator of a strongly continuous analytic semigroup on $X$, denoted by $\{e^{-tA}; t \geq 0\}$. It is well known that $A$ is a densely defined closed operator. Hence, the domain of $A$, $D(A)$, equipped with the graph norm becomes a Banach space. Throughout, we will change slightly the notation and write

$$X_1 := D(A) \ , \quad X_0 := X \ . \tag{2.2}$$

It follows that $(X_0, X_1)$ forms a pair of densely embedded Banach spaces, i.e. $X_1 \overset{d}{\hookrightarrow} X_0$. Note that $A \in \mathcal{H}(X_1, X_0)$, cf. (1.13). Now assuming that $f$ is a continuous function on a given time interval $I$, say $I := [0, T]$ for a fixed $T > 0$, we may ask, whether the Cauchy problem (2.1) has a solution

$$u := u(\cdot, x) := u(\cdot, x, f) \in C^1(I, X_0) \cap C(I, X_1) \ . \tag{2.3}$$

In general, this does not hold unless some more regularity assumptions are posed on $f$. In fact, it is known that (2.1) has a solution whenever

$$f \in C^\beta(I, X_0) + C(I, X_\gamma) \quad , \quad \beta, \gamma \in (0, 1] \ .$$

Hereby, $X_\gamma$ denotes an arbitrary interpolation space between $X_1$ and $X_0$ . We refer here to [**18**, Theorem 5.9] or [**3**, Theorem 8.2]. But there do exist spaces where (2.1) can indeed be solved for each function $f$ belonging merely to $C(I, X_0)$. We define

$$\begin{aligned} \mathcal{M}_1(X_1, X_0) := &\{A \in \mathcal{H}(X_1, X_0) \ ; \\ &(\partial_t + A, R_1) \in Isom\,(C^1(I, X_0) \cap C(I, X_1)\,,\, C(I, X_0) \times X_1)\}\,, \end{aligned} \tag{2.4}$$

where $R_1 u := u(0)$ . Since solutions of the Cauchy problem $(CP)_{(A,f,x)}$ necessarily are unique, the only hard requirement in this definition is that the mapping is surjective. Here, the continuous interpolation spaces come into play guaranteeing that there exist pairs $(X_0, X_1)$ of densely embedded Banach spaces and $A \in \mathcal{H}(X_1, X_0)$ such that $A \in \mathcal{M}_1(X_1, X_0)$. Observe that $A \in \mathcal{M}_1(X_1, X_0)$ means that the Cauchy problem has a unique solution which depends continuously on $(f, x)$. In addition, note that $\dot{u}, Au$ have the same regularity as $f$. This justifies and explains the words **maximal regularity**. This property is very important in connection with nonlinear equations, (i.e. 'fully' nonlinear evolution equations), cf. [**19**,**20**,**10**,**26**]. It should be mentioned that the assumption (2.4) implies a hard restriction on the geometry of the space $X_0$. In fact, it has been proven in [**11**] that spaces with maximal regularity contain a copy of the sequence space $c_0$, i.e. contain a closed subspace isomorphic to $c_0$. In particular, if $X_0$ is reflexive, $\mathcal{M}_1(X_0, X_1)$ certainly is empty. Whenever we would like to use maximal regularity results we have to work in non reflexive spaces.

It was observed in [**10**] that we can do better in (2.4), allowing functions $f$ with a singularity in 0. It is this improvement which allows to take care of the smoothing property of quasilinear parabolic evolution equations (in the framework of maximal regularity), cf. section 3 and [**10**,**35**].

In order to formulate these results, we introduce the following function spaces. These are important for two reasons. First, they describe some more general function spaces where results on maximal regularity can be stated, see (2.16) below. Second, these are useful in order to describe the continuous interpolation spaces.

Let $X_0$ and $X_1$ be Banach spaces with $X_1 \overset{d}{\hookrightarrow} X_0$. Then, for $\alpha \in (0,1)$ and $J := (0,T]$ we define :

$$\begin{aligned} V_\alpha\,(J; X_0, X_1) := \{u \in C^1(J, X_0) \cap C(J, X_1);& \\ \lim_{t\to 0} t^{1-\alpha}(\|u'(t)\|_{X_0} + \|u(t)\|_{X_1}) = 0\}&\ , \end{aligned} \tag{2.5}$$

equipped with the norm

$$\|u\|_{V_\alpha\,(J;X_0,X_1)} := \sup_{t\in J}\ t^{1-\alpha}(\|u'(t)\|_{X_0} + \|u(t)\|_{X_1})\ . \tag{2.6}$$

This definition can be extended to $\alpha = 1$ by setting

$$V_1\,(J; X_0, X_1) := C^1([0,T], X_0) \cap C([0,T], X_1)\ . \tag{2.7}$$

Given any $u \in V_\alpha\,(J; X_0, X_1)$ and $0 < s < t \le T$ we obtain

$$\|u(t) - u(s)\|_{X_0} \le \|u\|_{V_\alpha\,(J;X_0,X_1)} \int_s^t \frac{d\tau}{\tau^{1-\alpha}} \le \|u\|_{V_\alpha\,(J;X_0,X_1)} \int_0^{t-s} \frac{d\tau}{\tau^{1-\alpha}}\ ,$$

which shows that $V_\alpha\,(J; X_0, X_1)$ is continuously embedded in $UC^\alpha((0,T], X_0)$, the space of uniformly $\alpha$-Hölder continuos functions on the interval $(0,T]$ with values in $X_0$. Thus, each function $u \in V_\alpha\,(J; X_0, X_1)$ can be extended to $[0,T]$. It is not difficult to see that the mapping

$$R_\alpha : \quad V_\alpha\,(J; X_0, X_1) \to X_0\ , \quad u \mapsto R_\alpha u := u(0) \tag{2.8}$$

defines a bounded linear operator. Now, we can define the following space consisting of the traces of functions belonging to (2.5):

$$X_\alpha := R_\alpha\left(\,V_\alpha(J; X_0, X_1)\,\right)\ . \tag{2.9}$$

The space $X_\alpha$ is equipped with the norm

$$\|x\|_\alpha := \inf\{\|u\|\,;\ u \in V_\alpha(J; X_0, X_1),\ x = u(0)\} \tag{2.10}$$

which turns it into a Banach space. It can be shown that the mapping

$$(X_0, X_1) \to X_\alpha\ , \quad \alpha \in (0,1)\ , \tag{2.11}$$

assigning to each pair $(X_0, X_1)$ the intermediate space $X_\alpha$, defines an exact interpolation method of exponent $\alpha$. This interpolation method was introduced in [**20**], cf. also [**27**,**17**,**10**], and is called the **continuos interpolation method**. Besides this definition, the continuous interpolation spaces can also be introduced in another way. Indeed, due to [**22**], we have

$$(X_0, X_1)^0_{\alpha,\infty} \doteq X_\alpha\ . \tag{2.12}$$

Here, $(X_0, X_1)^0_{\alpha,\infty}$ is obtained by assigning to each pair $(X_0, X_1)$ (of densely embedded Banach spaces) the closure of $X_1$ in $(X_0, X_1)_{\alpha,\infty}$, where $(\cdot,\cdot)_{\alpha,\infty}$ denotes the real interpolation method. The second definition has the advantage that the continuous interpolation spaces can be related to the real (and also the complex) interpolation spaces, cf. section 6. Moreover, duality and reiteration results for the continuous interpolation method can be proved by using known results for the real interpolation method.

The situation where $X_1$ and $X_0$ are given by (2.2) is of particular interest, since the continuous interpolation spaces can then be characterized with the help of the semigroup $\{e^{-tA}; t \geq 0\}$ generated by $-A$. We assume that $type(-A) < 0$. Then

$$D_A(\alpha) := (X, D(A))^0_{\alpha,\infty} = \{x \in X\ ;\ \lim_{t\to 0} t^{1-\alpha}\|Ae^{-tA}x\|_X = 0\} \tag{2.13}$$

and

$$\|x\|_\alpha := \sup_{t\in J} t^{1-\alpha}\|Ae^{-tA}x\|_X \tag{2.14}$$

is an equivalent norm, cf. [**36**], [**35**, Corollary 3.8]. Therefore, the elements of the continuous interpolation spaces $X_\alpha \doteq D_A(\alpha)$ can be characterized by

$$x \in X_\alpha \iff (t \mapsto e^{-tA}x) \in V_\alpha(J; X, D(A)), \quad 0 < \alpha < 1\ . \tag{2.15}$$

Now, we can go back to the linear Cauchy problem $(CP)_{(A,f,x)}$. Following [**10**] we define

$$\begin{aligned}\mathcal{M}_\alpha(X_1, X_0) := \{&A \in \mathcal{H}(X_1, X_0)\ ;\\ &(\partial_t + A, R_\alpha) \in Isom\big(V_\alpha(J; X_0, X_1),\, C_\alpha(J, X_0) \times X_\alpha\big)\}\ .\end{aligned} \tag{2.16}$$

Here we have set

$$C_\alpha(J, X_0) := \{f \in C(J, X_0);\ \lim_{t\to 0} t^{1-\alpha}\|f(t)\|_{X_0} = 0\}\ ,\ \|f\|_{C_\alpha(J,X_0)} := \sup_{t\in J} t^{1-\alpha}\|f(t)\|_{X_0}$$

for $0 < \alpha < 1$ and $C_1(J, X_0) := C([0,T], X_0)$.

Observe that $A \in \mathcal{M}_\alpha(X_1, X_0)$ means that the Cauchy problem $(CP)_{(A,f,x)}$ has, for each $(f, x) \in C_\alpha(J, X_0) \times X_\alpha$, a unique solution

$$u := (\partial_t + A, R_\alpha)^{-1}(f, x) \in V_\alpha(J; X_0, X_1) \ .$$

$\dot{u}$ and $Au$ then have the same regularity as $f$. This is a **maximal regularity** result which extends (2.4). Of course, we have to show that the set $\mathcal{M}_\alpha(X_1, X_0)$ is nonvoid.

### 2.1 Remarks

a) For $A \in \mathcal{M}_\alpha(X_1, X_0)$, let $(f, x) \in C_\alpha(J, X_0) \times X_\alpha$ be given and set

$$\begin{aligned} J_A f &:= J_{A,T} f := (\partial_t + A, R_\alpha)^{-1}(f, 0) \ , \\ x(\cdot) &:= (\partial_t + A, R_\alpha)^{-1}(0, x) \ . \end{aligned}$$

We then have

$$\begin{aligned} (J_A f)(t) &= \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau \ , \\ x(t) &= e^{-tA} x \ , \qquad\qquad t \in (0, T] \ . \end{aligned} \tag{2.17}$$

(Each solution of the Cauchy problem $(CP)_{(A,f,x)}$ necessarily satisfies the variation of constants formula). Together with (2.15) we obtain the following characterization

$$A \in \mathcal{M}_\alpha(X_1, X_0) \iff J_A\big(C_\alpha(J, X_0)\big) \subset C_\alpha(J, X_1) \ . \tag{2.18}$$

Indeed, take $f \in C_\alpha(X_0)$. This certainly implies $f \in L_1((0,T), X_0) \cap C((0,T], X_0)$. Now, [**31**, Theorem 4.2.4] shows that $J_A f$ belongs to $C^1((0,T], X_0)$ and moreover that $(J_A f)' = f - A J_A f$ holds. Together with the assumption $J_A f \in C_\alpha(J, X_1)$ this implies $J_A f \in V_\alpha(J; X_0, X_1)$. For any $(f, x) \in C_\alpha(J, X_0) \times X_\alpha$, the function $J_A f + x(\cdot)$ - which belongs to $V_\alpha(J; X_0, X_1)$ by the given argument and by (2.15) - is the unique solution of the Cauchy problem. Now the open mapping theorem gives that the bounded linear operator in (2.16) is indeed an isomorphism.

b) Let $X_\gamma := (X_0, X_1)_\gamma$, $\gamma \in (0,1)$, be an interpolation space given by an arbitrary interpolation method $(\cdot,\cdot)_\gamma$ of exponent $\gamma$. Suppose $A \in \mathcal{M}_\alpha(X_1, X_0)$ and $B \in \mathcal{L}(X_\gamma, X_0)$ for $0 \le \gamma < 1$, where $X_\gamma := X_0$ for $\gamma = 0$. Then we have the following perturbation result

$$A + B \in \mathcal{M}_\alpha(X_1, X_0) \ , \tag{2.19}$$

cf. [**10**, Lemma 2.5].

c) The definitions and results in (2.9)-(2.19) are independent of $T$, except that some constants may change.

**2.2 Theorem**

*Suppose we have two Banach spaces* $E_1, E_0$ *with* $E_1 \overset{d}{\hookrightarrow} E_0$ *and an* $A \in \mathcal{H}(E_1, E_0)$. *Let* $X_0 := D_A(\theta)$ *be a continuous interpolation space for an arbitrary* $\theta \in (0,1)$. *Let* $X_1 := D_A(1+\theta)$ *be the domain of definition of* $A_{X_0}$, *the part of* $A$ *in* $X_0$. *Then*

*a)* $A_{X_0} \in \mathcal{M}_\alpha(X_1, X_0)$ *for each* $\alpha \in (0,1]$ .

*b) If* $type(-A) < 0$, *the following estimate holds*

$$\|J_{A_{X_0},T}\|_{\mathcal{L}(C_\alpha((0,T],X_0),V_\alpha((0,T];X_0,X_1)} \le c(\alpha,\theta) ,$$

*with a continuous function* $c \in C((0,1] \times (0,1), \mathbb{R}^+)$ *which does not depend on the length of the interval* $(0,T]$.

**Proof** a) has been proven in [**10**, Theorem 2.14]. Another proof, working with the semigroup $e^{-tA}$ rather than with the resolvent of $-A$, is given in [**35**, Theorem 5.4]. There, we also pay attention to the estimate in b). □

When showing the exponential attractivity for center manifolds we will have to use estimates which do not depend on the length of some fixed time intervals. In fact, we will need the following result

**2.3 Proposition**

*Let the assumptions of Theorem 2.2 be satisfied and fix* $\omega_0$ *with* $type(-A) < \omega_0 < 0$. *Then, there exist a continuous function*

$$k := k_\theta \in C((-\infty, |\omega_0|\,) \times (0,1], \mathbb{R}^+) \tag{2.20}$$

*with*

$$t^{1-\alpha} e^{\omega t} \, \| \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau \, \|_{X_1} \le k(\omega,\alpha) \sup_{\tau\in(0,t]} \tau^{1-\alpha} e^{\omega\tau} \, \|f(\tau)\|_{X_0} , \quad 0 < t \le T$$

*for each function* $f \in C_\alpha(J, X_0)$ .

**Proof** We refer to [**35**, Proposition 5.6]. Here, we stated the result for $\theta \in (0,1)$ fixed. It can be shown that the function $k$ depends continuously on $(\omega, \alpha, \theta)$, cf. [**35**, Proposition 5.6]. It should be observed that $type(-(A-\omega)) = type(-A) + \omega < 0$ for each $\omega \in (-\infty, |\omega_0|\,]$. □

Using the property of maximal regularity again, we can state a result on the existence of bounded global solutions for the linear Cauchy problem. It is this result which leads to the existence of center manifolds.

**2.4 Theorem**

*Let the assumptions of Proposition 2.3 be given. Then, the inhomogenous Cauchy problem*

$$\dot{u}(t) + Au(t) = f(t) \ , \quad t \in \mathbb{R} \ , \tag{2.21}$$

*has, for each $f \in BC(\mathbb{R}, X_0)$, a unique bounded solution*

$$u \in BC(\mathbb{R}, X_1) \cap BC^1(\mathbb{R}, X_0)$$

*given by*

$$u(t) = (Kf)(t) := \int_{-\infty}^{t} e^{-(t-\tau)A} f(t) \, d\tau \ . \tag{2.22}$$

*Moreover,*

$$K \in \mathcal{L}(BC_\eta(\mathbb{R}, X_0) \, , \, BC_\eta(\mathbb{R}, X_1)) \ , \quad \eta \in [\, 0, \, |\omega_0| \,) \tag{2.23}$$

*with*

$$\|K\|_{\mathcal{L}(BC_\eta(\mathbb{R},X_0)\, , \, BC_\eta(\mathbb{R},X_1))} \leq \overline{k}(\eta) \ , \tag{2.24}$$

*and*

$$\overline{k} \in C(\, [0, \, |\omega_0| \,), \mathbb{R}^+) \ .$$

*Hereby, $BC_\eta(\mathbb{R}, X)$, $\eta \geq 0$ , denotes the function space*

$$BC_\eta(\mathbb{R}, X) := \{ \ g \in C(\mathbb{R}, X) \ ; \ \|g\|_\eta := \sup_{t \in \mathbb{R}} e^{-\eta|t|} \ \|g(t)\|_X < \infty \ \} \ .$$

**Proof** See [**35**, Theorem 5.7] and also [**21**, Proposition 1.2] for a related statement. □

**2.5 Corollary**

*Let $f \in BC_\eta((-\infty, 0], X_0)$ be given with $0 \leq \eta < |\omega_0|$. Then*

$$\| \int_{-\infty}^{0} e^{\tau A} f(\tau) \, d\tau \ \|_{X_1} \leq \overline{k}(\eta) \ \sup_{t \leq 0} e^{-\eta|t|} \ \|f(t)\|_{X_0} \ .$$

**2.6 Remarks**

a) Theorem 2.4 can be generalized to the case that $\sigma(-A) \cap i\mathbb{R} = \emptyset$. (I.e. $e^{-tA}$ is hyperbolic). Then, the inhomogenous Cauchy problem (2.21) has, for $f \in BC(\mathbb{R}, X_0)$, a unique bounded solution $u \in BC(\mathbb{R}, X_1) \cap BC^1(\mathbb{R}, X_0)$ given by

$$u(t) = (Kf)(t) := \int_{-\infty}^{t} e^{-(t-\tau)A} \, \pi^s f(t) \, d\tau - \int_{t}^{\infty} e^{-(t-\tau)A} \, \pi^u f(t) \, d\tau \ .$$

Here, $\pi^s$ denotes the projection onto the stable subspace and $\pi^u$ the projection onto the unstable subspace.

b) For some remarks in connection with the results of this section and the existence of center manifolds for quasilinear parabolic equations see the discussion at the end of section 5.

## 3. Maximal regularity and quasilinear equations

In this section we are concerned with abstract quasilinear parabolic equations. We collect here some statements and facts we will need to carry through our arguments on the existence and attractivity of center manifolds.

To fix the notation, assume that there are given two Banach spaces $X_1$ and $X_0$ with $X_1 \overset{d}{\hookrightarrow} X_0$ . Let $U$ be a nonempty subset of $X_0$ and

$$(A,F):\ U \longrightarrow \mathcal{L}(X_1,X_0)\times X_0\ . \tag{3.1}$$

We then consider the **autonomous quasilinear Cauchy problem**

$$\dot{u}+A(u)u=F(u)\ ,\quad t>0\ ,\quad u(0)=x\ . \tag{3.2}$$

$u:J_u\to X_0$ is called a solution of (3.2) on (the nontrivial interval) $J_u$ if $0\in J_u$ and

$$u\in C(J_u,U)\cap C^1(J_u\setminus\{0\},X_0)\cap C(J_u\setminus\{0\},X_1) \tag{3.3}$$

and $u$ satisfies

$$\dot{u}(t)+A(u(t))u(t)=F(u(t))\ ,\ t\in J_u\setminus\{0\}\ ,\quad u(0)=x\ .$$

$u$ is called a maximal solution if there does not exist a solution of (3.2) which is a proper extension of $u$ . Of course we assume that the set $U$ carries a topology (which is not necessarily the topology of $X_0$). Now, we give conditions which guarantee that the quasilinear Cauchy problem (3.2) indeed has solutions. We then state that the equation (3.2) defines a smooth semiflow on an appropriate space. Let

$$X_\theta\ ,\quad 0<\theta<1\ , \tag{3.4}$$

be the continuous interpolation spaces introduced in section 2. We fix two reals $\alpha$ and $\beta$ with

$$0<\beta<\alpha\le 1\ . \tag{3.5}$$

Let

$$U_\beta\quad \text{be an open subset of } X_\beta \tag{3.6}$$

and

$$U_\alpha:=U_\beta\cap X_\alpha \tag{3.7}$$

be equipped with the topology of the space $X_\alpha$ . It is known that $X_\alpha\hookrightarrow X_\beta$ whenever $\beta<\alpha$ and this immediately implies that $U_\alpha$ is a well-defined open subset of $X_\alpha$ . We assume that

$$(A,F)\in C^k(U_\beta\ ,\ \mathcal{L}(X_1,X_0)\times X_0)\ ,\quad k\in\mathbb{N}^*\cup\{\infty,\omega\}\ , \tag{3.8}$$

and

$$A(x)\in\mathcal{M}_\alpha(X_1,X_0)\ ,\quad x\in U_\alpha\ . \tag{3.9}$$

Then we obtain the following result on the existence and smooth dependence of solutions

**3.1 Theorem** *The quasilinear Cauchy problem*

$$\dot{u} + A(u)u = F(u)\ , \quad t > 0\ , \quad u(0) = x \tag{3.10}$$

*has for each $x \in U_\alpha$ a unique maximal solution $u(\cdot, x)$, defined on the maximal interval of existence $[0, t^+(x))$, with*

$$u(\cdot, x) \in C([0, t^+(x)), U_\alpha)\ . \tag{3.11}$$

*Moreover,*

$$u(\cdot, x) \in V_\alpha\left((0, T]; X_0, X_1\right) \quad \text{for each} \quad 0 < T < t^+(x)\ . \tag{3.12}$$

$\mathcal{D} := \bigcup\limits_{x \in U_\alpha} [0, t^+(x)) \times \{x\}$ *is open in $\mathbb{R}^+ \times U_\alpha$ and*

$$(t, x) \mapsto u(t, x) \in C^{0,k}(\mathcal{D}, U_\alpha)\ , \tag{3.13}$$

*i.e. the map defines a $C^k$-semiflow on $U_\alpha$ .*

**Proof** For $\alpha < 1$ we indicate a proof which works for $k = 1$- too, i.e. for the case where the functions $(A, F)$ in (3.8) are merely Lipschitz continuous. We show that for each $x_0 \in U_\alpha$ there exist positive constants $\tau = \tau(x_0)$ and $\epsilon = \epsilon(x_0)$ such that the quasilinear Cauchy problem (3.10) has a unique local solution $u(\cdot, x)$ with

$$u(\cdot, x) \in V_\alpha((0, \tau]; X_1, X_0) \tag{3.14}$$

for each $x \in \mathbb{B}_\alpha(x_0, \epsilon)$ (for $\epsilon$ and $\tau$ being sufficiently small). In fact, for $x_0 \in U_\alpha$ we rewrite (3.10) as

$$\dot{u} + A(x_0)u = B(u)u + F(u)\ , \quad t > 0\ , \quad u(0) = x \tag{3.15}$$

with $B(u) := A(x_0) - A(u)$. Now, we obtain a local solution of (3.15) in the function space (3.14) by a fixed point argument. Indeed, for each $v \in V_\alpha((0, \tau]; X_1, X_0)$ we have

$$[\, s \mapsto B(v(s))v(s) + F(v(s))\,] \in C_\alpha\left((0, \tau], X_0\right)\ . \tag{3.16}$$

Thanks to the property of maximal regularity in (2.16), the linear Cauchy problem

$$\dot{u} + A(x_0)u = B(v(t))v(t) + F(v(t))\ , \quad u(0) = x\ ,$$

has a unique solution

$$\phi_x(v) := u \in V_\alpha((0, \tau]; X_0, X_1)\ . \tag{3.17}$$

Since $V_\alpha((0,\tau];X_1,X_0) \hookrightarrow C([0,\tau],X_\alpha)$, cf. [**35**, Proposition 5.1], (3.11), in fact, is a consequence of (3.12). We refrain from giving more details and refer to [**35**], [**10**]. □

**3.2 Corollary**

*Assume $0 \in U_\alpha$ and denote the linearization of* (3.10) *at* 0 *by* $L$, *i.e.*

$$L := A(0) - \partial F(0) \ . \tag{3.18}$$

*Then, each solution* $u(\cdot,x)$ *of the quasilinear Cauchy problem* (3.10) *satisfies*

$$u(t,x) = e^{-tL}x + \int_0^t e^{-(t-\tau)L} g(u(\tau,x))\, d\tau \ , \quad t \in [0,t^+(x)) \tag{3.19}$$

*with*

$$g(u) := (A(0) - A(u))u + F(u) - \partial F(0)u \ , \quad u \in U_\alpha \cap X_1 \ , \tag{3.20}$$

*and hence solves the Cauchy problem*

$$\dot u + Lu = g(u) \ , \quad t \in (0,t^+(x)) \ , \quad u(0) = x \ . \tag{3.21}$$

*Each solution of* (3.21) *also solves* (3.10).

**Proof** Indeed, let $u(\cdot,x)$ be a solution of (3.10), defined on the maximal existence interval $(0,t^+(x))$. Fix $T \in (0,t^+(x))$. Thanks to (3.11) and (3.12) we obtain

$$[\tau \mapsto g(u(\tau,x))] \in C_\alpha((0,T],X_0) \tag{3.22}$$

In fact, from (3.8) we get in particular that $F \in C^1(U_\alpha\,,X_0)$ and (3.22) then easily follows from (3.11) and (3.12). (3.9) shows that $A(0) \in \mathcal{M}_\alpha(X_1,X_0)$ and (3.8) gives that $C := -\partial F(0) \in \mathcal{L}(X_\beta,X_0)$. We infer from Remark 2.1 b) that

$$L := A(0) + C \ \in \mathcal{M}_\alpha(X_1,X_0) \ . \tag{3.23}$$

Now, it follows from (3.22)-(3.23), (2.15) and (2.16) that the function $v$ defined by

$$v(t) := e^{-tL}x + \int_0^t e^{-(t-\tau)L} g(u(\tau,x))\, d\tau \tag{3.24}$$

belongs to the function space $V_\alpha((0,T];X_0,X_1)$ and solves the linear Cauchy problem

$$\dot v(t) + Lv(t) = g(u(t,x)) \ , \quad 0 < t \le T \ , \quad v(0) = x \ . \tag{3.25}$$

But then, $v$ clearly also solves the Cauchy problem (3.10) and we conclude that

$$v = u(\cdot,x)|[0,T] \ .$$

Since $T$ can be chosen arbitrarily, the assertions follow. It is clear that each solution of (3.21) solves (3.10). (Note that (3.21) has a solution. This can be shown by similar arguments as in Theorem 3.1). □

**3.3 Remark** Observe that Theorem 3.1 expresses a smoothing property for solutions of the quasilinear parabolic equation (3.10). Indeed, $u(t,u_0)$ belongs to $X_1$ for each positive time $t$, even if the initial values are in $X_\alpha$. (Solutions immediately become more regular than the initial values are). In section 5, we will work simultaneously with the topologies of $X_\alpha$ and $X_1$.

## 4. Existence of invariant manifolds for quasilinear parabolic equations

In this section we shall show the existence of locally invariant manifolds for the abstract autonomous quasilinear equation

$$\dot{u} + A(u)u = F(u) \ , \quad t > 0 \ . \tag{4.1}$$

We assume that there are given two Banach spaces $X_1$ and $X_0$ with $X_1 \overset{d}{\hookrightarrow} X_0$. Let $\alpha\,,\beta$ be two fixed reals with $0 < \beta < \alpha < 1$ and let $X_\alpha\,,\ X_\beta$ be the continuous interpolation spaces. Let $U_\beta$ be an open subset of $X_\beta$ and set $U := U_\alpha := U_\beta \cap X_\alpha$. It follows that $U \subset X_\alpha$ is an open subset (where $U$ inherits the topology of $X_\alpha$). Moreover, we require

$$(A, F) \in C^k(U_\beta\,, \mathcal{L}(X_1, X_0) \times X_0) \ , \quad k \in \mathbb{N}\ ,\ k \geq 1 \ , \tag{4.2}$$

and

*There exists a pair* $(E_0, E_1)$ *of Banach spaces with* $E_1 \overset{d}{\hookrightarrow} E_0$ *and an extension* $\tilde{A}(\cdot)$ *of* $A(\cdot)$ *so that the following conditions hold for each* $x \in U$*:*

- (*i*) $\tilde{A}(x) \in \mathcal{H}(E_1, E_0)$ ,
- (*ii*) $X_0 \doteq D_{\tilde{A}(x)}(\theta)\,, \quad X_1 \doteq D_{\tilde{A}(x)}(1+\theta)$ *for some* $\theta \in (0,1)$ ,
- (*iii*) $A(x)$ *is the* $X_0$*-realization of* $\tilde{A}(x)$ ,
- (*iv*) $E_1 \hookrightarrow X_\beta \hookrightarrow E_0$ *and*
  *there exists* $c > 0$ *and* $\delta \in (0,1)$ *with* $\|e\|_{X_\beta} \leq c\|e\|_{E_0}^{1-\delta}\|e\|_{E_1}^{\delta}\ ,\ e \in E_1$ .

These technical assumptions will allow us to use maximal regularity results. Indeed, Theorem 2.2 together with (i)-(iii) implies that

$$A(x) \in \mathcal{M}_\alpha(X_1, X_0) \ , \quad \text{for each} \ \ x \in U \ . \tag{4.3}$$

Moreover, we have $(\tilde{A}(x) - \partial F(x)) \in \mathcal{H}(E_1, E_0)$ by (iv), Young's inequality and a well-known perturbation result, cf. [**23**, Theorem 5.3.6]. Finally, $F \in C^k(U_\beta, X_0)$ gives

$$A(x) - \partial F(x) = \textit{the } X_0\textit{-realization of } \tilde{A}(x) - \partial F(x) \ , \quad x \in U \ . \tag{4.4}$$

Assume that

$$0 \in U \ , \quad F(0) = 0 \tag{4.5}$$

such that 0 is an equilibrium for the semiflow generated by (4.1). We study the behavior of the semiflow in a neighborhood of this equilibrium under some suitable assumptions on the spectrum of the linearized equation. Set

$$L := A(0) - \partial F(0) \ . \tag{4.6}$$

It follows from (4.3)-(4.4) that $-L$ generates an analytic $C_0$-semigroup on $X_0$. Suppose that the spectrum of $-L$ admit the following decomposition:

$$\sigma(-L) = \sigma_s \cup \sigma_c \quad \text{with} \quad \sigma_s \subset [Re z < 0] \ , \quad \sigma_c \subset i\mathbb{R} \tag{4.7}$$

and

$$\sigma_c \quad \textit{consists of finitely many eigenvalues with finite multiplicity} \ . \tag{4.8}$$

We set $\lambda_s := \sup \{Re\, \sigma_s\}$ and choose $\omega_s$ and $\omega_c$ such that

$$\lambda_s < \omega_s < \omega_c < 0 \ . \tag{4.9}$$

Let $\pi^c$ be the spectral projection for the spectral set $\sigma_c$ , $\pi^s := id_{X_0} - \pi^c$ and $X^c := \pi^c(X_0)$. Then, there exists a decomposition of $X_0$ and $X_1$ into a direct topological sum which reduces $L$ and the analytic semigroup $(e^{-tL})_{t\geq 0}$ generated by $-L$, i.e.

$$\begin{aligned} X_1 &= X^c \oplus X_1^s \ , \qquad & X_0 &= X^c \oplus X_0^s \ , \\ L &= L_c \oplus L_s \ , \qquad & e^{-tL} &= e^{-tL_c} \oplus e^{-tL_s} \ , \quad t \geq 0 \ , \end{aligned}$$

where $L_c$ denotes the part of $L$ in $X^c$ and $L_s$ the part in $X_0^s$. We refrain from giving an additional index to the finite dimensional space $X^c$. It follows (from [**37**, p.118] for example) that there is also a decomposition

$$X_\gamma = X^c \oplus X_\gamma^s \qquad \text{with} \quad X_\gamma^s \doteq (X_0^s \, , X_1^s)_\gamma \ , \tag{4.10}$$

where $(\cdot \, , \cdot)_\gamma$ denotes the continuous interpolation method in our context. It can be shown that the spectrum of

$$\tilde{L} := \tilde{A}(0) - \partial F(0)$$

and the operator $L$ coincide, cf. [**9**]. Hence, there exist also decompositions of $E_0$ and $E_1$ and an analogous of (4.10) holds as well. We conclude that $L_s$ is the $X_0^s$-realization of $\tilde{L}_s$ , the part of $\tilde{L}$ in $E_0^s$ . Then

$$L_s \in \mathcal{M}_\alpha (X_1^s \, , X_0^s) \quad \text{and} \quad type(-L_s) = \lambda_s$$

i.e. $-L_s$ generates an analytic $C_0$-semigroup on $X_0^s$ and the property of maximal regularity holds. (This follows again from Theorem 2.2). Finally, let $M_c$ and $M_s$ be positive constants with

$$\|e^{-tL_c}\|_{\mathcal{L}(X^c)} \leq M_c e^{t\,\omega_c} \ , \ t \leq 0 \ , \quad \|e^{-tL_s}\|_{\mathcal{L}(X_j^s)} \leq M_s e^{t\,\omega_s} \ , \ t \geq 0 \ , \ j = 0, 1 \ . \tag{4.11}$$

Theorem 2.4 and Corollary 2.5 can now be applied to the spaces $X_0^s$ and $X_1^s$ with $\omega_0 = \omega_s$ and $A = L_s$ :

The linear Cauchy problem $\dot{v}(t) + L_s\, v(t) = f(t)$ , $t \in \mathbb{R}$ has, for each $\eta \in [0, |\omega_s|)$ and $f \in BC_\eta(\mathbb{R}, X_0^s)$, a unique solution $v \in BC_\eta(\mathbb{R}, X_1^s)$ which is given by

$$v(t) = (K_s f)(t) := \int_{-\infty}^t e^{-(t-\tau)L_s} f(\tau)\, d\tau \ .$$

Moreover,

$$\|K_s\|_{\mathcal{L}(BC_\eta(\mathbb{R},X_0^s)\,,\,BC_\eta(\mathbb{R},X_1^s))} \leq \overline{k}(\eta)\ , \quad \text{with} \quad \overline{k} \in C([0,|\omega_s|))\ . \tag{4.12}$$

Now using results of [**21**] and [**16**], (cf. also [**25**]) we can establish the existence of locally invariant $C^k$-manifolds, $\mathcal{M}^c_{loc} \subset X_1$ , for the quasilinear equation (4.1). We will show that these are exponentially attractive in the norm of the space $X_1$ for solutions with small initial data belonging to the interpolation space $X_\alpha$. This is an optimal result which pays attention to the smoothing property of solutions of quasilinear parabolic equations. Moreover, we are not loosing any invariant manifold by looking for them in the more regular space $X_1$. (Note that (4.1) defines a $C^k$-smooth semiflow on the space $X_\alpha$. Nevertheless, each invariant manifold lies in the smaller space $X_1$, again due to the smoothing property.) This gives us the possibility to get results in a larger range of spaces which do not have the property of maximal regularity. We collect here some notation and the existence results we will need. In particular we have to take care of a 'cutting' trick.

We may write the quasilinear problem (4.1) as a semilinear equation, i.e.

$$\dot{u}(t) + Lu(t) = g(u(t))\ , \quad u(0) = u_0 \tag{4.13}$$

with

$$g(z) := (A(0) - A(z))z + F(z) - \partial F(0)z\ , \quad z \in X_1\ . \tag{4.14}$$

It is this step which requires the property of maximal regularity. The function $g$ then has the following properties:

$$g \in C^k(U_1\,,\,X_0) \quad \text{and} \quad g(0) = 0\ , \quad \partial g(0) = 0\ . \tag{4.15}$$

Next, we modify the mapping $g$ in a neighborhood of zero of the finite dimensional space $X^c$. We may assume without loss of generality that $U$ is given by

$$U = U^c \times U^s \quad \text{with} \quad U^c \subset X^c\ ,\ U^s \subset X^s_\alpha\ ,$$

where $U^c$ and $U^s$ are neighborhoods of zero in the indicated spaces. Let $\rho_0$ be chosen such that

$$\begin{aligned} &W_1\,(2\rho_0) := \mathbb{B}_{X^c}(0, 2\rho_0) \times \mathbb{B}_{X_1}(0, 2\rho_0) \subset U^c \times U^s_1\ , \\ &(g|\,W_1\,(2\rho_0)) \in BC^k(W_1\,(2\rho_0)\ , X_0) \end{aligned} \tag{4.16}$$

holds, where $U^s_1 := U^s \cap X_1$ equipped with the topology of $X_1$. For $\rho > 0$ let $r_\rho \in C^\infty(X^c \times X^s_1, X_1)$ be given by

$$r_\rho(x,y) := \chi(\rho^{-1}x)x + y\ , \quad (x,y) \in X^c \times X^s_1\ ,$$

where $\chi \in C^\infty(X^c, [0,1])$ denotes a smooth cutoff function for the closed ball $\overline{\mathbb{B}}_{X^c}(0,1)$ of $X^c$ with support in $\mathbb{B}_{X^c}(0,2)$. Now we set

$$g_\rho := g \circ r_\rho\ , \quad 0 < \rho \leq \rho_0\ . \tag{4.17}$$

We then have for the modified mapping $g_\rho$

$$\begin{aligned} &g_\rho \in C^k(X^c \times U_1^s\,,\, X_0) \quad \text{and} \quad g_\rho(0) = 0\ , \quad \partial g_\rho(0) = 0 \quad , \\ &g_\rho \in BC^k(V_{2\rho_0}\,, X_0) \quad \text{with} \quad V_{2\rho_0} := X^c \times \overline{\mathbb{B}}_{X_1^s}(0, 2\rho_0) \quad \text{and} \end{aligned} \tag{4.18}$$

$$g_\rho = g \quad \text{in} \quad \overline{\mathbb{B}}_{X^c}(0,\rho) \times U_1^s\ . \tag{4.19}$$

Hence, the solutions of (4.13) remaining in $\overline{\mathbb{B}}_{X^c}(0,\rho) \times U_1^s$ coincide with the solutions of

$$\dot{u}(t) + Lu(t) = g_\rho(u(t))\ , \quad u(0) = u_0\ . \tag*{$(4.20)_\rho$}$$

Note that solutions indeed exist because of the property of maximal regularity. It is clear that the modified equation $(4.20)_\rho$ is equivalent to the coupled system

$$\begin{aligned} \dot{x}(t) + L_c\, x(t) &= \pi^c g_\rho\,(x(t)\,,\, y(t))\ , \quad x(0) = x_0\ , \\ \dot{y}(t) + L_s\, y(t) &= \pi^s g_\rho\,(x(t)\,,\, y(t))\ , \quad y(0) = y_0\ , \end{aligned} \tag*{$(4.21)_\rho$}$$

with $x_0 = \pi^c u_0,\ y_0 = \pi^s u_0$ .
We can now state the following result on the existence and smoothness of invariant manifolds:

**4.1 Theorem** *( Existence of center manifolds )*

*Let the assumptions* (4.1) – (4.8) *be satisfied. Then there exists a* $\rho_k \in (0,\rho_0]$ *and for each* $\rho \in (0,\rho_k]$ *there is a unique mapping*

$$\sigma = \sigma_\rho = \sigma_{\,k,\rho} \in BC^k(X^c\,,\, X_1^s) \tag{4.22}$$

*with the properties*

$$\sigma(0) = 0\ , \quad \partial\sigma(0) = 0\ . \tag{4.23}$$

*In addition*

$$\|\sigma(x) - \sigma(x')\|_{X_1^s} \le b\,\|x - x'\|_{X^c} \tag{4.24}$$

*for a suitable positive constant* $b$ *and*

$$im(\sigma) \subset \overline{\mathbb{B}}_{X_1^s}(0,\rho)\ . \tag{4.25}$$

*The following holds for the graph of* $\sigma$

*a)*

$$\mathcal{M}^c := \mathcal{M}^c(k,\rho) := graph\,(\sigma) \ \subset X_1$$

*is a globally invariant* $C^k$*-manifold for the equation* $(4.21)_\rho$ *or* $(4.20)_\rho$ *respectively, i.e. the solution* $(x_\rho\,, y_\rho)$ *of* $(4.21)_\rho$ *exists globally for each initial value* $(x_0, y_0) \in \mathcal{M}^c$ *and* $(x_\rho\,(t), y_\rho\,(t)) \in \mathcal{M}^c$ *for* $t \in \mathbb{R}$ .

*Let*

$$z(\cdot) := z(\cdot, x) := z(\cdot, x, \sigma, \rho) \tag{4.26}$$

*be the (global) solution of the* <u>*reduced*</u> *ordinary differential equation*

$$\dot{z}(t) + L_c\, z(t) = \pi^c g_\rho\,(\, z(t), \sigma(z(t))\,)\ , \quad t \in \mathbb{R}\ , \quad z(0) = x\ . \tag{4.27}$$

*Then $\sigma$ satisfies the (fixed point) equation*

$$\sigma(x) = \int_{-\infty}^{0} e^{\tau L_s}\, \pi^s g_\rho\,(\, z(\tau, x), \sigma(z(\tau, x))\,)\, d\tau\ . \tag{4.28}$$

*b) (i)*

$$\mathcal{M}^c_{loc} := M^c_{loc}(k, \rho) := graph\,(\sigma \,|\, \mathbb{B}_{X^c}(0, \rho)) \ \subset X_1 \tag{4.29}$$

*is a locally invariant $C^k$-manifold for the equation (4.13) relative to the set*

$$W_1\,(\rho) := \mathbb{B}_{X^c}(0, \rho) \times \mathbb{B}_{X^s_1}(0, \rho)\ ,$$

*i.e. $\mathcal{M}^c_{loc}$ is invariant for solutions of the equation (4.13) as long as they remain in $W_1\,(\rho)$.*

*(ii) If $u(\cdot) : \mathbb{R} \to X_1$ is a global solution of (4.13) with $u(t) \in W_1\,(\rho)$ for all $t \in \mathbb{R}$, then*

$$\pi^s u(t) = \sigma(\pi^c u(t))\ , \quad t \in \mathbb{R}\ , \tag{4.30}$$

*and $\pi^c u(\cdot)$ is the solution of the ordinary differential equation*

$$\dot{z}(t) + L_c\, z(t) = \pi^c g\,(z(t), \sigma(z(t)))\ , \quad t \in \mathbb{R}\ , \quad z(0) = \pi^c u(0)\ , \tag{4.31}$$

*i.e. $\mathcal{M}^c_{loc}$ contains all small global solutions.*

**Proof** Let $\mathcal{S}_{k,\rho}$ be the set

$$\mathcal{S}_{k,\rho} := \{\ \sigma : X^c \to X^s_1\ ;\ \sigma(0) = 0\ ,\ \|\sigma(x)\| \le \rho\ ,\ \|\partial^j \sigma(x)\| \le b_j\ ,\ j = 1, ..., k-1\ , \\ [\partial^{(k-1)}\sigma]_{1-}\ \le b_k\ \}\ ,$$

equipped with the topology of bounded functions, i.e. with the norm

$$\|\sigma\|_\infty := \sup_{x \in X^c} \|\sigma(x)\|_{X_1}\ .$$

For $\sigma \in \mathcal{S}_{k,\rho}$ let $G$ be the mapping defined by

$$G(\sigma)(x) := \int_{-\infty}^{0} e^{\tau L_s}\, \pi^s g_\rho\,(\, z(\tau, x), \sigma(z(\tau, x))\,)\, d\tau\ , \tag{4.32}$$

where $z(\cdot, x)$ denotes the solution of (4.27).
It follows from [**21**, Theorem 3.2] that there exists a $\rho_k > 0$ such that the mapping $G$ has a unique fixed point

$$\sigma_{k,\rho} \in \mathcal{S}_{k,\rho}$$

for a suitable choice of $b_j,\ j = 1, ..., k$. Here, the reals $b_j$ do not depend on $\rho \in (0, \rho_k]$. The assertion (4.24) then follows by taking $b := b_1$ and (4.25) follows from

$$im\,(\sigma) \subset \overline{\mathbb{B}}_{X_1^s}(0, \rho) \quad \text{for} \quad \sigma = \sigma_{k,\rho} \in \mathcal{S}_{k,\rho} \ .$$

The results in [**21**] guarantee the existence of a mapping $\sigma \in BC^{k-}(X^c, X_1^s)$, (i.e. $\sigma$ has continuous and bounded derivatives up to the order $(k-1)$ and $\partial^{(k-1)}\sigma$ is uniformly Lipschitz continuous), such that the graph, $graph\,(\sigma)$, is globally invariant for the system $(4.21)_\rho$.
In addition, we can deduce with [**16**, Theorem 6.2] that $\sigma$ has bounded and continuous derivatives up to the order of $k$.
It follows from (4.12) that the key assumption **(H)** in [**25**] is satisfied. Then it is not difficult to see that the assumptions of [**25**, Theorem 2.2] hold. The only slight difficulty lies in the fact that they use a different 'cutting' function $r_\rho$. However, their modified function corresponds to our $g_\rho$ on the set $V_\rho$, defined in (4.18). We can conclude with a little effort that $\sigma$ coincides with the mapping $\psi$ in [**25**, Theorem 2.2]. Now the assertions (4.23), (4.30) and (4.31) follow from [**25**, Theorem 2.3]. $\square$

### 4.2 Remarks

a) $\mathcal{M}^c_{loc}$ is, as the graph of a $C^k$-function defined on an open subset of a finite dimensional space, a finite dimensional $C^k$-manifold of dimension $dim\,(X^c)$ and

$$x \mapsto (x, \sigma(x))$$

is a parametrization. Hence, the tangential space of $\mathcal{M}^c_{loc}$ at 0 is given by

$$T_0\,(\mathcal{M}^c_{loc}) = im\,(id_{X^c}, \partial\sigma(0)) \ .$$

(4.23) then implies

$$T_0\,(\mathcal{M}^c_{loc}) = X^c \times \{0\} \equiv X^c$$

which says that the space $X^c$, the center space, is tangential to $\mathcal{M}^c_{loc}$ at 0. $\mathcal{M}^c_{loc}$ is a (local) **center manifold** in $X_1$ for (4.1).
The local center manifolds, $\mathcal{M}^c_{loc}$, are not uniquely determined, in contrast to the global center manifold described in Theorem 4.1. Note that we obtain center manifolds which are as smooth as the functions are. But in general, we can not guarantee the existence of smooth $C^\infty$ manifolds if the functions $(A, F)$ are smooth. In fact, the reals $\rho_k$ may shrink to 0 when $k$ increases. For a detailed analysis we refer to [**39**] and the references given there.

b) If we replace (4.7) with

$$\sigma(-L) = \sigma_s \cup \sigma_{cu} \quad \text{where} \quad \sigma_s \subset [Re z < 0]\ , \quad \sigma_{cu} \subset [Re z \geq 0]\ , \ \sigma_{cu} \cap i\mathbb{R} \neq \emptyset\ .$$

and then substitute $c$ with $cu$ at each place where this is meaningful, we obtain the existence of a **center unstable manifold** $\mathcal{M}^{cu}_{loc}$, which is tangential to $X^{cu}$.

## 5. Attractivity

We now prove that the center manifolds obtained in Theorem 4.1 attract solutions at an exponential rate. It is essential that we can prove that this happens in the topology of $X_1$ even if the initial values are in the weaker interpolation space $X_\alpha$. This is an optimal result which takes care of the smoothing property of quasilinear parabolic equations. We first state that solutions of $(4.20)_\rho$ with small initial data in $X_\alpha$ remain in a small neighborhood of $X_\alpha$. Due to the definition of $g_\rho$ we only have to prove this for the part of the solution in the stable subspace $X_\alpha^s$.

**5.1 Lemma**

*Let $u := u(\cdot, u_0) := u(\cdot, u_0, \rho)$ be the solution of the equation $(4.20)_\rho$ and let $t^+(u_0) := t^+(u_0, \rho)$ be the positive escape time of the initial value $u_0$.*
*Then, there exists a $\rho' \in (0, \rho_0]$ and for each $\rho \in (0, \rho']$ there exists a neighborhood $\mathcal{U}_\alpha(\rho)$ of 0 in $X_\alpha$ with the following properties:*

*a)* $t^+(u_0) > 1$ *for each initial value* $u_0 \in \mathcal{U}_\alpha(\rho)$ ,

*b)* $\pi^s u([0, t^+(u_0)), u_0) \subset \overline{\mathbb{B}}_{X_\alpha^s}(0, 2\rho)$ , $\quad u_0 \in \mathcal{U}_\alpha(\rho)$ .

**Proof** We refrain from proving this here and refer to [**35**, Lemma 9.1]. □

**5.2 Lemma**

*Let $\sigma = \sigma_\rho = \sigma_{k,\rho} \in BC^k(X^c, X_1^s)$ be as in Theorem 4.1 and let*

$$z(\cdot) := z(\cdot, \pi^c u_0) := z(\cdot, \pi^c u_0, \sigma, \rho) \tag{5.1}$$

*again denote the (global) solution of the ordinary differential equation*

$$\dot{z}(\tau) + L_c\, z(\tau) = \pi^c g_\rho(\, z(\tau), \sigma(z(\tau))\,) \ , \quad \tau \in \mathbb{R} \ , \quad z(0) = \pi^c u_0 \ . \tag{5.2}$$

*Moreover, define*

$$w(\tau, t) := z(\tau - t, \pi^c u(t)) \quad \text{for} \quad \tau \in \mathbb{R} \quad \text{and} \quad t \in [0, t^+(u_0)) \ ,$$

*i.e. $w(\cdot, t)$ solves the differential equation*

$$\dot{w}(\tau) + L_c\, w(\tau) = \pi^c g_\rho(\, w(\tau), \sigma(w(\tau))\,) \ , \quad \tau \in \mathbb{R} \ , \quad w(t) = \pi^c u(t) \ , \tag{5.3}$$

*where $u := u(\cdot, u_0) := u_\rho(\cdot, u_0)$ is the solution of equation $(4.20)_\rho$.*

*Finally, set*

$$\xi(t) := \pi^s u(t) - \sigma(\pi^c u(t)) \ , \qquad t \in [0, t^+(u_0)) \ . \tag{5.4}$$

*Then*

$$\xi(t) = e^{-tL_s}\xi(0) + \int_0^t e^{-(t-\tau)L_s}\, h_1(\tau,t)\,d\tau + e^{-tL_s}\int_{-\infty}^0 e^{\tau L_s}\, h_2(\tau,t)\,d\tau \tag{5.5}$$

*where the functions $h_1$ and $h_2$ are given by*

$$h_1(\tau,t) := \pi^s[\, g_\rho\,(\,\pi^c u(\tau), \pi^s u(\tau)\,) - g_\rho\,(\,w(\tau,t), \sigma(w(\tau,t))\,)\,] \ , \quad 0 < \tau \le t < t^+(u_0) \ ,$$

$$h_2(\tau,t) := \pi^s[\, g_\rho\,(\,z(\tau), \sigma(z(\tau))\,) - g_\rho\,(\,w(\tau,t), \sigma(w(\tau,t))\,)\,] \ , \qquad \tau \le 0 \le t < t^+(u_0) \ .$$

**Proof** Due to the property of maximal regularity, the solutions of $(4.20)_\rho$ are given by the variation of constants formula, cf. Corollary 3.2 and in particular (3.19). Hence,

$$\pi^s u(t) = e^{-tL_s}\pi^s u_0 + \int_0^t e^{-(t-\tau)L_s}\, \pi^s g_\rho\,(\pi^c u(\tau), \pi^s u(\tau))\,d\tau \ . \tag{5.6}$$

It then follows from (4.28) that

$$\begin{aligned}
\xi(t) = e^{-tL_s}\pi^s u_0 + \int_0^t e^{-(t-\tau)L_s}\, \pi^s g_\rho\,(\,\pi^c u(\tau), \pi^s u(\tau)\,)\,d\tau \\
- \int_{-\infty}^0 e^{\tau L_s}\, \pi^s g_\rho\,(\,z(\tau, \pi^c u(t)), \sigma(z(\tau, \pi^c u(t)))\,)\,d\tau \ .
\end{aligned}$$

Using the substitution $\tau \mapsto t + \tau$ we obtain for the second integral

$$\begin{aligned}
\int_{-\infty}^0 e^{\tau L_s}\, \pi^s g_\rho\,(\,z(\tau, \pi^c u(t)), \sigma(z(\tau, \pi^c u(t)))\,)\,d\tau \\
= \int_{-\infty}^t e^{-(t-\tau)L_s}\, \pi^s g_\rho\,(\,w(\tau,t), \sigma(w(\tau,t))\,)\,d\tau \ .
\end{aligned}$$

Now, the assertion follows from

$$\sigma(\pi^c u_0) = \int_{-\infty}^0 e^{\tau L_s}\, \pi^s g_\rho\,(\,z(\tau), \sigma(z(\tau))\,)\,d\tau \ . \quad \square$$

**5.3 Proposition**

*Set $x(\cdot) := \pi^c u$ and $y(\cdot) := \pi^s u$. Then, for each $\rho \in (0, \rho']$ there exists a $L_\alpha(\rho) > 0$ with*

$$\lim_{\rho \to 0} L_\alpha(\rho) = 0 \tag{5.7}$$

*such that, given any initial value $u_0 \in \mathcal{U}_\alpha(\rho)$, the following holds:*

*(i)* $\| g_\rho ( x(\tau), y(\tau) ) - g_\rho ( w(\tau,t), \sigma(w(\tau,t)) ) \|_{X_0}$

$$\leq L_\alpha(\rho) \| x(\tau) - w(\tau,t) \| + L_\alpha(\rho) \| y(\tau) - \sigma(x(\tau)) \|_{X_1} \ , \quad \tau, t \in (0, t^+(u_0)) \ ,$$

*(ii)* $\| g_\rho ( z(\tau), \sigma(z(t)) ) - g_\rho ( w(\tau,t), \sigma(w(\tau,t)) ) \|_{X_0}$

$$\leq L_\alpha(\rho) \| z(\tau) - w(\tau,t) \| \ , \quad \tau \in \mathbb{R} \ , \ t \in (0, t^+(u_0)) \ .$$

**Proof**

(i) We define

$$W_\alpha(2\rho) := \overline{\mathbb{B}}_{X^c}(0, 2\rho) \times \overline{\mathbb{B}}_{X^s_\alpha}(0, 2\rho) \ . \tag{5.8}$$

Moreover, we set $B(z) := A(0) - A(z)$ for $z \in U = U_\alpha$. Lemma 4.1 and the definiton of the function $r_\rho$ then imply that

$$v_\rho(\tau) := r_\rho(x(\tau), y(\tau)) \in W_\alpha(2\rho) \ , \quad \tau \in [0, t^+(u_0)) \tag{5.9}$$

for $\rho \in (0, \rho']$ and for each initial value $u_0 \in \mathcal{U}_\alpha(\rho)$. We conclude from (4.25) that

$$w_\rho(\tau, t) := r_\rho(w(\tau,t), \sigma(w(\tau,t))) \in \overline{\mathbb{B}}_{X^c}(0, 2\rho) \times \overline{\mathbb{B}}_{X^s_1}(0, \rho) \tag{5.10}$$

for each $\tau \in \mathbb{R}$. We can assume that the norm of the inclusion $i : X_1 \to X_\alpha$ is bounded by 1. (If not, we can replace the norm $\| \cdot \|_\alpha$ of $X_\alpha$ by $( \|i\|_{\mathcal{L}(X_1, X_\alpha)} )^{-1} \| \cdot \|_\alpha$ and use Lemma 4.1 with this norm). Due to (5.9)-(5.10) and (4.17) we have to estimate

$$g(v_\rho(\tau)) - g(w_\rho(\tau,t)) \tag{5.11}$$

for $\tau, t \in (0, t^+(u_0))$. With (4.14) we estimate:

$$\begin{aligned} &B(v_\rho(\tau)) v_\rho(\tau) - B(w_\rho(\tau,t)) w_\rho(\tau,t) \\ &\qquad = B(v_\rho(\tau)) \big( v_\rho(\tau) - w_\rho(\tau,t) \big) + \big( B(v_\rho(\tau)) - B(w_\rho(\tau,t)) \big) w_\rho(\tau,t) \ . \end{aligned}$$

The first term can be estimated by

$$\|B(v_\rho(\tau))(v_\rho(\tau)-w_\rho(\tau,t))\|_{X_0} \le \sup_{z\in W_\alpha(2\rho)} \|B(z)\|_{\mathcal{L}(X_1,X_0)}\, \|v_\rho(\tau)-w_\rho(\tau,t)\|_{X_1} \ . \quad (5.12)$$

Using the mean value theorem we obtain for the second term

$$\begin{aligned}
&\|\big(B(v_\rho(\tau)) - B(w_\rho(\tau,t))\big)w_\rho(\tau,t)\|_{X_1} \\
&\le \sup_{z\in W_\alpha(2\rho)} \|\partial B(z)\|_{\mathcal{L}(X_\alpha,\mathcal{L}(X_1,X_0))}\, \|v_\rho(\tau)-w_\rho(\tau,t)\|_{X_\alpha}\, \|w_\rho(\tau,t)\|_{X_1} \\
&\le \sup_{z\in W_\alpha(2\rho)} \|\partial B(z)\|_{\mathcal{L}(X_\alpha,\mathcal{L}(X_1,X_0))}\, \|v_\rho(\tau)-w_\rho(\tau,t)\|_{X_1}\cdot 3\rho\ , \quad \tau>0\ ,
\end{aligned} \quad (5.13)$$

where

$$\|w_\rho(\tau)\|_{X_1} \le 3\rho \quad \text{for} \quad \tau\in\mathbb{R}$$

is a consequence of (5.10) (or (4.25)). Again using the mean value theorem we also obtain

$$\begin{aligned}
&\|F(v_\rho(\tau)) - \partial F(0)\, v_\rho(\tau) - [\,F(w_\rho(\tau,t)) - \partial F(0)\, w_\rho(\tau,t)\,]\,\|_{X_0} \\
&\le \sup_{z\in W_\alpha(2\rho)} \|\partial F(z)-\partial F(0)\|_{\mathcal{L}(X_1,X_0)}\, \|v_\rho(\tau)-w_\rho(\tau,t)\|_{X_1}\ , \quad \tau>0\ .
\end{aligned} \quad (5.14)$$

It follows from the definiton of $r_\rho$ given in (4.17) that

$$\|v_\rho(\tau)-w_\rho(\tau,t)\|_{X_1} \le \|\chi\,(\rho^{-1}x(\tau))\,x(\tau)-\chi\,(\rho^{-1}w(\tau,t))\,w(\tau,t)\|+\|y(\tau)-\sigma(w(\tau,t))\|_{X_1}$$

for $\tau>0$. If $[\chi]_{1-}$ denotes the Lipschitz seminorm of the cutoff function $\chi$ we obtain (by considering the cases $w(\tau,t)\le 2\rho$ resp. $w(\tau,t)>2\rho$ and by a symmetry argument):

$$\|\chi\,(\rho^{-1}x(\tau))\,x(\tau)-\chi\,(\rho^{-1}w(\tau,t))\,w(\tau,t)\| \le (1+2[\chi]_{1-})\,\|x(\tau)-w(\tau,t)\|\ ,\ \tau>0 \quad (5.15)$$

and with (4.24)

$$\begin{aligned}
\|y(\tau)-\sigma(w(\tau,t))\|_{X_1} &\le \|y(\tau)-\sigma(x(\tau))\|_{X_1} + \|\sigma(x(\tau))-\sigma(w(\tau,t))\|_{X_1} \\
&\le \|y(\tau)-\sigma(x(\tau))\|_{X_1} + b\,\|x(\tau)-w(\tau,t)\|_{X_1}\ ,\ \tau>0\ .
\end{aligned} \quad (5.16)$$

By collecting the results in (5.12)-(5.16) we get the assertion (i) where $L_\alpha(\rho)$ is given by $\big(1+b+2[\chi]_{1-}\big)\ \times$

$$\sup_{z\in W_\alpha(2\rho)} \big(\|B(z)\|_{\mathcal{L}(X_1,X_0)} + 3\rho\,\|\partial B(z)\|_{\mathcal{L}(X_\alpha,\mathcal{L}(X_1,X_0))} + \|\partial F(z)-\partial F(0)\|_{\mathcal{L}(X_1,X_0)}\big)\ .$$

Observe that $L_\alpha$ can be made small by decreasing $\rho$.

For proving (ii) we replace $v_\rho$ given in (5.9) by

$$v_\rho(\tau) := r_\rho\,(z(\tau), \sigma(z(\tau))\ , \quad \tau \in \mathbb{R}\ .$$

Now, all conclusions of (5.9)-(5.16) hold for this situation as well. $\square$

For the next proposition we also refer to [**21**, p.131] for the case $\alpha = 1$ and to [**24**] p.148 for the easier case of semilinear equations.

**5.4 Proposition**

*Set* $p := \max\{\ \|\pi^c\|_{\mathcal{L}(X_0)}\ , \|\pi^s\|_{\mathcal{L}(X_0)}\ \}$. *Then*

*a)* $$\|x(\tau) - w(\tau,t)\| \le pM_cL_\alpha(\rho) \int_\tau^t e^{\mu(\vartheta-\tau)}\ \|\xi(\vartheta)\|_1\, d\vartheta\ , \qquad 0 < \tau \le t < t^+(u_0)$$

$$\textit{with} \qquad \mu := \mu(\rho) := pM_cL_\alpha(\rho) - \omega_c\ . \tag{5.17}$$

*b)* $$\|z(\tau) - w(\tau,t)\| \le pM_cL_\alpha(\rho)\ e^{-\mu\tau} \int_0^t e^{\mu\vartheta}\ \|\xi(\vartheta)\|_1\, d\vartheta\ , \quad \tau \le 0\ ,\ t \in [0, t^+(u_0))\ .$$

**Proof**

a) Let $t \in (0, t^+(u_0))$ be fixed. It follows from (5.3) that $w(\cdot,t)$ is given by

$$w(\tau,t) = e^{-(\tau-t)L_c}x(t) + \int_t^\tau e^{-(\tau-s)L_c}\,\pi^c\, g_\rho\,(w(s,t), \sigma(w(s,t)))\, ds\ . \tag{5.18}$$

Moreover, we infer from $(4.21)_\rho$ that

$$x(\tau) = e^{-(\tau-t)L_c}x(t) + \int_t^\tau e^{-(\tau-s)L_c}\,\pi^c\, g_\rho\,(x(s), y(s))\, ds\ ,\ \tau \in (0, t^+(u_0))\ . \tag{5.19}$$

(4.11) and Proposition 5.3 then imply

$$\|x(\tau) - w(\tau,t)\| \le pM_c\,|\int_t^\tau e^{(\tau-s)\omega_c}\,\|\,g_\rho\,(\,x(s), y(s)\,) - g_\rho\,(\,w(s,t), \sigma(w(s,t))\,)\,\|_0\, ds\,|$$
$$\le pM_cL_\alpha(\rho)\,|\int_t^\tau e^{(\tau-s)\omega_c}\,\|\xi(s)\|_1\, ds\,| + pM_cL_\alpha(\rho)\,|\int_t^\tau e^{(\tau-s)\omega_c}\,\|x(s) - w(s,t)\|\, ds\,|\ .$$

By multiplying this inequality with $e^{-\tau\omega_c}$ we obtain

$$e^{-\tau\omega_c}\|x(\tau) - w(\tau,t)\| \le a(\tau,t) + k\,|\int_t^\tau e^{-s\omega_c}\,\|x(s) - w(s,t)\|\, ds\,|\ ,$$

where we have set

$$k := pM_cL_\alpha(\rho)\ , \quad a(\tau,t) := k \,|\int_t^\tau e^{-s\omega_c}\,\|\xi(s)\|_1\,ds\,|\ . \quad 0<\tau\le t \tag{5.20}$$

For $t\in(0,t^+(u_0))$ fixed we get $a(\cdot,t)\in C([0,t],\mathbb{R}^+)$, due to

$$\xi(\cdot)\in C_\alpha\left((0,t],X_1^s\right)\hookrightarrow L_1((0,t),X_1^s)\ .$$

By applying Gronwall's Lemma (cf. [**5**, Lemma 6.1]) we obtain

$$e^{-\tau\omega_c}\|x(\tau)-w(\tau,t)\| \le a(\tau,t) + k\,|\int_t^\tau a(s,t)e^{\,|\int_s^\tau k\,d\sigma|}\,ds\,|\ ,\ 0<\tau\le t\ . \tag{5.21}$$

Now, after plugging in $a(\tau,t)$, interchanging the order of integration and doing some computation we get

$$e^{-\tau\omega_c}\|x(\tau)-w(\tau,t)\| \le k\int_\tau^t e^{k(\vartheta-\tau)}e^{-\omega_c\vartheta}\,\|\xi(\vartheta)\|_1\,d\vartheta\ ,\ 0<\tau\le t$$

which gives the assertion in $a)$ .

b) $z(\cdot):=z(\cdot,x(0))$, being the solution of (5.2), is given by

$$z(\tau)=e^{-\tau L_c}x(0)+\int_0^\tau e^{-(\tau-s)L_c}\,\pi^c g_\rho\left(\,z(s),\sigma(z(s))\,\right)ds\ , \quad \tau\in\mathbb{R}\ .$$

On the other hand we have

$$x(t)=e^{-tL_c}x(0)+\int_0^t e^{-(t-s)L_c}\,\pi^c g_\rho\left(x(s),y(s)\right)ds\ .$$

This together with (5.18) implies

$$z(\tau)-w(\tau,t)=\int_0^t e^{-(\tau-s)L_c}\,k_1(s,t)\,ds+\int_\tau^0 e^{-(\tau-s)L_c}k_2(s,t)\,ds \tag{5.22}$$

with

$$k_1(s,t) := \pi^c\left[\ g_\rho\left(\,w(s,t),\sigma(w(s,t))\,\right)-g_\rho\left(\,x(s),y(s)\,\right)\ \right]\ , \quad 0<s\le t$$

and

$$k_2(s,t) := \pi^c\left[\ g_\rho\left(\,w(s,t),\sigma(w(s,t))\,\right)-g_\rho\left(\,z(s),\sigma(z(s))\,\right)\ \right]\ , \quad s\le 0\ .$$

Now, Proposition 5.3 and the first part of the proof show

$$\begin{aligned}\|k_1(s,t)\| &\le pL_\alpha(\rho)\,\|w(s,t)-x(s)\|+pL_\alpha(\rho)\,\|y(s)-\sigma(x(s))\|_1\\ &\le M_c(pL_\alpha(\rho))^2\int_s^t e^{(\vartheta-s)\mu}\,\|\xi(\vartheta)\|_1\,d\vartheta+pL_\alpha(\rho)\|\xi(s)\|_1\end{aligned} \tag{5.23}$$

for $0 < s \leq t$ and

$$\|k_2(s,t)\| \leq pL_\alpha(\rho)\,\|w(s,t) - z(s)\|_1 \;, \quad s \leq 0 \;. \tag{5.24}$$

We then obtain from (5.22)-(5.24) (by interchanging the order of integration and a forward calculation)

$$\|z(\tau) - w(\tau,t)\| \leq k e^{\tau\omega_c} \int_0^t e^{\vartheta\mu}\,\|\xi(\vartheta)\|_1\,d\vartheta + k \,|\int_0^\tau e^{(\tau-s)\omega_c}\,\|z(s) - w(s,t)\|\,ds\;| \;.$$

with $k$ given in (5.20). Hence

$$\begin{aligned} e^{-\tau\omega_c}\|z(\tau) - w(\tau,t)\| &\leq k\int_0^t e^{\vartheta\mu}\,\|\xi(\vartheta)\|_1\,d\vartheta + k\,|\int_0^\tau e^{-s\omega_c}\,\|z(s) - w(s,t)\|\,ds\;| \\ &= a(t) + k\,|\int_0^\tau e^{-s\omega_c}\|z(s) - w(s,t)\|\,ds\;| \end{aligned}$$

with

$$a(t) := k\int_0^t e^{\vartheta\mu}\;\|\xi(\vartheta)\|_1\,d\vartheta \;. \tag{5.25}$$

Using Gronwall's Lemma, cf. [**5**, Corollary 6.2], we obtain:

$$e^{-\tau\omega_c}\|z(\tau) - w(\tau,t)\| \leq a(t)\,e^{\;|\int_0^\tau k\,d\sigma\,|} = a(t)\,e^{-k\tau} \;, \quad \tau \leq 0 \;.$$

Now, assertion b) follows. □

**5.5 Corollary**

*a)* $\|h_1(\tau,t)\|_{X_0^s} \leq n\,(\rho)\int_\tau^t e^{(\vartheta-\tau)\mu}\,\|\xi(\vartheta)\|_1\,d\vartheta \;+\; pL_\alpha(\rho)\,\|\xi(\tau)\|_1 \;,\; 0 < \tau \leq t < t^+(u_0) \;,$

*b)* $\|h_2(\tau,t)\|_{X_0^s} \leq n\,(\rho)\;e^{-\mu\tau}\int_0^t e^{\mu\vartheta}\;\|\xi(\vartheta)\|_1\,d\vartheta \;, \qquad \tau \leq 0 \leq t < t^+(u_0) \;,$

*with*

$$n\,(\rho) := n\,(\rho,\alpha) := M_c(pL_\alpha(\rho))^2 \;. \tag{5.26}$$

**Proof** Proposition 5.3 and Proposition 5.4. □

We can assume that $\rho$' is sufficiently small such that

$$\mu = \mu(\rho) := pM_cL_\alpha(\rho) - \omega_c < |\omega_s| \qquad \rho \in (0, \rho'\,] \tag{5.27}$$

holds. (Note that $L_\alpha(\rho)$ can be made small by decreasing $\rho$ and $\omega_c, \omega_s$ have been chosen with $\omega_s < \omega_c < 0$). Moreover, we fix $\omega$ sufficiently close to $\omega_s$ with

$$\omega_s < \omega < \omega_c < 0 \quad \text{and} \quad \mu + \omega := \mu(\rho) + \omega < 0 \ , \quad \rho \in (0, \rho'] \ . \tag{5.28}$$

**5.6 Lemma** *For each $t \in (0, t^+(u_0))$ the following holds:*

*a)* $h_1(\cdot, t) \in C_\alpha\left((0,t], X_0^s\right)$ *and*

$$\sup_{\tau \in (0,t]} \tau^{1-\alpha}\, e^{|\omega|\tau}\, \|h_1(\tau,t)\|_{X_0^s} \le \left( \tfrac{n(\rho)}{|\mu(\rho)+\omega|} + pL_\alpha(\rho) \right) \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \ .$$

*b)* $h_2(\cdot, t) \in BC_\mu((-\infty, 0], X_0^s)$ *and*

$$\sup_{\tau \le 0} e^{\mu\tau}\, \|h_2(\tau,t)\|_{X_0^s} \le n(\rho)\, \tfrac{\Gamma(\alpha)}{|\mu(\rho)+\omega|^\alpha} \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta} \|\xi(\vartheta)\|_1 \ .$$

**Proof**

a) It follows from Corollary 3.2 that the function $h_1(\cdot,t)$ belongs to $C_\alpha\left((0,t], X_0^s\right)$. (Observe that this has been used several times throughout this section). From Corollary 5.5 we obtain for $\tau \in (0,t]$ :

$$\tau^{1-\alpha}\, e^{|\omega|\tau}\, \|h_1(\tau,t)\|_{X_0^s} \le \tau^{1-\alpha}\, e^{|\omega|\tau} \left( n(\rho) \int_\tau^t e^{(\vartheta-\tau)\mu}\, \|\xi(\vartheta)\|_1 \, d\vartheta + pL_\alpha(\rho)\, \|\xi(\tau)\|_1 \right) .$$

For the first term we get, keeping in mind the second part of (5.28),

$$\begin{aligned}
&\tau^{1-\alpha}\, e^{|\omega|\tau} \int_\tau^t e^{(\vartheta-\tau)\mu}\, \|\xi(\vartheta)\|_1 \, d\vartheta \\
&\quad \le \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \int_\tau^t e^{-|\mu+\omega|\,(\vartheta-\tau)} (\tau/\vartheta)^{1-\alpha}\, d\vartheta \\
&\quad \le \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \int_\tau^t e^{-|\mu+\omega|\,(\vartheta-\tau)}\, d\vartheta \\
&\quad \le \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \, |\mu+\omega|^{-1} \ .
\end{aligned}$$

Now, the assertion a) follows.

b) Corollary 5.5 b) immediately implies

$$\sup_{\tau \le 0} e^{\mu\tau}\, \|h_2(\tau,t)\|_{X_0^s} \le n(\rho) \int_0^t e^{\mu\vartheta}\, \|\xi(\vartheta)\|_1 \, d\vartheta \ .$$

Furthermore, we have

$$\int_0^t e^{\mu\vartheta}\,\|\xi(\vartheta)\|_1\,d\vartheta \;\le\; \sup_{\vartheta\in(0,t]} \vartheta^{1-\alpha}\,e^{|\omega|\vartheta}\,\|\xi(\vartheta)\|_1 \;\int_0^t e^{-|\mu+\omega|\,\vartheta}\,\vartheta^{\alpha-1}\,d\vartheta$$

and with the substitution $\vartheta \mapsto |\mu+\omega|\,\vartheta$ we can estimate

$$\int_0^t e^{-|\mu+\omega|\vartheta}\vartheta^{\alpha-1}\,d\vartheta \le |\mu+\omega|^{-\alpha}\,\Gamma(\alpha)\ ,$$

where $\Gamma(\alpha)$ denotes the Gamma function. This gives the assertion in b) . $\square$

**5.7 Proposition**

*Let $\omega$ be given by (5.28). Then there exist continuous functions*

$$\overline{k}\ :[0,|\omega_s|)\to\mathbb{R}^+\quad \textit{and}\quad k\ :[0,|\omega_s|)\times(0,1]\to\mathbb{R}^+$$

*such that the following holds for each $\rho\in(0,\rho'\,]$ and $t\in(0,t^+(u_0))$ :*

$$t^{1-\alpha}\,e^{|\omega|t}\,\|\int_0^t e^{-(t-\tau)L_s}\,h_1(\tau,t)\,d\tau\|_{X_1^s}$$
$$\le k(|\omega|,\alpha)\,\Big(\frac{n(\rho)}{|\mu+\omega|}+pL_\alpha(\rho)\Big)\sup_{\vartheta\in(0,t]}\vartheta^{1-\alpha}\,e^{|\omega|\vartheta}\;\|\xi(\vartheta)\|_1\ ,$$

*and*

$$\|\int_{-\infty}^0 e^{\tau L_s}h_2(\tau,t)\,d\tau\ \|_{X_1^s}\le\ \overline{k}(\mu)n(\rho)\ \frac{\Gamma(\alpha)}{|\mu+\omega|^\alpha}\ \sup_{\vartheta\in(0,t]}\vartheta^{1-\alpha}\,e^{|\omega|\vartheta}\ \|\xi(\vartheta)\|_1\ .$$

**Proof**

The first assertion follows from our assumptions in section 4, from Proposition 2.3 (by replacing $X_j$ with $X_j^s$ , $j=0,1$ , and $A$ with $L_s$) and from Lemma 5.6. Corollary 2.5 (with the same modification), (5.28) and Lemma 5.6 give the second assertion. $\square$

We now give the main result of this section which states the exponential attractivity for the invariant $C^k$-manifolds constructed in Theorem 4.1.

**5.8 Theorem** ( *exponential attractivity* )
*Let $\omega$ be given by (5.28). Then there exists a $\overline{\rho} \in (0, \rho']$ such that*

$$\|\pi^s u(t) - \sigma(\pi^c u(t))\|_1 \le \frac{N_\alpha}{t^{1-\alpha}}\, e^{-|\omega|\, t}\, \|\pi^s u_0 - \sigma(\pi^c u_0)\|_\alpha\ , \quad t \in (0, t^+(u_0))$$

*holds for each $\rho \in (0, \overline{\rho}]$ and each initial value $u_0 \in \mathcal{U}_\alpha(\rho)$. Here, the constant $N_\alpha$ does not depend on the initial values $u_0 \in \mathcal{U}_\alpha(\rho)$ .*

**Proof** According to Lemma 5.2, $\xi(t)$ is given by

$$\xi(t) = e^{-tL_s}\xi(0) + \int_0^t e^{-(t-\tau)L_s}\, h_1(\tau, t)\, d\tau + e^{-tL_s} \int_{-\infty}^0 e^{\tau L_s}\, h_2(\tau, t)\, d\tau\ . \tag{5.29}$$

Let $u_0 \in \mathcal{U}_\alpha(\rho)$ and $t \in (0, t^+(u_0))$ be fixed. For the first term in (5.29) we obtain by using 2.14 (or a general result from interpolation theory)

$$t^{1-\alpha} e^{|\omega| t}\, \|e^{-tL_s}\xi(0)\|_1 \le c_\alpha\, \|\xi(0)\|_\alpha\ . \tag{5.30}$$

Proposition 5.7 gives

$$t^{1-\alpha}\, e^{|\omega| t}\, \| \int_0^t e^{-(t-\tau)L_s}\, h_1(\tau, t)\, d\tau \|_{X_1^s} \le c_1(\rho, \alpha) \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \tag{5.31}$$

with

$$c_1(\rho, \alpha) := k(|\omega|, \alpha)\Big( \frac{n(\rho)}{|\mu(\rho) + \omega|} + pL_\alpha(\rho)\Big)\ . \tag{5.32}$$

Finally, using (4.11) and again Corollary 5.7 we get

$$t^{1-\alpha} e^{|\omega| t}\, \|e^{-tL_s} \int_{-\infty}^0 e^{\tau L_s} h_2(\tau, t)\, d\tau\, \|_{X_1^s} \le c_2(\rho, \alpha) \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha}\, e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1 \tag{5.33}$$

with

$$c_2(\rho, \alpha) := M_s\, \overline{k}\, (\mu(\rho))\, n(\rho)\, \frac{\Gamma(\alpha)}{|\mu(\rho) + \omega|^\alpha} \sup_{t \ge 0} t^{1-\alpha} e^{(\omega_s + |\omega|)t}\ . \tag{5.34}$$

Now, (5.30)-(5.33) yield

$$t^{1-\alpha} e^{|\omega| t}\, \|\xi(t)\|_1 \le c_\alpha\, \|\xi(0)\|_\alpha + (c_1(\rho, \alpha) + c_2(\rho, \alpha)) \sup_{\vartheta \in (0,t]} \vartheta^{1-\alpha} e^{|\omega|\vartheta}\, \|\xi(\vartheta)\|_1\ . \tag{5.35}$$

We choose $\overline{\rho} \in (0, \rho']$ such that

$$c_1(\rho, \alpha) + c_2(\rho, \alpha) \le 1/2\ , \quad \rho \in (0, \overline{\rho}]\ . \tag{5.36}$$

Fix an arbitrary $T \in (0, t^+(u_0))$. Then, we can conclude with (5.35) and (5.36) that

$$t^{1-\alpha} e^{|\omega| t} \, \|\xi(t)\|_1 \leq c_\alpha \, \|\xi(0)\|_\alpha + (1/2) \sup_{\vartheta \in (0,T]} \vartheta^{1-\alpha} e^{\,|\omega|\vartheta} \, \|\xi(\vartheta)\|_1$$

for $t \in (0, T]$ and hence

$$t^{1-\alpha} e^{|\omega| t} \, \|\xi(t)\|_1 \leq 2c_\alpha \, \|\xi(0)\|_\alpha \;, \qquad t \in (0, T] \;.$$

Since $T$ can be chosen arbitrarily the statement follows. $\square$

**5.9 Remarks**

a) Let $M \subset X_1$ be a given subset and denote by $d_{X_1}(z, M) := \inf_{m \in M} \|z - m\|_{X_1}$ the distance of a point $z \in X_1$ from $M$. Then, Theorem 5.8 gives that

$$d_{X_1}(u(t), \mathcal{M}^c) \leq \frac{N_\alpha}{t^{1-\alpha}} \, e^{-|\omega| t} \, \|\xi(0)\|_\alpha \;, \; t \in (0, t^+(u_0)) \;, \; u_0 \in \mathcal{U}_\alpha(\rho) \;,$$

i.e. the set $\mathcal{M}^c$ is exponentially attracting solutions of $(4.20)_\rho$ with initial values in $\mathcal{U}_\alpha(\rho)$. Remark that the solutions of $(4.20)_\rho$ coincide with the solutions of (4.1), or (4.13) respectively, as long as these remain small. Therefore, the set $\mathcal{M}^c$ is also exponentially attractive for solutions of (4.1) as long as they are small.

b) Let $X$ be a Banach space with

$$X_1 \hookrightarrow X \hookrightarrow X_\alpha \;. \tag{5.37}$$

Suppose that

$$(t, x) \mapsto u(t, x) \tag{5.38}$$

generates a semiflow on $U \cap X$ , where $u(\cdot, x)$ denotes the solution of (4.13), or $(4.18)_\rho$ respectively, with $x \in U \cap X$ . Then the center manifolds constructed in Theorem 4.1 are invariant for the semiflow (5.38) and Theorem 5.8 gives

$$\|\pi^s u(t) - \sigma(\pi^c u(t))\|_X \leq c \, \frac{N_\alpha}{t^{1-\alpha}} \, e^{-|\omega| \, t} \, \|\pi^s u_0 - \sigma(\pi^c u_0)\|_X \tag{5.40}$$

for $t \in (0, t^+(u_0))$ and $u_0 \in U_\alpha(\rho) \cap X$ .

This last remark deserves some comments. We obtain the exponential attractivity of the center manifolds for each space which is 'sandwiched' by $X_1$ and $X_\alpha$ . For the quasilinear reaction-diffusion system (1.1) described in the introduction, $X$ is the space $W_p^1(\Omega, \mathbb{R}^N)$. Note that we already established that (1.1) generates a (smooth) semiflow on $X$. We keep in mind that we had to work in spaces with **maximal regularity** to carry through our arguments. In particular we had to use the property of being able to linearize the quasilinear equation (4.1) and then represent the solutions by the standard variation of constant formula. Moreover, the existence result in 4.1 relies on maximal regularity. We should mention again that this result was obtained, essentially, in [**21**]. We also would like to draw attention to [**39**]. In fact, after establishing (4.12) (for which we invoked maximal regularity), the assumption **(H)** in their work is satisfied and the general existence results apply. It is this property, namely that solutions of the linear problem

$$\dot{v}(t) + L_s\, v(t) = f(t)\ , \quad t \in \mathbb{R}$$

are given by

$$v(t) = (K_s f)(t) := \int_{-\infty}^{t} e^{-(t-\tau)L_s} f(\tau)\, d\tau\ ,$$

which leads to the fixed point equation (4.32). We do not know of a possibility to omit the use of maximal regularity results. What is left to show is that we are able to verify (5.38) and all the assumptions of section 4 in the context of partial differential equations, say for the quasilinear reaction-diffusion system (1.1).

Theorem 5.8 gives the best possible estimate. In fact, it gives an estimate in the 'better' spaces $X_1$ for solutions with initial values in the weaker spaces $X_\alpha$, taking into consideration the smoothing property of quasilinear equations. It is the refinement in [**10**] which allows to handle this smoothing property in the context of maximal regularity.

(5.40) is not quite optimal for the space $X$. It is to be expected that the center manifolds attract with exponential rate without the factor $t^{\alpha-1}$ which appears due to our method. However, the result in (5.40) is perfectly good enough. It gives the exponential attractivity for $t$ being bounded away from zero, say for $t \geq 1$. Since $0 \in X$ is an equilibrium for the semiflow we can be sure, that solutions with sufficiently small initial data exist in a time interval which is larger than $[0, 1]$. Remark that this was already incorporated in Lemma 5.1. But in the time interval $[0, 1]$ the solutions can not leave a small neighborhood of $X$ (since they depend continuously on $t$ and the initial values and $[0, 1]$ is compact). So in fact, (5.40) gives the desired result on the exponential attractivity.

## 6. Some function spaces

In this section we introduce some function spaces which turn out to be important for the study of quasilinear reaction-diffusion equations. These are the Sobolev and Bessel potential spaces, the Besov spaces and in particular the Nikol'skii and the so called little Nikol'skii spaces. All of these are intimately connected with various interpolation methods. The use of the little Nikol'skii spaces appears in the context of maximal regularity. We collect some basic facts and refer mainly to [**13**,**37**,**38**,**9**,**35**].

In the following, let $E := (\mathbb{R}^N, |\cdot|)$ be the euclidean space of dimension $N$. Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n, E)$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$ with values in $E$ and let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^n, E)$ denote its dual, the space of tempered $E$-valued distributions, endowed with the strong topology. Then it is well-known that the Fourier transform satisfies $\mathcal{F} \in Isom(S(\mathbb{R}^n, E)) \cap Isom(\mathcal{S}'(\mathbb{R}^n, E))$. Finally, let $L_p(\mathbb{R}^n, E) := \big(L_p(\mathbb{R}^n, E)\,;\, \|\cdot\|_p\big)$ denote the Lebesgue spaces of $E$-valued functions for $p \in [1, \infty]$.

In the following, we mostly suppress $E$ but we always mean $E$-valued functions and distributions. For convenience, we assume further on that $p \in (1, \infty)$. Then, for $s \in \mathbb{R}$, the **Bessel potential spaces** $H_p^s(\mathbb{R}^n)$ are defined by

$$H_p^s(\mathbb{R}^n) := \big(\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \mathcal{F}^{-1}\Lambda^{s/2}\mathcal{F}u \in L_p(\mathbb{R}^n)\}\,,\, \|\cdot\|_{H_p^s(\mathbb{R}^n)}\big)\ , \tag{6.1}$$

where $\Lambda(\xi) := (1 + |\xi|^2)$ and $\|u\|_{H_p^s(\mathbb{R}^n)} := \|\mathcal{F}^{-1}\Lambda^{s/2}\mathcal{F}u\|_p$ for $u \in H_p^s(\mathbb{R}^n)$. It is well-known that

$$H_p^k(\mathbb{R}^n) \doteq W_p^k(\mathbb{R}^n)\ , \quad p \in (1, \infty)\ ,\ k \in \mathbb{N}\ , \tag{6.2}$$

i.e. the Bessel potential spaces with integer exponents coincide with the Sobolev spaces $W_p^k(\mathbb{R}^n)$ of order $k$. We mention that the Bessel potential spaces are stable under complex interpolation:

$$[H_p^{s_0}(\mathbb{R}^n), H_p^{s_1}(\mathbb{R}^n)]_\theta \doteq H_p^{(1-\theta)s_0+\theta s_1}(\mathbb{R}^n)\ , \quad \theta \in (0,1)\ ,\ s_0, s_1 \in \mathbb{R}\ . \tag{6.3}$$

We further note the duality property

$$[H_p^s(\mathbb{R}^n)]' \doteq H_{p'}^{-s}(\mathbb{R}^n)\ , \quad p' := p/(p-1)\ , \tag{6.4}$$

where the duality pairing is induced by the standard $L_{p'} \times L_p$ pairing, i.e. by

$$< v, u >:= \int_{\mathbb{R}^n} < v(x), u(x) > \,dx\ , \quad (v, u) \in \mathcal{S}(\mathbb{R}^n, E) \times \mathcal{S}(\mathbb{R}^n, E)\ . \tag{6.5}$$

Here, $< \eta, \xi >$ denotes for $\eta, \xi \in E$ the pairing in $E$. We will also use another class of function spaces.

For

$$p \in (1, \infty) , \quad q \in [1, \infty] , \quad s \in \mathbb{R}$$

we define the **Besov spaces**

$$B^s_{p,q}(\mathbb{R}^n) := \begin{cases} (H^k_p(\mathbb{R}^n), H^{k+1}_p(\mathbb{R}^n))_{s-k,q} & \text{if } s \in (k, k+1), \ k \in \mathbb{Z} \\ (H^{k-1}_p(\mathbb{R}^n), H^{k+1}_p(\mathbb{R}^n))_{1/2,q} & \text{if } s = k \ , \ k \in \mathbb{Z} \ , \end{cases} \tag{6.6}$$

where $(\cdot,\cdot)_{\theta,q}$ denotes the continuous interpolation method. In the case of $q = \infty$, $B^s_{p,\infty}(\mathbb{R}^n)$ are the **Nikol'skii spaces**.

It is well-known that these spaces satisfy

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,1}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow B^t_{p,1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \tag{6.7}$$

for $t < s$. Moreover,

$$B^s_{p,1}(\mathbb{R}^n) \hookrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^n) \ . \tag{6.8}$$

Finally,

$$\mathcal{S}(\mathbb{R}^n) \overset{d}{\hookrightarrow} B^s_{p,1}(\mathbb{R}^n) \overset{d}{\hookrightarrow} H^s_p(\mathbb{R}^n) \overset{d}{\hookrightarrow} B^t_{p,q}(\mathbb{R}^n) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n) \tag{6.9}$$

for $t < s$ and $q \in [1, \infty)$. Note that all of these spaces are embedded in the space of (tempered) distributions. For the distributional derivatives $\partial^\alpha$, with $\alpha \in \mathbb{N}^n$, we have

$$\partial^\alpha \in \mathcal{L}(B^s_{p,q}(\mathbb{R}^n), B^{s-|\alpha|}_{p,q}(\mathbb{R}^n)) \cap \mathcal{L}(H^s_p(\mathbb{R}^n), H^{s-|\alpha|}_p(\mathbb{R}^n)) \tag{6.10}$$

for $q \in [1, \infty]$. We mention that the Besov spaces can be equipped with several equivalent norms. In fact, although the Besov are introduced as interpolation spaces, there also exist other descriptions of them. But for our purposes, the definition in (6.6) together with the quoted properties are sufficient.
Let us also note the following duality property of the Besov spaces

$$[B^s_{p,q}(\mathbb{R}^n)]' \doteq B^{-s}_{p',q'}(\mathbb{R}^n) \ , \quad q \in [1, \infty) \tag{6.11}$$

with respect to the duality pairing induced by (6.5). Finally, the **little Nikol'skii spaces** can be introduced as

$$b^s_{p,\infty}(\mathbb{R}^n) := \begin{cases} (H^k_p(\mathbb{R}^n), H^{k+1}_p(\mathbb{R}^n))^0_{s-k,\infty} & \text{if } s \in (k, k+1), \ k \in \mathbb{Z} \\ (H^{k-1}_p(\mathbb{R}^n), H^{k+1}_p(\mathbb{R}^n))^0_{1/2,\infty} & \text{if } s = k \, , \ k \in \mathbb{Z} \end{cases} \tag{6.12}$$

Here, $(\cdot,\cdot)^0_{\theta,\infty}$ denotes the continuous interpolation method, cf. (2.12). Now, we can complete (6.7) and (6.9) by

$$\mathcal{S}(\mathbb{R}^n) \overset{d}{\hookrightarrow} B^s_{p,1}(\mathbb{R}^n) \overset{d}{\hookrightarrow} H^s_p(\mathbb{R}^n) \overset{d}{\hookrightarrow} b^s_{p,\infty}(\mathbb{R}^n) \overset{d}{\hookrightarrow} B^t_{p,q}(\mathbb{R}^n) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n) \tag{6.13}$$

for $t < s,\ q \in [1,\infty)$ and

$$B^s_{p,q}(\mathbb{R}^n) \overset{d}{\hookrightarrow} b^s_{p,\infty}(\mathbb{R}^n)\ , \quad q \in [1,\infty)\ . \tag{6.14}$$

Moreover,

$$\partial^\alpha \in \mathcal{L}(b^s_{p,\infty}(\mathbb{R}^n), b^{s-|\alpha|}_{p,\infty}(\mathbb{R}^n))\ , \quad \alpha \in \mathbb{N}^n\ . \tag{6.15}$$

The little Nikol'skii spaces also enjoy the following duality property

$$[b^s_{p,\infty}(\mathbb{R}^n)]' \doteq B^{-s}_{p',1}(\mathbb{R}^n)\ , \quad p' := p/(p-1)\ , \tag{6.16}$$

with respect to the duality pairing introduced in (6.5). We also note the following properties of the little Nikol'skii spaces

$$(H^{s_0}_p(\mathbb{R}^n), H^{s_1}_p(\mathbb{R}^n))^0_{\theta,\infty} \doteq b^s_{p,\infty}(\mathbb{R}^n) \tag{6.17}$$

and

$$(b^{s_0}_{p,\infty}(\mathbb{R}^n), b^{s_1}_{p,\infty}(\mathbb{R}^n))^0_{\theta,\infty} \doteq b^s_{p,\infty}(\mathbb{R}^n) \tag{6.18}$$

for $s = (1-\theta)s_0 + \theta s_1$ and $s_0 < s_1$ . Moreover, we note

$$(H^{s_0}_p(\mathbb{R}^n), H^{s_1}_p(\mathbb{R}^n))_{\theta,q} \doteq B^s_{p,q}(\mathbb{R}^n)\ , \tag{6.19}$$

where $(\cdot,\cdot)_{\theta,q}$ denote the real interpolation methods. The little Nikol'skii spaces have been introduced in [**20**] (and denoted by $h^s_p(\mathbb{R}^n)$). We refer to [**35**] and [**9**] for proofs and additional results.

Now, we briefly indicate how these spaces can be defined on an open subset $\Omega \subset \mathbb{R}^n$. Let

$$r_\Omega : \mathcal{D}'(\mathbb{R}^n, E) \to \mathcal{D}'(\Omega, E)\ , \tag{6.20}$$

be the restriction mapping, where $\mathcal{D}'$ denotes the space of $E$-valued distributions on $\mathbb{R}^n$ or $\Omega$, respectively, i.e.

$$< r_\Omega u\,, \phi > := < u\,, \phi >\ , \quad \phi \in \mathcal{D}(\Omega, E) \tag{6.21}$$

for $u \in \mathcal{D}'(\mathbb{R}^n, E)$. From now on, we suppress the space $E$ (noting that all functions and distributions are $E$-valued). For $s \in \mathbb{R}$ and $p \in (1,\infty)$ let

$$F^s_p(\mathbb{R}^n) := \{H^s_p(\mathbb{R}^n), B^s_{p,q}(\mathbb{R}^n), b^s_{p,\infty}(\mathbb{R}^n)\,;\ q \in [1,\infty]\}\ . \tag{6.22}$$

The local versions of the spaces $F^s_p(\mathbb{R}^n)$ are defined by

$$F^s_p(\Omega) := r_\Omega(F^s_p(\mathbb{R}^n))\ , \tag{6.23}$$

equipped with the natural topology of a quotient space. Then the spaces $F^s_p(\Omega)$ are well defined Banach spaces. If we assume that $\Omega$ is a smooth (bounded) domain in $\mathbb{R}^n$, it is well-known that there exists a linear, bounded (total) extension operator

$$ext \in \mathcal{L}(F^s_p(\Omega), F^s_p(\mathbb{R}^n)) \quad \text{with} \quad r_\Omega \circ ext = id_{F^s_p(\Omega)}\ , \tag{6.24}$$

cf. [**1**,**34**] and [**37**, Theorem 4.2.2]. Hence, $r_\Omega$ is a retraction and all interpolation results apply to the local spaces, see [**37**, Theorem 1.2.4]. We note the following results and properties for the local spaces $F_p^s(\Omega)$.

Fix $p \in (1,\infty)$, $q \in [1,\infty)$ and $-\infty < t < s < \infty$. Then

$$C^\infty(\bar\Omega) \overset{d}{\hookrightarrow} B^s_{p,1}(\Omega) \overset{d}{\hookrightarrow} H^s_p(\Omega) \overset{d}{\hookrightarrow} b^s_{p,\infty}(\Omega) \overset{d}{\hookrightarrow} B^t_{p,1}(\Omega) \overset{d}{\hookrightarrow} \mathcal{D}'(\Omega) \ , \tag{6.25}$$

$$\mathcal{D}(\Omega) \overset{d}{\hookrightarrow} F^s_p(\Omega) \quad \text{if} \quad -\infty < s < \frac{1}{p} \ . \tag{6.26}$$

If $-1+1/p < s < 1/p$, we have the following duality properties:

$$[H^s_p(\Omega)]' \doteq H^{-s}_{p'}(\Omega) \ , \quad [B^s_{p,q}(\Omega)]' \doteq B^{-s}_{p',q'}(\Omega) \ , \quad [b^s_{p,\infty}(\Omega)]' \doteq B^{-s}_{p',1}(\Omega) \ , \tag{6.27}$$

with respect to the duality pairing

$$< u,v > = \int_\Omega < u(x), v(x) > \, dx \ , \quad u,v \in \mathcal{D}(\Omega,E) \ . \tag{6.28}$$

The distributional derivative is bounded and continuous, i.e

$$\partial^\alpha \in \mathcal{L}(F^s_p(\Omega), F^{s-|\alpha|}_p(\Omega)) \ , \quad \alpha \in \mathbb{N}^n \ . \tag{6.29}$$

We also note that

$$\begin{aligned} (H^{s_0}_p(\Omega), H^{s_1}_p(\Omega))^0_{\theta,\infty} &\doteq b^s_{p,\infty}(\Omega) \ , \\ (b^{s_0}_{p,\infty}(\Omega), b^{s_1}_{p,\infty}(\Omega))^0_{\theta,\infty} &\doteq b^s_{p,\infty}(\Omega) \ , \end{aligned} \tag{6.30}$$

for $s = (1-\theta)s_0 + \theta s_1$ , $s_0 < s_1$ . Moreover,

$$(H^{s_0}_p(\Omega), H^{s_1}_p(\Omega))_{\theta,q} \doteq B^s_{p,q}(\Omega) \ . \tag{6.31}$$

Finally, we will use the following result on pointwise multipliers for the little Nikol'skii spaces: The mapping

$$C^\rho(\bar\Omega, \mathcal{L}(E)) \times b^t_{p,\infty}(\Omega,E) \to b^t_{p,\infty}(\Omega,E) \ , \quad (m,u) \mapsto mu \ , \tag{6.32}$$

is bilinear and continuous if $|t| < \rho < 1$.

The following results will be used in section 7 when studying the Dirichlet form of a boundary value system on some function spaces. Assume that $p_1, q_0 \in [1,\infty)$, $p \in (1,\infty)$ and $s \in \mathbb{R}$ are given reals with

$$1/p < s < 1 + 1/p \ , \qquad p_1 > n/2 \, , \ q_0 > n-1 \ . \tag{6.33}$$

For each $s$ fix $\rho(s)$ such that

$$\rho(s) > |s-1| \ . \tag{6.34}$$

Finally, set $\partial := \partial_j\ ,\ j \in \{1, \dots, n\}$ and $p' = (p-1)/p$ . Then

**6.1 Lemma**

*a)* $[(a,u,w) \mapsto < \partial w, a\partial u >] \in \mathcal{L}^3(C^{\rho(s)}(\bar{\Omega}, \mathcal{L}(E)) \times b^s_{p,\infty}(\Omega, E) \times B^{2-s}_{p',1}(\Omega, E); \mathbb{R})$ ,

*b)* $[(a,u,w) \mapsto < w, a\partial u >] \in \mathcal{L}^3(C^{\rho(s)}(\bar{\Omega}, \mathcal{L}(E)) \times b^s_{p,\infty}(\Omega, E) \times B^{2-s}_{p',1}(\Omega, E); \mathbb{R})$ ,

*c)* $[(a,u,w) \mapsto < w, au >] \in \mathcal{L}^3(L_{p_1}(\Omega, \mathcal{L}(E)) \times b^s_{p,\infty}(\Omega, E) \times B^{2-s}_{p',1}(\Omega, E); \mathbb{R})$ ,

*d)* $[(b,u,w) \mapsto < \gamma w, b\gamma u >_\partial] \in \mathcal{L}^3(L_{q_0}(\partial\Omega, \mathcal{L}(E)) \times b^s_{p,\infty}(\Omega, E) \times B^{2-s}_{p',1}(\Omega, E); \mathbb{R})$ .

*Here, $\mathcal{L}^m(F_1 \times \cdots \times F_m; \mathbb{R})$ denotes the linear space of all continuous $\mathbb{R}$-valued $m$-multilinear forms on $\prod_1^m F_i$ . Moreover, $< \cdot, \cdot >$ always denotes the duality pairing in various spaces, induced by* (6.28).

**Proof** (6.29) gives

$$\partial \in \mathcal{L}(B^{2-s}_{p',1}(\Omega), B^{1-s}_{p',1}(\Omega)) \cap \mathcal{L}(b^s_{p,\infty}(\Omega), b^{s-1}_{p,\infty}(\Omega)) \ . \tag{6.35}$$

Using (6.27), (6.32)-(6.35) we get

$$|< \partial w, a\partial u >| \leq c\,\|w\|_{B^{2-s}_{p',1}(\Omega)} \|a\|_{C^{\rho(s)}} \|u\|_{b^s_{p,\infty}(\Omega)}$$

and hence the assertion in a). The remaining assertions follow from Sobolev type embedding theorems, the trace theorem, and Hölder's inequality. We refrain from giving more details and refer to [**35**, Lemma 4.6]. □

## 7. Normally elliptic boundary value problems

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain in $\mathbb{R}^n$. We denote the tangent bundle of $\partial\Omega$ by $T(\partial\Omega)$ and the outer unit normal field on $\partial\Omega$ by $\nu = (\nu^1, \dots, \nu^n)$. We then consider the (formal) differential operator

$$\mathcal{A}u := -\partial_j(a_{jk}\partial_k u) + a_j u + a_0 u \ . \tag{7.1}$$

For the moment, we only state the following regularity assumption for the coefficients of the principal part of $\mathcal{A}$

$$a_{jk} \in C(\bar{\Omega}, \mathcal{L}(\mathbb{R}^N)) \ , \quad 1 \leq j,k \leq n \ . \tag{7.2}$$

Let $a_\pi \in C(\bar{\Omega} \times \mathbb{R}^n, \mathcal{L}(\mathbb{R}^N))$ denote the symbol of the principal part, that is,

$$a_\pi(x,\xi) := a_{jk}(x)\xi^j\xi^k \ , \quad (x,\xi) \in \bar{\Omega} \times \mathbb{R}^n \ . \tag{7.3}$$

Then, $\mathcal{A}$ is called **normally elliptic** if

$$\sigma(a_\pi(x,\xi)) \subset [Re\, z > 0] \ , \quad (x,\xi) \in \bar{\Omega} \times (\mathbb{R}^n \setminus \{0\}) \ , \tag{7.4}$$

where $\sigma(a_\pi(x,\xi))$ denotes the spectrum (i.e. the eigenvalues) of the $N \times N$- Matrix $a_\pi(x,\xi)$.

For $r \in \{1, \dots, N\}$, let $\delta^r$ be a function defined on $\partial\Omega$ and satisfying $\delta^r \in C(\partial\Omega, \{0,1\})$. This implies that $\delta^r$ either vanishes or equals 1 on a component (of connectedness) of $\partial\Omega$. With this we set

$$\delta := \mathrm{diag}\,[\,\delta^1, \dots, \delta^N\,] \in C(\partial\Omega, \mathcal{L}(\mathbb{R}^N)) \ . \tag{7.5}$$

We then define a general boundary operator by

$$\mathcal{B}u := \delta\,(a_{jk}\nu^j\gamma\partial_k u + b_0\gamma u) + (1-\delta)\,\gamma u \ , \tag{7.6}$$

where $\gamma$ denotes the trace operator. Note that $\mathcal{B}$ acts on vector valued functions $u = (u^1, \dots, u^N)$, assigning to each $u^r, 1 \leq r \leq N$, a Dirichlet condition on components $\Gamma \subset \partial\Omega$ with $\delta^r(\Gamma) = 0$ and a 'Neumann type' condition on components with $\delta^r(\Gamma) = 1$. It should be observed that every system of $N$ linear differential operators of order at most 1 on $\partial\Omega$ can be written in the form (7.6). Let

$$b_\pi(x,\xi) := \delta(x)\,(a_{jk}(x)\nu^j(x)\xi^k) + (1-\delta(x)) \ , \quad (x,\xi) \in \partial\Omega \times \mathbb{R}^n \tag{7.7}$$

be the associated principal boundary symbol.

Then $\mathcal{B}$ is said to satisfy the **normal complementing condition with respect to** $\mathcal{A}$, if zero is, for each $(x,\xi) \in T(\partial\Omega)$ and $\lambda \in [Re\, z \geq 0]$ with $(\xi,\lambda) \neq (0,0)$, the only exponentially decaying solution of the boundary value problem on $\mathbb{R}^+$ :

$$[\lambda + a_\pi(x, \xi + \nu(x) i\partial_t)]u = 0 \ , \ t > 0 \ , \quad b_\pi(x, \xi + \nu(x) i\partial_t) u(0) = 0 \ .$$

Finally, $(\mathcal{A},\mathcal{B})$ is a **normally elliptic boundary value problem** on $\Omega$ if $\mathcal{A}$ is normally elliptic and $\mathcal{B}$ satisfies the normal complementing condition with resect to $\mathcal{A}$.

**7.1 Remarks** We note some conditions guaranteeing that $(\mathcal{A},\mathcal{B})$ is normally elliptic. For a detailed discussion we refer to [**6**, section 4].

*a)* Let $\mathcal{A}$ be *uniformly strongly elliptic*, i.e.

$$(a_\pi(x,\xi)\eta \,|\, \eta) > 0 \ , \quad (x,\xi,\eta) \in \bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^N \ , \ \xi \neq 0, \ , \eta \neq 0 \ ,$$

where $(\cdot\,|\,\cdot)$ denotes the euclidean product in $\mathbb{R}^N$, and let $\mathcal{B} = \gamma$ be the Dirichlet operator. Then $(\mathcal{A},\mathcal{B})$ is normally elliptic.

*b)* $\mathcal{A}$ is called *uniformly very strongly elliptic* if the *uniform Legendre condition* is satisfied, i.e. if

$$\sum_{r,s=1}^{N} a_{jk}^{rs}\, \zeta_r^j\, \zeta_s^k > 0 \ , \quad x \in \bar{\Omega} \ , \ \zeta \in \mathbb{R}^{nN} \setminus \{0\} \ .$$

Then $(\mathcal{A},\mathcal{B})$ is normally elliptic for each boundary operator $\mathcal{B} = \delta(a_{jk}\nu^j\gamma\partial_k + b_0\gamma) + (1-\delta)\gamma$ .

*c)* We consider the special case of *separated divergence-form systems*, i.e. we assume

$$a_{jk} = A\,\alpha_{jk} \ , \quad 1 \leq j,k \leq n$$

with

$$A \in C(\bar{\Omega}, \mathcal{L}(\mathbb{R}^N)) \ , \quad [\alpha_{jk}] \in C(\bar{\Omega}, \mathcal{L}(\mathbb{R}^n)) \ ,$$
$$[\alpha_{jk}] \quad \text{is symmetric and uniformly positive definite} \quad .$$

If

$$\sigma(A(x)) \subset [Re\, z > 0] \ , \ x \in \bar{\Omega} \ , \quad (1-\delta(x))A(x)\delta(x) = 0 \ , \ x \in \partial\Omega \ ,$$

then $(\mathcal{A},\mathcal{B})$ is normally elliptic.

*d)* In the case of $N = 1$ the definitions of normally elliptic, uniformly strongly elliptic and uniformly very strongly elliptic coincide. However, the notion of normally elliptic, which was introduced by Amann, really is more general for systems and is optimal in some sense, see [**6**, Theorem 2.4].

We will now study linear boundary value problems of second order in some function spaces. We will do this under very weak regularity assumptions. Although we do not explicitly introduce this concept, we are working in the setting of extrapolation spaces, see [**2**,**3**,**6**,**9**]. This will finally render the possibility to study the reaction-diffusion system (1.1) as an abstract evolution equation in appropriate spaces.

It turns out to be very convenient to define a topology on the set of all second order normally elliptic boundary value systems.

To do so, we fix from now on $p \in (n, \infty)$. For each $s \in (1/p, 1 + 1/p)$, let $\rho(s)$ be chosen such that

$$\rho(s) > |s - 1| \ . \tag{7.8}$$

Set $X := \mathcal{L}(\mathbb{R}^N)$. Then we define

$$\mathbb{M}_p^s(\Omega) := \mathbb{M}_p^{\rho(s)}(\Omega) := C^{\rho(s)}(\bar{\Omega}, X)^{n^2} \times C^{\rho(s)}(\bar{\Omega}, X)^n \times L_p(\Omega, X) \times L_p(\partial\Omega, X) \tag{7.9}$$

with a general element

$$m := ((a_{jk}), (a_j), a_0, b_0) \ . \tag{7.10}$$

We now identify each of the elements $m \in \mathbb{M}_p^s(\Omega)$ with the (formal) boundary value problem

$$\begin{aligned} \mathcal{A} &:= \mathcal{A}(m) := -\,\partial_j(a_{jk}\partial_k\ \cdot) + a_j\partial_j + a_0 \\ \mathcal{B} &:= \mathcal{B}(m) := \delta\,(a_{jk}\nu^j\gamma\partial_k + b_0\gamma) + (1-\delta)\gamma \ . \end{aligned} \tag{7.11}$$

It is clear that there is a one-one correspondence between (7.10) and (7.11). In (7.9) we give a precise statement on the regularity of the coefficients and moreover give a topology to the (linear) space $\mathbb{M}_p^s(\Omega)$.

For $m \in \mathbb{M}_p^s(\Omega)$, the **Dirichlet form** of the boundary value problem $(\mathcal{A}(m), \mathcal{B}(m))$ can be defined, say for $(w, u) \in H_{p'}^2(\Omega) \times H_p^2(\Omega)$, by

$$\begin{aligned} a(m)(w,u) &:= <\partial_j w, a_{jk}\partial_k u> + <w, a_j\partial_j u + a_0 u> + <\gamma w, b_0\gamma u>_\partial \\ &:= \int_\Omega \{<\partial_j w, a_{jk}\partial_k u> + <w, a_j\partial_j u + a_0 u>\}\, dx + \int_{\partial\Omega} <\gamma w, b_0\gamma u>\, d\sigma \, , \end{aligned} \tag{7.12}$$

We now show that the Dirichlet form can be extended to some of the spaces introduced in section 6.

**7.2 Lemma** *Let $s \in (1/p, 1 + 1/p)$. Then the mapping*

$$\mathbb{M}_p^s(\Omega) \to \mathcal{L}^2(B_{p',1}^{2-s}(\Omega) \times b_{p,\infty}^s(\Omega), \mathbb{R}) \ , \quad [m \mapsto a(m)] \tag{7.13}$$

*is well-defined, continuous and linear. Here, $\mathcal{L}^2(X \times Y, \mathbb{R})$ denotes the space of all continuous bilinear forms on $X \times Y$ for two Banach spaces $X, Y$.*

**Proof** The statement is an immediate consequence of Proposition 6.1. $\square$

**7.3 Corollary** *The mapping*

$$\mathbb{M}_p^s(\Omega) \to \mathcal{L}^2(B^{2-s}_{p',1,\mathcal{B}}(\Omega) \times b^s_{p,\infty,\mathcal{B}}(\Omega), \mathbb{R}) \ , \quad [\, m \mapsto a(m)\,] \tag{7.14}$$

*is continuous and linear, where*

$$B^{2-s}_{p',1,\mathcal{B}}(\Omega) := \{u \in B^{2-s}_{p',1}(\Omega)\, ; \, (1-\delta)\gamma u = 0\} \ , \tag{7.15}$$

$$b^s_{p,\infty,\mathcal{B}}(\Omega) := \{u \in b^s_{p,\infty}(\Omega)\, ; \, (1-\delta)\gamma u = 0\} \ . \tag{7.16}$$

**Proof** Note that $2-s \in (1/p', 1+1/p')$ for $s \in (1/p, 1+1/p)$. Hence, the trace operator $\gamma$ is well-defined and continuous, i.e

$$\gamma \in \mathcal{L}(B^{2-s}_{p',1}(\Omega), B^{2-s-1/p'}_{p',1}(\partial\Omega)) \cap \mathcal{L}(b^s_{p,\infty}(\Omega), b^{s-1/p}_{p,\infty}(\partial\Omega)) \ .$$

(This follows from [**37**, Theorem 4.7.1] and a density argument, cf. (6.25)). Now we can conclude that

$$B^{2-s}_{p',1,\mathcal{B}}(\Omega) \subset B^{s-2}_{p',1}(\Omega) \ , \quad b^s_{p,\infty,\mathcal{B}}(\Omega) \subset b^s_{p,\infty}(\Omega)$$

are closed subspaces and the assertion follows immediately from Lemma 7.2. □

We now use the fact that each continuous bilinear form induces a continuous linear operator. In fact, $a \in \mathcal{L}^2(X \times Y, \mathbb{R})$ iff there exists $A \in \mathcal{L}(Y, X')$ with

$$a(x,y) = < Ay, x > \ , \quad (x,y) \in X \times Y \quad \text{and} \quad \|a\|_{\mathcal{L}^2(X\times Y,\mathbb{R})} = \|A\|_{\mathcal{L}(Y,X')} \ . \tag{7.17}$$

Obviously, the map $[\, a \mapsto A\,]$ is linear (and continuous).

**7.4 Corollary** *Let $A(m)$ be the linear operator induced by the Dirichlet form $a(m)$, i.e.*

$$a(m)(w,u) = < A(m)u, w > \ , \quad (w,u) \in B^{2-s}_{p',1,\mathcal{B}}(\Omega) \times b^s_{p,\infty,\mathcal{B}}(\Omega) \ . \tag{7.18}$$

*Then*

$$[\, m \mapsto A(m)\,] \in \mathcal{L}\big(\mathbb{M}_p^s(\Omega), \mathcal{L}(b^s_{p,\infty,\mathcal{B}}(\Omega), (B^{2-s}_{p',1,\mathcal{B}}(\Omega))')\big) \ .$$

**7.5 Remark** Using similar estimates as in Lemma 7.2 it can be shown that

$$[\, m \mapsto A(m)\,] \in \mathcal{L}(\mathbb{M}_p^s(\Omega), \mathcal{L}(H^s_{p,\mathcal{B}}(\Omega), H^{s-2}_{p,\mathcal{B}}(\Omega))) \ , \ 1/p < s < 1+1/p \ , \tag{7.19}$$

where

$$H^s_{p,\mathcal{B}}(\Omega) = \{u \in H^s_p(\Omega); (1-\delta)\gamma u = 0\} \ , \quad H^{s-2}_{p,\mathcal{B}}(\Omega) := (H^{2-s}_{p',\mathcal{B}}(\Omega))' \ , \tag{7.20}$$

the duality pairing being induced by the standard $L_{p'} \times L_p$ pairing.

**7.6 Lemma** *Let $1/p < s < 1 + 1/p$. Then the spaces*

$$b^{s-2}_{p,\infty,\mathcal{B}}(\Omega) := cl(H^{s-2}_{p,\mathcal{B}})' \ in \ \ (B^{2-s}_{p',1,\mathcal{B}}(\Omega))' \tag{7.21}$$

*are well-defined and $H^{s-2}_{p,\mathcal{B}}(\Omega) \overset{d}{\hookrightarrow} b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$.*

**Proof** The assertion certainly follows from (the first part of)

$$B^t_{q,1,\mathcal{B}}(\Omega) \overset{d}{\hookrightarrow} H^t_{q,\mathcal{B}}(\Omega) \overset{d}{\hookrightarrow} b^t_{q,\infty,\mathcal{B}}(\Omega)\ , \quad 1/q < t < 1 + 1/q\ . \tag{7.22}$$

Indeed, we then have $B^{2-s}_{p',1,\mathcal{B}}(\Omega) \overset{d}{\hookrightarrow} H^{2-s}_{p',\mathcal{B}}(\Omega)$ for $s \in (1/p, 1 + 1/p)$. Hence, each continuous linear form on $H^{2-s}_{p',\mathcal{B}}$ induces a (unique) continuous linear form on $B^{2-s}_{p',1,\mathcal{B}}(\Omega)$ (by restriction) and

$$H^{s-2}_{p,\mathcal{B}}(\Omega) = (H^{2-s}_{p',\mathcal{B}}(\Omega))' \hookrightarrow (B^{2-s}_{p',1,\mathcal{B}}(\Omega))'\ .$$

Now, $H^{s-2}_{p,\mathcal{B}}(\Omega) \overset{d}{\hookrightarrow} b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$ obviously follows from the definition. Let

$$\mathcal{R} \in \mathcal{L}(\partial B^t_{q,1}, B^t_{q,1}(\Omega)) \cap \mathcal{L}(\partial B^t_{q,q}, H^t_q(\Omega)) \cap \mathcal{L}(\partial b^t_{q,\infty}, b^t_{q,\infty}(\Omega)) \tag{7.23}$$

be a coretraction for the operator $(1-\delta)\gamma$. (For the existence, cf. [**35**, Lemma 13.3] and [**37**, Theorem 4.7.1]. We have set $\partial B^t_{q,1} := \{z \in B^{t-1/q}_{q,1}(\partial\Omega)\,;\,(1-\delta)z = 0\}$ and an analogous definition is used for the other spaces). To show the first dense embedding in (7.22) we pick $u \in H^t_{q,\mathcal{B}}(\Omega)$ and fix $\varepsilon > 0$ arbitrarily. Thanks to (6.25), there exists $v \in B^t_{q,1}(\Omega)$ such that

$$\|u - v\|_{H^t_q(\Omega)} < \varepsilon\ .$$

Now we set $w := \mathcal{R}\,(1-\delta)\gamma v$. Since $\mathcal{R}$ is a coretraction for $(1-\delta)\gamma$ we have $(1-\delta)\gamma w = (1-\delta)\gamma v$. We conclude that $v - w \in B^t_{q,1,\mathcal{B}}(\Omega)$. It follows that

$$\begin{aligned}\|u-(v-w)\|_{H^s_p(\Omega)} &\le \|u-v\|_{H^s_p} + \|w\|_{H^s_p} = \|u-v\|_{H^s_p} + \|\mathcal{R}\,(1-\delta)\gamma(u-v)\|_{H^s_p}\\ &\le (1 + \|\mathcal{R}\,(1-\delta)\gamma\|_{\mathcal{L}(H^s_p)}\,)\,\|u-v\|_{H^s_p}\ .\end{aligned}$$

The same arguments also give the remaining assertions. $\square$

**7.7 Proposition** *Let $s \in (1/p, 1+1/p)$. Then*

$$[\,m \mapsto A(m)\,] \in \mathcal{L}(\mathbb{M}^s_p(\Omega),\, \mathcal{L}(b^s_{p,\infty,\mathcal{B}}(\Omega)\,,\, b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)))$$

*and $a(m)(w,u) =< A(m)u, w >$ for each $(w,u) \in B^{2-s}_{p',1,\mathcal{B}}(\Omega) \times b^s_{p,\infty,\mathcal{B}}(\Omega)$.*

**Proof** By collecting the results in Corollary 7.4 and in (7.19) we have that

$$
\begin{aligned}
A(m):\ & H^s_{p,\mathcal{B}}(\Omega) \longrightarrow H^{s-2}_{p,\mathcal{B}}(\Omega)\ , \\
A(m):\ & b^s_{p,\infty,\mathcal{B}}(\Omega) \longrightarrow (\, B^{2-s}_{p',1,\mathcal{B}}(\Omega)\,)'
\end{aligned}
$$

are continuous and linear. Since these linear mappings are induced by the same form, we may use the same notation. Loosely expressed, $A(m)$ is the realization of the same operator in different spaces. In fact, $A(m)$ acting on the first space is the restriction of $A(m)$ acting on $b^s_{p,\infty,\mathcal{B}}(\Omega)$. This follows by a density argument and the fact that $A(m)$ is induced by the same form. The assertion now follows from Lemma 7.6 and (7.22). □

Now, we define the set $\mathcal{M}^{\rho(s)}_p(\Omega) := \mathcal{M}^s_p(\Omega)$ of all **normally elliptic boundary value problems in** $\mathbb{M}^s_p(\Omega)$ by

$$
\begin{aligned}
\mathcal{M}^s_p(\Omega) := \{\, m \in \mathbb{M}^s_p(\Omega)\ :\ & (\mathcal{A}(m), B(m))\ ,\ (\mathcal{A}_\pi(m(x_0)), \mathcal{B}_\pi(m(x_0))) \\
& \textit{are normally elliptic for each } x_0 \textit{ in } \bar{\Omega}\,\}\ ,
\end{aligned}
\tag{7.24}
$$

where $(A_\pi(m), \mathcal{B}_\pi(m)) := \big(-\partial_k(a_{jk}\partial_j\ )\,,\ \delta a_{jk}\nu^j\gamma\partial_k + (1-\delta)\gamma\big)$ denotes the principal part of $(\mathcal{A}, \mathcal{B})$. Then (cf. [**6**,**9**])

$$
\mathcal{M}^s_p(\Omega) \subset \mathbb{M}^s_p(\Omega) \quad \text{is open}\ . \tag{7.25}
$$

Due to the fact that linear mappings, restricted to an open subset of a linear space, are analytic, we can note the following immediate consequence of Proposition 7.7

**7.8 Corollary** $\quad [\, m \mapsto A(m)\,] \in C^\omega(\mathcal{M}^s_p(\Omega),\ \mathcal{L}(b^s_{p,\infty,\mathcal{B}}(\Omega), b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)))\ .$

We now state the following very important result

**7.9 Theorem** (*Generation Theorem*)

*Let* $1/p < s < 1 + 1/p$. *Then*

$$
A(m) \in \mathcal{H}\,(H^s_{p,\mathcal{B}}(\Omega)\,,\ H^{s-2}_{p,\mathcal{B}}(\Omega)) \quad \textit{for each} \quad m \in \mathcal{M}^s_p(\Omega)\ , \tag{7.26}
$$

*where* $\mathcal{H}(E_1, E_0)$ *has been defined in* (1.13).

**Proof** For a proof and many additional and more general results we refer to [**9**]. □

**7.10 Lemma** *Let* $s \in (1/p, 1+1/p)$. *Then, for each* $m \in \mathcal{M}^s_p(\Omega)$, *there exists an* $\omega_0 = \omega_0(m) \in \mathbb{R}$ *such that*

$$
(\omega + A(m)) \in Isom\,(\, b^s_{p,\infty,\mathcal{B}}(\Omega)\,,\ b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)\,)\ , \quad \omega > \omega_0\ .
$$

**Proof** Fix $t_0, t_1 \in (1/p, 1+1/p)$ with

$$t_0 < s < t_1 \quad \text{and} \quad \rho(s) > |t_1 - 1| \vee |t_0 - 1| \ .$$

(This is always possible, of course, for $t_0, t_1$ sufficiently close to $s$). Pick $m \in \mathcal{M}_p^s(\Omega)$. Thanks to Theorem 7.9 there exists an $\omega_0 = \omega_0(m)$ such that the mappings

$$\begin{aligned}(\omega + A(m)) \,:\, H^{t_0}_{p,\mathcal{B}}(\Omega) &\longrightarrow H^{t_0-2}_{p,\mathcal{B}}(\Omega)\\ (\omega + A(m)) \,:\, H^{t_1}_{p,\mathcal{B}}(\Omega) &\longrightarrow H^{t_1-2}_{p,\mathcal{B}}(\Omega)\end{aligned}$$

are isomorphisms for $\omega > \omega_0$. (We use that these mappings coincide for $u \in H^{t_0}_{p,\mathcal{B}}(\Omega) \cap H^{t_1}_{p,\mathcal{B}}(\Omega)$ ). It then follows from interpolation theory that

$$(\omega + A(m)) \,:\, (H^{t_0}_{p,\mathcal{B}}\,,\, H^{t_1}_{p,\mathcal{B}})^0_{\theta,\infty} \longrightarrow (H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})^0_{\theta,\infty}\ , \quad \theta \in (0,1)\ , \tag{7.27}$$

is an isomorphism between the interpolation spaces for $\omega > \omega_0$. ($(\cdot,\cdot)^0_{\theta,\infty}$ denotes the continuous interpolation method, cf. (2.12)). We show that

$$(H^{t_0}_{p,\mathcal{B}}\,,\, H^{t_1}_{p,\mathcal{B}})^0_{\frac{s-t_0}{t_1-t_0},\infty} \doteq b^s_{p,\infty,\mathcal{B}}\ , \quad (H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})^0_{\frac{s-t_0}{t_1-t_0},\infty} \doteq b^{s-2}_{p,\infty,\mathcal{B}}\ . \tag{7.28}$$

Note that the first assertion follows from (6.30), from [**37**, p.118] and the fact that

$$H^{t_0}_{p,\mathcal{B}} \subset H^{t_0}_p(\Omega)\ , \quad H^{t_1}_{p,\mathcal{B}} \subset H^{t_1}_p(\Omega)$$

are complemented subspaces with $(id - \mathcal{R}(1-\delta)\gamma)$ being a projection, where $\mathcal{R}$ is given in (7.23). The same argument together with (6.31) shows

$$(H^{2-t_0}_{p',\mathcal{B}}\,,\, H^{2-t_1}_{p',\mathcal{B}})_{(s-t_0)/(t_1-t_0),1} \doteq B^{2-s}_{p',1,\mathcal{B}}\ . \tag{7.29}$$

Using the definition of $(\cdot,\cdot)^0_{\theta,\infty}$ in (2.12) we have

$$(H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})^0_{\theta,\infty} := cl\,(H^{t_1-2}_{p,\mathcal{B}}) \quad \text{in} \quad (H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})_{\theta,\infty}\ . \tag{7.30}$$

Due to

$$H^{t_1-2}_{p,\mathcal{B}} \overset{d}{\hookrightarrow} [H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}}]_\theta \hookrightarrow (H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})_{\theta,\infty}\ ,$$

which holds for the complex interpolation method $[\cdot,\cdot]_\theta$, cf. [**37**,**13**], we obtain

$$(H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})^0_{\theta,\infty} = cl\,([H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}}]_\theta) \quad \text{in} \quad (H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})_{\theta,\infty}\ . \tag{7.31}$$

Let $\theta := (s-t_0)/(t_1-t_0)$.
It follows from the duality theorem, cf. [**37**, Theorem 1.11.2] or [**35**, Theorem 1.3], from (7.20) and (7.29) that

$$(H^{t_0-2}_{p,\mathcal{B}}\,,\, H^{t_1-2}_{p,\mathcal{B}})_{\theta,\infty} \doteq (\,(H^{2-t_0}_{p',\mathcal{B}}\,,\, H^{2-t_1}_{p',\mathcal{B}})_{\theta,1}\,)' \doteq (\,B^{2-s}_{p',1,\mathcal{B}}\,)'\ . \tag{7.32}$$

Moreover, [**13**, Corollary 4.5.2] and (7.20) give

$$[H^{t_0-2}_{p,\mathcal{B}}\,,\,H^{t_1-2}_{p,\mathcal{B}}]_\theta \doteq (\,[H^{2-t_0}_{p',\mathcal{B}}\,,\,H^{2-t_1}_{p',\mathcal{B}}]_\theta\,)' \doteq (\,H^{2-s}_{p',\mathcal{B}}\,)'\;, \tag{7.33}$$

where we have used (6.3), (6.24) and the fact that the spaces $H^{2-t_0}_{p,\mathcal{B}}$ and $H^{2-t_1}_{p,\mathcal{B}}$ are complemented to obtain

$$[H^{2-t_0}_{p',\mathcal{B}}\,,\,H^{2-t_1}_{p',\mathcal{B}}]_{(s-t_0)/(t_1-t_0)} \doteq H^{2-s}_{p',\mathcal{B}}\;.$$

Finally, we collect (7.31)-(7.33) and conclude with $H^{s-2}_{p,\mathcal{B}} := (\,H^{2-s}_{p',\mathcal{B}}\,)'$ and (7.21) that

$$(H^{t_0-2}_{p,\mathcal{B}}\,,\,H^{t_1-2}_{p,\mathcal{B}})^0_{(s-t_0)/(t_1-t_0)\,,\infty} \doteq b^{s-2}_{p,\infty,\mathcal{B}}\;. \qquad \square$$

We now fix $t_0 \in (1/p, 1+1/p)$ and set

$$E_0 := H^{t_0-2}_{p,\mathcal{B}}(\Omega)\;,\;\; E_1(m) := D(A(m)) \quad \text{for} \;\; m \in \mathcal{M}^{t_0}_p(\Omega)\;. \tag{7.34}$$

Thanks to Theorem 7.9 we know that $-A(m)$ generates an analytic $C_0$-semigroup on $E_0$ with

$$E_1(m) \doteq H^{t_0}_{p,\mathcal{B}}(\Omega) \quad \text{for each} \;\; m \in \mathcal{M}^{t_0}_p(\Omega)\;. \tag{7.35}$$

Given $A \in \mathcal{H}(E_1, E_0)$, $D_A(\theta)$ has been defined in (2.13) as being the continuous interpolation space between $E_0$ and $D(A) \doteq E_1$. Moreover, $D_A(1+\theta)$ has been defined as the domain of definition of the $D_A(\theta)$-realization of $A$. Note that this implies $A \in \mathcal{H}(D_A(1+\theta), D_A(\theta))$, where we use the same notation for $A$ and the $D_A(\theta)$-realization of $A$. Now we are ready to show

**7.11 Theorem** *For $s \in (t_0, 1+1/p)$ let $\theta := (s-t_0)/2$. Then*

*(i)* $D_{A(m)}(\theta) \doteq b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)\;,\;\; D_{A(m)}(1+\theta) \doteq b^{s}_{p,\infty,\mathcal{B}}(\Omega)$ *for each* $m \in \mathcal{M}^s_p(\Omega)$ .

*(ii)* $[\,m \mapsto A(m)\,] \in C^{\omega}\big(\mathcal{M}^s_p(\Omega),\, \mathcal{H}(b^s_{p,\infty,\mathcal{B}}(\Omega)\,,\, b^{s-2}_{p,\infty,\mathcal{B}}(\Omega))\big)$ .

**Proof** It follows from (7.35) that

$$E_1(m) \doteq E_1(m') \quad \text{for} \quad m,\, m' \in \mathcal{M}^s_p(\Omega)\;. \tag{7.36}$$

Hence, it suffices to identify the interpolation spaces $(E_0, E_1(m_0))^0_{(s-t_0)/2,\infty}$ for a particular $m_0 \in \mathcal{M}^s_p(\Omega)$. We choose

$$m_0 := ((a_{jk}(x_0)), 0, 0, 0))\;. \tag{7.37}$$

Then

$$a_{jk}(x_0) \in C^\infty(\bar{\Omega}, \mathcal{L}(\mathbb{R}^N))\;, \quad a_{jk}(x_0)\nu^j \in C^\infty(\partial\Omega, \mathcal{L}(\mathbb{R}^N))$$

and it follows from (7.24) that

$$(\mathcal{A}_\pi(m_0), \mathcal{B}_\pi(m_0)) := (-\partial_j(a_{jk}(x_0)\partial_k\,),\ \delta a_{jk}(x_0)\nu^j\gamma\partial_k + (1-\delta)\gamma) \tag{7.38}$$

is normally elliptic. We set

$$\mathbb{A}_0(m_0) := \mathcal{A}_\pi(m_0)|H^2_{p,\mathcal{B}_\pi(m_0)}\ :\ H^2_{p,\mathcal{B}_\pi(m_0)} \subset L_p(\Omega) \to L_p(\Omega)$$

and

$$\mathbb{E} := \mathbb{E}_0 := L_p(\Omega)\ ,\quad \mathbb{E}_1(m_0) := H^2_{p,\mathcal{B}_\pi(m_0)}\ .$$

We conclude with [**6**], Theorem 5.2 and Corollary 5.3, that $\mathbb{A}_0(m_0) \in \mathcal{H}(\mathbb{E})$. Let

$$\{\,(\mathbb{E}_\beta(m_0), \mathbb{A}_\beta(m_0))\ ;\ -1 \le \beta \le 1\,\}\ , \tag{7.39}$$

be the scale of interpolation and extrapolation spaces, constructed with the complex interpolation method. Thanks to [**6**, Proposition 5.4], we get

$$\mathbb{E}_\beta(m_0) \doteq H^{2\beta}_{p,\mathcal{B}}\ ,\quad 2\beta \in (-2+\frac{1}{p}, 1+\frac{1}{p}),\ 2\beta \notin \mathbb{Z}+\frac{1}{p}\ , \tag{7.40}$$

and in particular

$$E_0 \doteq \mathbb{E}_{t_0/2-1}(m_0),\ E_1(m_0) \doteq \mathbb{E}_{t_0/2}(m_0),\ A(m_0) = \mathbb{A}_{t_0/2-1}(m_0)\,. \tag{7.41}$$

This implies

$$(E_0, E_1(m_0))^0_{\theta,\infty}\ \doteq\ (\mathbb{E}_{t_0/2-1}(m_0), \mathbb{E}_{t_0/2}(m_0))^0_{\theta,\infty} \tag{7.42}$$

for $\theta := (s-t_0)/2$. Now, we will use the reiteration theorem for the continuous interpolation method, cf. [**20**,**9**] or [**35**, Theorem 1.3]. Fix $t_1 \in (1/p, 1+1/p)$ with $t_0 < s < t_1$. Then, due to [**6**, Proposition 5.5]),

$$\mathbb{E}_{t_1/2-1}(m_0) \doteq [\mathbb{E}_{t_0/2-1}(m_0), \mathbb{E}_{t_0/2}(m_0)]_{(t_1-t_0)/2}\ .$$

Hence, $\mathbb{E}_{t_1/2-1}(m_0)$ is an intermediate space of class $(t_1-t_0)/2$ and the reiteration theorem gives

$$(\mathbb{E}_{t_0/2-1}(m_0), \mathbb{E}_{t_0/2}(m_0))^0_{\theta,\infty} \doteq (\mathbb{E}_{t_0/2-1}(m_0), \mathbb{E}_{t_1/2-1}(m_0))^0_{\mu,\infty}\ ,\ \mu := \frac{s-t_0}{t_1-t_0}\,. \tag{7.43}$$

We conclude with (7.40) and (7.28) that $D_{A(m_0)}(\theta) \doteq b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$. Now, it follows from (7.36) and the definition of $D_{A(m)}(\theta)$ that

$$D_{A(m)}(\theta) \doteq b^{s-2}_{p,\infty,\mathcal{B}}(\Omega) \quad \text{for each} \quad m \in \mathcal{M}^s_p(\Omega)\ . \tag{7.44}$$

Since (the $D_{A(m)}(\theta)$-realization of) $A(m)$ generates an analytic strongly continuous semigroup on $D_{A(m)}(\theta)$ as well, we have that $(\omega + A(m)) : D_{A(m)}(1+\theta) \to D_{A(m)}(\theta)$ is an isomorphism for $\omega \in \mathbb{R}$ sufficiently large. On the other hand,

$$(\omega + A(m)) \in Isom\,(b^s_{p,\infty,\mathcal{B}}(\Omega)\,,\ b^{s-2}_{p,\infty,\mathcal{B}}(\Omega))\ ,$$

thanks to Lemma 7.10. This observation together with (7.44) leads to

$$D_{A(m)}(1+\theta) \doteq b^s_{p,\infty,\mathcal{B}}(\Omega) \ .$$

(ii) is now a consequence of (i) and Corollary 7.8. One should observe that $\mathcal{H}(X_1, X_0)$ is an open subset of $\mathcal{L}(X_1, X_0)$ for two arbitrary Banach spaces $X_1 \overset{d}{\hookrightarrow} X_0$. $\square$

**7.12 Remark** Let $m_0$ be given by (7.37) and let

$$\{ (\mathbb{F}_\alpha(m_0), \mathbb{A}_\alpha(m_0)) \ ; \ -1 \le \alpha \le 1 \} \tag{7.45}$$

be the scale of interpolation and extrapolation spaces constructed with the continuous interpolation method, i.e.

$$\mathbb{F}_\alpha(m_0) := \begin{cases} (\mathbb{E}_0, \mathbb{E}_1(m_0))^0_{\alpha,\infty} \ , & 0 < \alpha < 1 \ , \\ (\mathbb{E}_{-1}(m_0), \mathbb{E}_0)^0_{\alpha+1,\infty} \ , & -1 < \alpha < 0 \ , \\ \mathbb{E}_i(m_0) \ , & \alpha = i \in \{-1, 0, 1\} \ . \end{cases} \tag{7.46}$$

Then

$$b^s_{p,\infty,\mathcal{B}}(\Omega) \doteq \mathbb{F}_{s/2}(m_0) \tag{7.47}$$

for $s \in (-2+\frac{1}{p}, 1+\frac{1}{p}) \setminus \{-1+\frac{1}{p}, 0, \frac{1}{p}\}$, where $b^s_{p,\infty,\mathcal{B}}(\Omega) := b^s_{p,\infty}(\Omega)$ for $s \in (-1+1/p, 1/p)$. Indeed, for $s \in (1/p, 1+1/p)$, we obtain from the first part of (7.28) and from (7.40) that

$$b^s_{p,\infty,\mathcal{B}}(\Omega) \doteq (\mathbb{E}_{t_0/2}(m_0), \mathbb{E}_{t_1/2}(m_0))^0_{(s-t_0)/(t_1-t_0),\infty} \ ,$$

where $t_0, t_1 \in (1/p, 1+1/p)$ and $t_0 < s < t_1$. Since

$$E_{t_0/2}(m_0) \doteq [E_0, E_1(m_0)]_{t_0/2} \ , \quad E_{t_1/2}(m_0) \doteq [E_0, E_1(m_0)]_{t_1/2}$$

we conclude with the reiteration theorem for the continuous interpolation method that

$$b^s_{p,\infty,\mathcal{B}}(\Omega) \doteq (\mathbb{E}_0, \mathbb{E}_1(m_0))^0_{s/2,\infty} = \mathbb{F}_{s/2}(m_0) \ .$$

Using similar arguments we get - from the second part of (7.28) - the assertion for $s \in (-2+1/p, -1+1/p)$. Finally, we use (6.30) instead of (7.28) if $s \in (-1+1/p, 1/p)\setminus\{0\}$ and get the assertion on the base of (7.40) and a reiteration argument.

**7.13 Proposition** *Let $1/p < s < 1+1/p$.*
*Set $X_0 := b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$ and $X_1 := b^s_{p,\infty,\mathcal{B}}(\Omega)$. Then*

$$(X_0, X_1)^0_{\mu,\infty} \doteq b^{s+2(\mu-1)}_{p,\infty,\mathcal{B}}(\Omega) \quad \textit{for} \ \ \mu > 1 - s/2 \ .$$

**Proof** Due to (7.47),

$$X_0 \doteq \mathbb{F}_{s/2-1}(m_0)\ , \quad X_1 \doteq \mathbb{F}_{s/2}(m_0)\ . \tag{7.48}$$

We claim that

$$(\mathbb{F}_{-1}, \mathbb{F}_1)_{1/2,1} \hookrightarrow \mathbb{F}_0 \hookrightarrow (\mathbb{F}_{-1}, \mathbb{F}_1)_{1/2,\infty}\ . \tag{7.49}$$

(We write $\mathbb{F}_\alpha := \mathbb{F}_\alpha(m_0)$). This follows as in [**2**, section 8], cf. also [**35**, Theorem 2.2]. Hence, $\mathbb{F}_0 \in \mathcal{C}(1/2; (\mathbb{F}_{-1}, \mathbb{F}_1))$ where $\mathcal{C}(\theta; (\mathbb{F}_{-1}, \mathbb{F}_1))$ denotes the intermediate spaces of class $\theta$ between $\mathbb{F}_1$ and $\mathbb{F}_{-1}$, cf. [**13**, section 3.5]. (7.49) and the reiteration theorem for the continuous interpolation method give

$$\mathbb{F}_{s/2-1} \doteq (\mathbb{F}_{-1}, \mathbb{F}_1)^0_{s/4,\infty} \quad \text{and} \quad \mathbb{F}_{s/2} \doteq (\mathbb{F}_{-1}, \mathbb{F}_1)^0_{1/2+s/4,\infty}\ . \tag{7.50}$$

Due to (7.50) and the reiteration theorem we see that

$$(\mathbb{F}_{s/2-1}, \mathbb{F}_{s/2})^0_{\mu,\infty} \doteq (\mathbb{F}_{-1}, \mathbb{F}_1)^0_{(1-\mu)s/4+\mu(1/2+s/4),\infty} = (\mathbb{F}_{-1}, \mathbb{F}_1)^0_{\mu/2+s/4,\infty}\ . \tag{7.51}$$

Note that $(\mu-1)+s/2>0$ by our assumption. Since $\mathbb{F}_0 \in \mathcal{C}(1/2; (\mathbb{F}_{-1}, \mathbb{F}_1))$, cf. (7.49), and $\mathbb{F}_1 \in \mathcal{C}(1; (\mathbb{F}_{-1}, \mathbb{F}_1))$, we use the reiteration theorem a last time and obtain

$$(\mathbb{F}_{-1}, \mathbb{F}_1)^0_{\mu/2+s/4,\infty} = (\mathbb{F}_0, \mathbb{F}_1)^0_{\mu-1+s/2,\infty}\ . \tag{7.52}$$

Now, the assertion follows from (7.51)-(7.52) and (7.47). □

### 7.14 Remarks

a) The main results of this section are given in Theorem 7.9 and Theorem 7.11 showing that the operators $-A(m)$ generate analytic $C_0$-semigroups on

$$H^{s-2}_{p,\mathcal{B}}(\Omega) \quad \text{and} \quad b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$$

under rather weak regularity assumptions on the coefficients of the 'boundary value problem' $(\mathcal{A}(m), \mathcal{B}(m))$.
We mention that the regularity assumptions on the coefficients in Theorem 7.9 can even be relaxed as is shown in [**9**]. If $s=1$, for example, it suffices to assume that $((a_{jk}),(a_j)) \in C(\bar{\Omega})^{n^2} \times L_p(\Omega)^n$ for $p \in (n,\infty)$.
We would also like to draw attention to the results of Vespri. We refer to [**40**], where additional references are quoted.

b) It should be observed that Theorem 7.11 contains a result on **maximal regularity**. Indeed, this follows immediately from Theorem 2.2. Roughly speaking, normally elliptic boundary problems generate analytic $C_0$-semigroups with the property of maximal regularity on the spaces $b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$.

c) Finally, we mention that Theorem 7.11 gives the analytic dependence of $A(\cdot)$ on the data, i.e on the coefficients. This turns out to be very important in order to show the smooth dependence of the mapping $v \mapsto A(v)$.

## 8. Quasilinear reaction-diffusion systems

We will now return to the **quasilinear reaction-diffusion system**

$$\begin{cases} \partial_t u + \mathcal{A}(u)u = f(\cdot,u) & \text{in} \quad \Omega \times (0,\infty)\ , \\ \mathcal{B}(u)u = g(\cdot,u) & \text{on} \quad \partial\Omega \times (0,\infty)\ , \\ u(0) = u_0 & \text{in} \quad \Omega\ , \end{cases} \tag{8.1}$$

where $\mathcal{A}(u)$ denotes a differential operator of second order and $\mathcal{B}(u)$ denotes a boundary operator as given in (7.6), i.e.

$$\begin{aligned} \mathcal{A}(u) &:= -\partial_j\big(a_{jk}(\cdot,u)\partial_k\ \big) + a_j(\cdot,u)\partial_j + a_0(\cdot,u) \\ \mathcal{B}(u) &:= \delta\big(a_{jk}(\cdot,u)\nu^j\gamma\partial_k\ + b_0(\cdot,u)\big) + (1-\delta)\gamma\ . \end{aligned} \tag{8.2}$$

We will associate with (8.1) a quasilinear evolution equation

$$\dot u + A(u)u = F(u)\ ,\ t>0\ , \quad u(0) = u_0\ ,$$

in appropriate spaces. One of the major difficulties in doing this stems from the nonlinear boundary conditions in (8.1). It forces us to choose the spaces very carefully. To do so we work in the extrapolation setting. Here we would like to refer to [**2**,**3**,**9**]. The advantage of working in the scale of extrapolation spaces has been exploited by Amann in a series of papers, see [**6**,**7**,**8**]. In fact, the advantage of this theory lies in the fact that we can work in rather 'weak spaces' and then use the smoothing property of the 'parabolic' semiflow generated by (1.8).

We will assume from now on that

- $(i)$ $\quad p \in (n,\infty)\ ,\ p \ge 2\ ,$
- $(ii)$ $\quad$ *$G$ is an open neighborhood of 0 in* $\mathbb{R}^N\ ,$
- $(iii)$ $\quad a_{jk}\ ,\ a_j\ ,\ a_0 \in C^\infty(\bar\Omega \times G\,,\,\mathcal{L}(\mathbb{R}^N))\,,\ b_0 \in C^\infty(\partial\Omega \times G\,,\,\mathcal{L}(\mathbb{R}^N))\ ,$
  $f \in C^\infty(\bar\Omega \times G\,,\,\mathbb{R}^N)\,,\ g \in C^\infty(\partial\Omega \times G\,,\,\mathbb{R}^N)\,,\ (1-\delta)\,g = 0\ ,$
- $(iv)$ $\quad n/p < r < 1 < s_0 < 1 + 1/p\ , \quad s_0 - 1 < \rho < (r - n/p) \wedge 1/p\ .$

Setting

$$U_r := \{\, u \in H^r_{p,\mathcal{B}}(\Omega)\ ;\ u(\bar\Omega) \subset G \,\}\ , \tag{8.3}$$

it follows from $r > n/p$ and Sobolev's embedding theorem (and the fact that $\bar\Omega$ is compact) that

$$U_r \quad \text{is an open neighborhood of 0 in} \ \ H^r_{p,\mathcal{B}}(\Omega)\ . \tag{8.4}$$

We now consider the mapping

$$\begin{aligned} &\hat m\ :\ U_r \longrightarrow \mathbb{M}^\rho_p(\Omega)\ , \\ &v \mapsto \hat m(v) := \big((a_{jk}(\cdot,v)),(a_j(\cdot,v)),a_0(\cdot,v),b_0(\cdot,v)\big)\ , \end{aligned} \tag{8.5}$$

where the set $\mathbb{M}_p^\rho(\Omega)$ is introduced in (7.9).

**8.1 Proposition** $\qquad \hat m \in C^\infty(U_r\,,\,\mathbb{M}_p^\rho(\Omega))\ .$

**Proof** Since $H^r_{p,\mathcal{B}}(\Omega) \hookrightarrow C^{r-n/p}(\bar\Omega)$, the assertion can be proven by studying the substitution operators

$$\Phi\,:\,C^{r-n/p}(\bar\Omega,\mathbb{R}^N) \to C^\rho(\bar\Omega,\mathcal{L}(\mathbb{R}^N))\ ,$$

induced by the coefficients $a_{jk}$, $a_j$, $a_0$, $b_0$ , $1 \le j,k \le n$ , i.e. $\Phi(v)(x) := a(x,v(x))$ for $v \in U_r$ and $a \in C^\infty(\bar\Omega \times G, \mathcal{L}(\mathbb{R}^N))$ or $a \in C^\infty(\partial\Omega \times G, \mathcal{L}(\mathbb{R}^N))$. For more details we refer to [**6**] and [**18**, chapter 15]. $\square$

We can now define the formal 'boundary value system' $(\mathcal{A}(v),\mathcal{B}(v))$ for each $v \in U_r$ by

$$\begin{aligned}\mathcal{A}(v) &:= \mathcal{A}(\hat m(v)) := -\partial_j\big(a_{jk}(\cdot,v)\partial_k\ \big) + a_j(\cdot,v)\partial_j + a_0(\cdot,v)\\ \mathcal{B}(v) &:= \mathcal{B}(\hat m(v)) := \delta\big(a_{jk}(\cdot,v)\nu^j\gamma\partial_k\ + b_0(\cdot,v)\big) + (1-\delta)\gamma\ .\end{aligned} \tag{8.6}$$

We require that

$$\begin{aligned}&\big(\,\mathcal{A}(\hat m(v)),\mathcal{B}(\hat m(v))\,\big)\ ,\ \ \big(\,\mathcal{A}_\pi(\hat m(v)(x_0)),\mathcal{B}_\pi(\hat m(v)(x_0))\,\big)\ ,\\ &\textit{are normally elliptic for each } v \in U_r \quad (\textit{ and each } x_0 \in \bar\Omega\ )\ .\end{aligned} \tag{8.7}$$

This requirement is certainly satisfied if the coefficients

$$\begin{aligned}&\big((a_{jk}(\cdot,\eta)),\ \delta(a_{jk}(\cdot,\eta)\nu^j) + (1-\delta)\big)\\ &\big((a_{jk}(x_0,\eta)),\ \delta(a_{jk}(x_0,\eta)\nu^j) + (1-\delta)\big)\end{aligned} \tag{8.8}$$

define a **normally elliptic boundary value problem** for each $\eta \in G$ and each $x_0 \in \bar\Omega$. Then, (8.5) and Proposition 8.1 imply (cf. (7.24))

**Corollary 8.2** $\qquad \hat m \in C^\infty(U_r\,,\,\mathcal{M}_p^\rho(\Omega))\ .$

We fix $1 < t_0 < s_0 < 1 + 1/p$ and set

$$E_1 := H^{t_0}_{p,\mathcal{B}}(\Omega)\ ,\ E_0 := H^{t_0-2}_{p,\mathcal{B}}(\Omega)\ , \tag{8.9}$$

$$X_1 := b^{s_0}_{p,\infty,\mathcal{B}}(\Omega)\ ,\ X_0 := b^{s_0-2}_{p,\infty,\mathcal{B}}(\Omega)\ . \tag{8.10}$$

**8.3 Proposition** *Assume that* (8.7) *holds. Then*

$$[v \mapsto A(v)] \in C^\infty(U_r\,,\,\mathcal{H}(X_1,X_0))\ , \tag{8.11}$$

*and moreover,*

$$A(v) \in \mathcal{H}(E_1, E_0) \quad \textit{for each} \quad v \in U_r \ , \tag{8.12}$$

*where* $A(v) := A(\hat{m}(v))$ *is given by* $< w, A(v)u >= a(v)(w,u)$ *for* $(w,u) \in X_0' \times X_1$ *resp.* $(w,u) \in E_0' \times E_1$ *and* $a(v) := a(\hat{m}(v))$ *denotes the Dirichlet form of the boundary value problem* $(\mathcal{A}(v), \mathcal{B}(v))$. *Note that the duality pairings are induced by* (6.28).

**Proof** An inspection of (7.27) shows that

$$b^{s_0-2}_{p,\infty,\mathcal{B}}(\Omega) \doteq (H^{t_0-2}_{p,\mathcal{B}}(\Omega)\,,\, H^{t_1-2}_{p,\mathcal{B}}(\Omega))^0_{(s-t_0)/(t_1-t_0),\infty} \ .$$

It then follows from the duality theorem for the continuous interpolation method, cf. [**37**, Theorem 1.11.2 (3b)], and (7.29) that $X_0' \doteq B^{2-s_0}_{p',1,\mathcal{B}}(\Omega)$. Hence, the Drichlet form is well-defined for $(w,u) \in X_0' \times X_1$ , cf. Corollary 7.3. We consider the mapping

$$U_r \to \mathcal{M}^\rho_p(\Omega) \to \mathcal{H}(X_1, X_0) \ , \ v \mapsto \hat{m}(v) \mapsto A(\hat{m}(v)) \ . \tag{8.13}$$

It follows from Theorem 7.11 that (8.13) is well-defined, since $\hat{m}(v) \in \mathcal{M}^\rho_p(\Omega)$ and $\rho > (s_0 - 1)$ (due to the assumption (iv)). Theorem 7.11, Corollary 8.2 and (8.13) immediately give the assertion in (8.11). Moreover, $\rho > (t_0 - 1)$ and we can infer from Theorem (7.9) that (8.12) holds as well. Although $A(m)$ acts as a linear operator in different spaces, we use the same notation. This is justified by the fact that $A(m)$ is the realization of the same bilinear form in different spaces. The remaining statements follow from section 7. $\square$

We now focus our attention on the functions $f, g$ in (8.1).

**8.4 Lemma** *Let*

$$F(v) := f(\cdot, v) + \gamma' g(\cdot, v) \ , \quad v \in U_r \ , \tag{8.14}$$

*where* $\gamma'$ *denotes the dual of the trace operator*

$$\gamma \in \mathcal{L}(H^\sigma_{p',\mathcal{B}}(\Omega)\,,\, B^{\sigma-1/p'}_{p',p'}(\partial\Omega)) \ , \quad 1 - 1/p < \sigma < 2 - t_0 \ . \tag{8.15}$$

*Then*

$$[\, v \mapsto F(v)\,] \in C^\infty(U_r\,,\, X_0) \ . \tag{8.16}$$

**Proof** We may consider the mapping

$$H^r_{p,\mathcal{B}}(\Omega) \overset{i}{\hookrightarrow} C(\bar{\Omega}) \overset{f(\cdot,v)}{\longrightarrow} C(\bar{\Omega}) \hookrightarrow L_p(\Omega) \hookrightarrow X_0 \ .$$

Here, the first inclusion follows from Sobolev's embedding theorem and the last embedding is a consequence of (7.46)-(7.47). It is well-known that the (substitution) mapping

$$[\, i\,(U_r)\,] \subset C(\bar{\Omega}) \to C(\bar{\Omega}) \ , \quad v \mapsto f(\cdot, v) \ ,$$

defined on the open subset $[\,i\,(U_r)] \subset C(\bar\Omega)$, is $C^\infty$, if $f$ satisfies the assumption (iii). Now the first part of the statement follows from our diagram. For the remaining part, we consider

$$H_p^r(\Omega) \overset{i}{\hookrightarrow} C(\bar\Omega) \overset{\gamma}{\to} C(\partial\Omega) \overset{g(\cdot,v)}{\longrightarrow} C(\partial\Omega) \hookrightarrow L_p(\partial\Omega) \hookrightarrow B_{p,p}^{1-\sigma-1/p}(\partial\Omega) \; .$$

Note that the mapping

$$[\,\gamma \circ i\,(U_r)] \subset C(\partial\Omega) \to C(\partial\Omega) \; , \quad v \mapsto g(\cdot, v)$$

is $C^\infty$. If follows from (8.15) that $1-\sigma-1/p<0$ such that $L_p(\partial\Omega) \hookrightarrow B_{p,p}^{1-\sigma-1/p}(\partial\Omega)$ indeed holds. Hence,

$$[\, v \mapsto \gamma' g(\cdot, v)] \in C^\infty(U_r\, , \, H_{p,\mathcal{B}}^{-\sigma}(\Omega)) \; ,$$

where $H_{p,\mathcal{B}}^{-\sigma}(\Omega) = (H_{p,\mathcal{B}}^{\sigma}(\Omega))'$, due to the definition in (7.20). We finally use

$$H_{p,\mathcal{B}}^{-\sigma}(\Omega) \hookrightarrow H_{p,\mathcal{B}}^{s_0-2}(\Omega) \hookrightarrow b_{p,\infty,\mathcal{B}}^{s_0-2}(\Omega) = X_0 \; ,$$

where the last inclusion is shown in (7.21). This proves the assertion in (8.16). $\square$

**8.5 Remark** We have succeed to obtain an abstract evolution equation

$$\dot u + A(u)u = F(u) \; , \; t>0, \quad u(0) = u_0 \tag{8.17}$$

in the Banach space $X_0$. Thanks to Proposition 8.3 and Lemma 8.4, we have

$$(A,F) \in C^k(U_r, \mathcal{H}(X_1, X_0) \times X_0) \quad \text{for each} \;\; k \in \mathbb{N}^* \; , \tag{8.18}$$

and

$$A(v) \in \mathcal{M}_\alpha(X_1, X_0) \quad \text{for each} \;\; v \in U_r \; . \tag{8.19}$$

Indeed, (8.19) follows from Theorem 2.2 and Theorem 7.11, cf. Remark 7.15 b). (Note that $\tilde A(v) := A(v) \in \mathcal{H}(E_1, E_0)$ is an extension of $A(v) \in (X_1, X_0)$, since these operators are induced by be the (same) Dirichlet form). (8.19) is the statement on maximal regularity we were looking for. The only thing which seems to be a little mystery is, how (8.17) reflects the quasilinear reaction-diffusion system (8.1). With the requirement that $a(v)(w,u) = < w, A(v)u >$ for $(w,u) \in X_0' \times X_1$ and the definition of $F$ in (8.14) we ensure that solutions of (8.17) are weak solutions of (8.1). Using the smoothing property of the quasilinear parabolic problem (8.17), it can be shown that solutions are in fact much more regular. We work in weak spaces in order to get, temporarily, rid of the nonlinear boundary conditions. Of course, they are always present and in some sense hidden in the spaces. The key observation is that weak solutions satisfy the boundary conditions as soon as they are regular enough, say as soon they belong to $H_p^2(\Omega)$. This idea was used by Amann in [**6**] to prove the results quoted in (1.6)-(1.8) (cf. also [**7**]).

We suppose that

$$(f(\cdot,0)\,,\ g(\cdot,0)) = (0,0)\ , \tag{8.20}$$

such that $u = 0$ is a solution of the reaction-diffusion system (8.1).

We can now state our main result. It reads as follows

**8.6 Theorem**

a) *Given any initial value* $u_0 \in U := U_r \cap H^1_{p,\mathcal{B}}(\Omega)$ *there exists a unique maximal classical solution*

$$u(\cdot,u_0) \in C([0,t^+(u_0)),U) \cap C^\infty(\bar{\Omega} \times (0,t^+(u_0),\mathbb{R}^N) \tag{8.21}$$

*to the quasilinear Cauchy problem* (8.1) *and the map*

$$(t,u_0) \mapsto u(t,u_0) \tag{8.22}$$

*defines a smooth semiflow on* $U$ *with* 0 *being an equilibrium.*

b) *Let the assumptions in* (1.10)$-$(1.11) *and* (8.20) *be satisfied and let* $X^c$ *be the center space introduced in Theorem* 1.1. *Then, for any* $k \in \mathbb{N}^*$, *there exists a mapping*

$$\sigma = \sigma_k \in BC^k(X^c, H^1_{p,\mathcal{B}}(\Omega))\ , \tag{8.23}$$

*such that the* $C^k$*-manifold*

$$\mathcal{M}^c := graph\,(\sigma) \subset H^1_{p,\mathcal{B}}(\Omega) \tag{8.24}$$

*is locally invariant for those solutions of the quasilinear reaction-diffusion system*

$$\begin{aligned} \partial_t u + \mathcal{A}(u)u &= f(\cdot,u) \quad \textit{in} \quad \Omega \times (0,\infty)\ , \\ \mathcal{B}(u)u &= g(\cdot,u) \quad \textit{on} \quad \partial\Omega \times (0,\infty)\ , \end{aligned} \tag{8.25}$$

*which remain in a sufficiently small neighborhood,* $\mathcal{U}(k)$, *of* 0 *in* $U$.

*Moreover*

$$\sigma(0) = 0\ , \quad \partial\sigma(0) = 0\ . \tag{8.26}$$

b) $\mathcal{M}^c$ *is exponentially attractive in* $H^1_{p,\mathcal{B}}(\Omega)$ *for all small solutions* $u(\cdot,u_0)$.

**Proof** The proof of a) is given in [**6**], cf. also [**7**]. We now consider the quasilinear parabolic evolution equation (8.17). We show first that (8.17) satisfies all of the assumptions we stated in section 3 and 4. We fix

$$\alpha := 1 - (s_0 - 1)/2 \quad \text{and} \quad \beta \in (1 - (s_0 - r)/2\,,\ \alpha)\ . \tag{8.27}$$

We then have for the continuous interpolation spaces

$$X_\alpha = b^1_{p,\infty,\mathcal{B}}(\Omega) \quad \text{and} \quad X_\beta = b^{s_0+2(\beta-1)}_{p,\infty,\mathcal{B}}(\Omega) \; , \tag{8.28}$$

thanks to Proposition 7.13. We infer from (6.25),(7.16), from (7.20) and (8.27) that $X_\beta \hookrightarrow H^r_{p,\mathcal{B}}(\Omega)$. Hence, $U_r \cap X_\beta =: U_\beta$ is an open subset of $X_\beta$. Let $U_\alpha \subset X_\alpha$ be the open subset $U_\alpha := U_\beta \cap X_\alpha$. Due to $U_\beta \hookrightarrow U_r$ we infer from (8.18) that

$$(A, F) \in C^\infty(U_\beta \, , \, \mathcal{H}(X_1, X_0) \times X_0) \; . \tag{8.29}$$

Moreover, $A(v) \in \mathcal{M}_\alpha(X_1, X_0)$ for each $v \in U_\alpha$ by (8.19). We can now conclude that the assumptions (3.4)-(3.9) and the assumptions (i)-(iii) of section 4 are satisfied. Due to (8.9), (7.40)-(7.41), [**6**, Proposition 5.5] and (7.22) we obtain

$$[E_0, E_1]_\delta \doteq H^{t_0-2+2\delta}_{p,\mathcal{B}} \hookrightarrow X_\beta \quad \text{for} \quad \delta > (s_0 - t_0)/2 + \beta \; ,$$

which gives assumption (iv) of section 8. Set

$$L := A(0) - \partial F(0) \; . \tag{8.30}$$

It is not difficult to see that the spectrum of $-L$, $\sigma(-L)$, coincides with the eigenvalues of the linearized elliptic problem (1.10) and hence admit a decomposition $\sigma(-L) = \sigma_c \cup \sigma_s$ with the properties (1.11). If $\pi^c$ denotes the spectral projection with respect to the spectral set $\sigma_c$ it follows that

$$\pi^c(X_0) \doteq X^c := \oplus_{\mu_j \in \sigma_c} N(\mu_j) \; , \quad N(\mu_j) \;\; \textit{the algebraic eigenspace of} \;\; \mu_j \; . \tag{8.31}$$

Theorem 4.1 now gives, for each $k \in \mathbb{N}^*$, the existence of a mapping $\sigma \in BC^k(X^c, X_1)$, such that

$$\mathcal{M}^c := \operatorname{graph}(\sigma) \subset X_1 \hookrightarrow H^1_{p,\mathcal{B}}(\Omega)$$

is locally invariant for small solutions of the quasilinear parabolic evolution equation (8.17). Here we should observe that solutions of (8.17) are in fact classical solutions of the quasilinear reaction-diffusion system (8.25). In fact, each solution $u(\cdot, u_0)$ of (8.17) is also a solution of the same evolution equation considered in the spaces $(E_0, E_1)$. It follows from the considerations in [**6**, section 9] that $u(\cdot, u_0)$ is the unique maximal classical solution of (8.25) satisfying $u(0, u_0) = u_0$. This proves b).

Finally, note that $H^1_{p,\mathcal{B}}(\Omega)$ is 'sandwiched' by the spaces $X_1$ and $X_\alpha$. Indeed,

$$b^{s_0}_{p,\infty,\mathcal{B}}(\Omega) \hookrightarrow H^1_{p,\mathcal{B}}(\Omega) \hookrightarrow b^1_{p,\infty,\mathcal{B}}(\Omega)$$

by (6.25),(7.16) and (7.20). The last assertion now follows from Theorem 5.8 and Remark 5.9 b). $\square$

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