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# INVARIANT MANIFOLDS AND BIFURCATION FOR QUASILINEAR REACTION-DIFFUSION SYSTEMS

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## 1. INTRODUCTION

IN THIS PAPER we consider parameter-dependent quasilinear reaction–diffusion equations on a bounded domain of $\mathbb{R}^n$. It is our purpose to study the dynamical behavior of solutions to these equations. In particular, we are interested in the behavior of the solution set if the parameter is changed. To be more precise, let us assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain having a smooth boundary. Then we consider the parameter-dependent equation

$$\begin{cases} \partial_t u + \mathcal{A}(\lambda, u)u = f(\lambda, \cdot, u) & \text{in } \Omega \times (0, \infty), \\ \mathcal{B}(\lambda, u)u = g(\lambda, \cdot, u) & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.1$_\lambda$}$$

Here, $u = (u^1, \ldots, u^N)$ is a $\mathbb{R}^N$-valued function and $\lambda$ varies in an open subset $\Lambda$ of some finite dimensional Banach space, say of $\mathbb{R}$. Moreover,

$$(\mathcal{A}(\lambda, u), \mathcal{B}(\lambda, u)) \tag{1.2}$$

denotes a very general boundary value system, where $\mathcal{A}(\lambda, u)$ stands for a system of second order quasilinear differential operators and $\mathcal{B}(\lambda, u)$ denotes a system of quasilinear first order boundary operators. Being more specific, we consider the general second order differential operator

$$\mathcal{A}(\lambda, u)u := \partial_j(a_{jk}(\lambda, \cdot, u)\partial_k u) + a_j(\lambda, \cdot, u)\partial_j u + a_0(\lambda, \cdot, u)u. \tag{1.3}$$

Since we allow $u$ to be a vector valued function, the coefficients take values in the space $\mathcal{L}(\mathbb{R}^N)$ of all (real) $N \times N$-matrices. Note that we use the summation convention, where $j$ and $k$ run from 1 to $n$. Moreover, the dot stands for the space variable. Throughout, we assume the coefficients to be smooth functions of the variables, i.e.

$$a_{jk}, a_j, a_0 \in C^\infty(\Lambda \times \bar{\Omega} \times G, \mathcal{L}(\mathbb{R}^N)), \tag{1.4}$$

where $G$ denotes an open subset in $\mathbb{R}^N$. $\mathcal{B}(\lambda, u)$ then stands for the boundary operator

$$\mathcal{B}(\lambda, u)u := \delta(a_{jk}(\lambda, \cdot, u)\nu^j\gamma\partial_k u + b_0(\lambda, \cdot, u)\gamma u) + (1 - \delta)\gamma u. \tag{1.5}$$

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Here, $\gamma$ is the trace operator and $\nu = (\nu^1, \ldots, \nu^n)$ denotes the outer unit field on $\partial\Omega$. Moreover, for

$$\delta := \mathrm{diag}[\delta^1, \ldots, \delta^N]: \partial\Omega \to \mathcal{L}(\mathbb{R}^N),$$

the boundary characterization map, we assume that $\delta^r \in C(\partial\Omega, \{0, 1\})$, $1 \le r \le N$. Hence, $\delta^r$ assigns to each component of connectedness of $\partial\Omega$ one of the values $\{0, 1\}$. Thus, the function $\delta$ characterizes whether we have "Neumann type" or Dirichlet boundary conditions on a given component of $\partial\Omega$. We assume that the function $b_0$ depends smoothly on the indicated variables. In addition, let

$$f \in C^\infty(\Lambda \times \bar{\Omega} \times G, \mathbb{R}^N), \qquad g \in C^\infty(\Lambda \times \partial\Omega \times G, \mathbb{R}^N). \tag{1.6}$$

In order to have a powerful theory we shall impose an ellipticity and complementing condition upon the boundary value systems $(\mathcal{A}(\lambda, u), \mathcal{B}(\lambda, u))$. We will require that

$$(\mathcal{A}(\lambda, u), \mathcal{B}(\lambda, u)) \text{ are normally elliptic for } (\lambda, u) \in \Lambda \times C(\bar{\Omega}, G). \tag{1.7}$$

We refer to [1], where the definition of normally elliptic boundary value problems is introduced. It should be noted that this definition weakens the ellipticity conditions usually imposed on systems. Moreover, the concept of normally elliptic boundary value systems is in a certain sense optimal, cf. [1, theorem 2.4; 2, theorem 4.1, remark 4.2c].

Note that $(1.1)_\lambda$ is a system of parameter-dependent, strongly coupled reaction–diffusion equations subject to nonlinear boundary conditions.

To give a simple example where our assumptions are met, we consider the special case $a_{jk}(\lambda, \cdot, u) = a(\lambda, \cdot, u)\delta_{jk}$ and $a_j = a_0 = b_0 = 0$ for $1 \le j, k \le n$, where $\delta_{jk}$ is the Kronecker symbol. Also, let $\delta^1 = \cdots = \delta^N = 1$ for the boundary characterization map. Then the quasilinear reaction–diffusion system $(1.1)_\lambda$ takes the form

$$\begin{cases} \partial_t u - \partial_j(a(\lambda, \cdot, u)\partial_j u) = f(\lambda, \cdot, u) & \text{in } \Omega \times (0, \infty), \\ a(\lambda, \cdot, u)\nu^j \gamma \partial u = g(\lambda, \cdot, u) & \text{on } \partial\Omega \times (0, \infty), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.8$_\lambda$}$$

We assume that

$$\sigma(a(\lambda, x, \eta)) \subset [\mathrm{Re}\, z > 0], \qquad (\lambda, x, \eta) \in \Lambda \times \bar{\Omega} \times G, \tag{1.9}$$

where $\sigma(a(\lambda, x, \eta))$ denotes the spectrum of the $N \times N$-matrix $a(\lambda, x, \eta)$. Then

$$(-\partial_j(a(\lambda, \cdot, u)\partial_j), a(\lambda, \cdot, u)\nu^j\gamma\partial_j)$$

defines a normally elliptic boundary value system, see [1, theorem 4.4]. If $N = 2$ and the matrix $a$ is given by

$$a(\lambda, x, \eta) = \begin{bmatrix} a_{11}(\lambda, x, \eta) & a_{12}(\lambda, x, \eta) \\ -a_{21}(\lambda, x, \eta) & 0 \end{bmatrix}, \tag{1.10}$$

condition (1.9) is satisfied whenever

$$a_{11}(\lambda, x, \eta) > 0, \qquad a_{12}(\lambda, x, \eta)a_{21}(\lambda, x, \eta) > 0, \qquad (\lambda, x, \eta) \in \Lambda \times \bar{\Omega} \times G. \tag{1.11}$$

First of all, we are interested in whether the quasilinear reaction–diffusion system $(1.1)_\lambda$ does have a unique solution for a given initial value $u_0$. After this very first question, we are concerned with the dynamical behavior of solutions. Thanks to the very general results proven in [1], we already know that the quasilinear reaction–diffusion system $(1.1)_\lambda$ generates a semiflow on an open subset of an appropriate function space. Note that this property stands at the beginning of a thorough development of a geometric (or dynamic) theory for quasilinear equations which contain $(1.1)_\lambda$ as a prominent and important prototype. In fact, let $p > n$ and set

$$H^1_{p,\mathcal{B}} := \{u \in H^1_p(\Omega, \mathbb{R}^N);\ (1-\delta)\gamma u = 0\}, \qquad V := \{u \in H^1_{p,\mathcal{B}};\ u(\bar{\Omega}) \subset G\}. \tag{1.12}$$

Owing to the trace theorem, $H^1_{p,\mathcal{B}}$ is a closed subspace of the Sobolev space $H^1_p(\Omega, \mathbb{R}^N)$. Moreover, it follows from Sobolev's embedding theorem and the fact that $\bar{\Omega}$ is compact that $V$ is a well-defined open subset of $H^1_{p,\mathcal{B}}$. If we assume that $(1-\delta)g = 0$, the following result is proven in [1, 3].

Given any initial value $u_0$, the parameter-dependent quasilinear reaction–diffusion system $(1.1)_\lambda$ has a unique maximal classical solution

$$u(\cdot, u_0, \lambda) \in C([0, t^+(u_0, \lambda)], V) \cap C^\infty(\bar{\Omega} \times (0, t^+(u_0, \lambda)), \mathbb{R}^N) \tag{1.13}$$

satisfying $u(0, u_0, \lambda) = u_0$. The mapping

$$(t, u_0) \mapsto u(t, u_0, \lambda) \tag{1.14}$$

defines a smooth semiflow on $V$ depending smoothly on $\lambda \in \Lambda$.

Important further questions are related to the existence of bifurcating solutions such as steady states and periodic solutions emerging from an equilibrium. We will now assume that $0 \in G$ and

$$(f(\cdot, 0, \lambda), g(\cdot, 0, \lambda)) = (0, 0), \qquad \lambda \in \Lambda. \tag{1.15}$$

Then 0, i.e. the solution $u \equiv 0$, is an equilibrium for the quasilinear system $(1.1)_\lambda$ and we may consider the linearization at this point. Thus, we consider the elliptic eigenvalue problem

$$\begin{aligned} &[-\mathcal{A}(\lambda, 0) + \partial_3 f(\lambda, \cdot, 0)]v = \mu(\lambda)v && \text{in } \Omega, \\ &[-\mathcal{B}(\lambda, 0) + \partial_3 g(\lambda, \cdot, 0)]v = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.16)$_\lambda$}$$

Observe that $(1.16)_\lambda$ is well-posed since the $L_p$-realization has a compact resolvent. Then let

$$\{\mu_k(\lambda);\ k \in \mathbb{N}\}, \qquad \lambda \in \Lambda, \qquad (1.17)_\lambda$$

denote the set of eigenvalues of $(1.16)_\lambda$, each one counted according to its multiplicity. If we assume for the first moment that

$$\{\mu_k(\lambda);\ k \in \mathbb{N}\} \subset [\operatorname{Re} z < 0], \qquad \lambda \in \Lambda,$$

then we infer from the principle of linearized stability that 0 is an asymptotically stable critical point for the semiflow induced by $(1.1)_\lambda$. For a proof, see [4, 5]. Indeed, in the case that there are no eigenvalues on the imaginary axis, the results in the latter paper imply in particular that 0 is exponentially stable. Hence, the dynamical behavior of small solutions to the parameter-dependent equation $(1.1)_\lambda$ remains essentially the same if $\lambda$ is changed. But there is a drastic change if some of the eigenvalues hit or cross the imaginary axis if $\lambda$ is varied. This situation occurs in the case of Hopf bifurcation, for example, where a pair of conjugate eigenvalues crosses the imaginary axis, causing bifurcation of periodic solutions. It is this situation we would like to investigate in detail.

We first establish the existence and attractivity of parameter-dependent, locally invariant manifolds, $\mathfrak{M}^c(\lambda)$, for the quasilinear reaction–diffusion system $(1.1)_\lambda$. Using invariant (center) manifolds is a well-known approach to study (local) bifurcation of small solutions, such as periodic solutions or steady states. The method of invariant manifolds is common in dynamical systems and in ordinary differential equations and has also been used in the context of semilinear evolution equations, cf. [6–8]. However, not much is done in the case of quasilinear equations. In fact, we shall use the results in [5] where the existence and attractivity of center manifolds in the case of quasilinear equations has been established for the first time. To be more precise, we were able to show that center manifolds exist and attract solutions in the topology of $V$. (Note that $V$ is a natural phase space for $(1.1)_\lambda$, due to (1.13) and (1.14).) In order to prove this, we had to use maximal regularity results, cf. [9, 10]. But while maximal regularity results are only obtained in some very special spaces (the continuous interpolation spaces), we were able to overcome this restriction, cf. [5]. A slight modification of the results in [5] will cover the parameter-dependent case. To be more specific, we suppose that there exists $\lambda_0$ in $\Lambda$ such that

$$\sigma_c := \{\mu_1(\lambda_0), \ldots, \mu_J(\lambda_0)\} \subset i\mathbb{R}, \qquad \sigma_s := \{\mu_j(\lambda_0); j > J\} \subset [\operatorname{Re} z < 0]. \tag{1.18}$$

Let

$$X^c := \oplus_{1 \le j \le J} N(\mu_j(\lambda_0)), \tag{1.19}$$

be the "center space", where $N(\mu_j(\lambda_0))$ denotes the algebraic eigenspace of the eigenvalue $\mu_j(\lambda_0)$. Note that each $N(\mu_j(\lambda_0))$ is finite dimensional since $(1.16)_\lambda$ has a compact resolvent.

With this we can state the following theorem.

THEOREM 1.1. Let $k \in \mathbb{N}^*$ be given. Then the parameter-dependent quasilinear reaction–diffusion system

$$\begin{aligned} \partial_t u + \mathcal{A}(\lambda, u)u &= f(\lambda, \cdot, u) && \text{in } \Omega \times (0, \infty), \\ \mathcal{B}(\lambda, u)u &= g(\lambda, \cdot, u) && \text{on } \partial\Omega \times (0, \infty), \end{aligned} \tag{1.20$_\lambda$}$$

has a parameter-dependent, finite dimensional, locally invariant $C^k$-manifold $\mathfrak{M}^c(\lambda) \subset V$, for $\lambda$ in a sufficiently small neighborhood $\Lambda = \Lambda(k)$ of $\lambda_0$. More precisely: there exists a mapping

$$\sigma = \sigma_k \in BC^k(\Lambda \times X^c, V)$$

with

$$\sigma(\lambda, 0) = 0 \qquad \text{for } \lambda \in \Lambda,\ \partial\sigma(0,0) = 0,$$

such that

$$\mathfrak{M}^c(\lambda) := \operatorname{graph} \sigma(\lambda, \cdot)$$

is invariant for small solutions of $(1.20)_\lambda$. The invariant manifolds $\mathfrak{M}^c(\lambda)$ attract small solutions at an exponential rate.

It is shown that the invariant (center) manifolds $\mathfrak{M}^c(\lambda)$ contain all the local recurrence. This allows the reduction of $(1.20)_\lambda$ to a finite dimensional system of differential equations. Using results on bifurcation for ordinary differential equations, we are able to study the problem of bifurcating solutions for the quasilinear reaction–diffusion equation $(1.1)_\lambda$. We focus our attention on Hopf bifurcation and give conditions, whether the bifurcating orbits are stable or unstable.

THEOREM 1.2. Let $\Lambda = (-\lambda_0, \lambda_0)$ for some $\lambda_0 > 0$ and assume that $\{\pm i\omega_0\}$ are simple eigenvalues of the eigenvalue problem $(1.16)_0$, where $\omega_0 > 0$, and all of the remaining eigenvalues are contained in $[\operatorname{Re} z < 0]$.

Let $\mu(\lambda)$ be the unique continuation of the eigenvalue $i\omega_0$ in a neighborhood of $\lambda = 0$ and suppose that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\operatorname{Re}\mu(\lambda))|_{\lambda=0} > 0. \tag{1.21}$$

Then there exists a unique one-parameter family $\{\Gamma(s); 0 < s < \varepsilon\}$ of nontrivial periodic orbits of the system $(1.1)_\lambda$ in a neighborhood of $(0, 0) \in \Lambda \times V$.

More precisely, there exists $\varepsilon > 0$ and a map

$$(\lambda(\cdot), T(\cdot), u(\cdot)) \in C^k((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times V)$$

with

$$(\lambda(0), T(0), u(0)) = (0, 2\pi/\omega_0, 0)$$

and such that

$$\Gamma(s) := \Gamma(u(s))$$

is a nontrivial $T(s)$-periodic orbit of the quasilinear reaction–diffusion system $(1.1)_{\lambda(s)}$ passing through $u(s) \in V$. If $0 < s_1 < s_2 < \varepsilon$, then $\Gamma(s_1) \neq \Gamma(s_2)$.

Moreover, the family $\{\Gamma(s); 0 < s < \varepsilon\}$ contains every nontrivial periodic orbit of $(1.1)_\lambda$ lying in a neighborhood of $(0, T(0), 0) \in \Lambda \times \mathbb{R} \times V$.

If

$$s\dot{\lambda}(s) > 0 \qquad \text{for } s > 0 \qquad (\text{``supercritical bifurcation''}),$$

then each orbit $\Gamma(s)$ is asymptotically stable in $V$.

In the case of

$$s\dot{\lambda}(s) < 0 \qquad \text{for } s > 0 \qquad (\text{``subcritical bifurcation''}),$$

each of the orbits $\Gamma(s)$, $0 < s < \varepsilon$, is unstable in $V$.

It is one of the main purposes of this paper to derive stability conditions for the bifurcating periodic orbits. In fact, we will give an algorithm which enables the determination of whether the *supercritical* or *subcritical* case occurs. This algorithm works in the (infinite dimensional) case of quasilinear reaction–diffusion systems. It involves derivatives up to the third order and the knowledge of the linear semigroup on the stable subspace, cf. Section 5. In some special cases, see remark 5.5(b), the stability analysis is rather simple. An example, based on the quasilinear reaction–diffusion system $(1.8)_\lambda$, will be given elsewhere.

The derivation of our algorithm is self-contained and it uses corresponding results for two-dimensional ordinary differential equations. We should mention that a similar algorithm has been described in [6]. However, there are some differences between our derivation and theirs. First, we have the feeling that our computation is much simpler than the one given in [6] (cf. [6, remark on p. 125]). Second, the results in proposition 5.4 are new and in some sense naturally related to the problem. Our results apply to ordinary differential equations, semilinear evolution equations and, of course, to quasilinear evolution equations. They use the existence of invariant manifolds and the stability result given in Section 3, which, in turn, is a consequence of the fact that the invariant manifolds attract solutions at an exponential rate. We should mention that the authors of [6] have included a proof for the existence of center manifolds, which only uses that a given evolution equation generates a smooth semiflow on a Banach space (which is supposed to admit a $C^\infty$-norm) (see [6, theorem 2.7]). But to our knowledge, this proof is only valid for the case of flows, and the extension to semiflows is incorrect [11, p. 36].

In the context of ordinary differential equations and semilinear evolution equations, Hopf bifurcation has been widely studied in recent years, see for instance the monographs [6, 7, 12–14] and also [15–18], to mention a few. Extensions to more general nonlinear equations have been given in [19, 20]. In fact, there are quite different approaches to the study of Hopf bifurcation. Instead of using reduction via invariant manifolds, one can also use the so-called Ljapunov–Schmidt reduction. This has been done in [19], where the existence of bifurcating solutions for quasilinear reaction–diffusion equations has been proven. However, the author does not study stability conditions, cf. the remarks at the end of his article. On the other hand, there is a functional analytic approach which was introduced in [15] and later also used by [20] for fully nonlinear equations. The authors in the latter paper obtain the existence of bifurcating periodic solutions even for fully nonlinear equations, but they do not consider stability. Finally, we also mention the work [10], where the authors prove existence of center manifolds for fully nonlinear equations and then state a result on Hopf bifurcation. Our results improve on theirs in several directions in the case of quasilinear equations, cf. the discussion in the introduction of [5]. Our results apply to the wide class of normally elliptic quasilinear reaction–diffusion systems subject to nonlinear boundary conditions.

In recent years, Amann has developed a dynamic theory for quasilinear parabolic equations which is able to cover the quasilinear reaction–diffusion systems introduced at the beginning of this section, cf. [1–3, 21]. Indeed, it is this approach which enables us to deal with quasilinear reaction–diffusion systems with nonlinear boundary conditions. We mention that the quasilinear systems studied by Amann, and considered in this paper, are indeed very general and cover many interesting equations governed by problems in physics, biology and chemistry, see [2, Section 1]. The ellipticity conditions (normal ellipticity) weaken the usual conditions imposed on systems (which is sometimes the demanding Legendre condition), cf. again [1, 3]. On the other side, the imposed conditions are still strong enough to render a very powerful theory (and are in some sense optimal). The equations under consideration include strongly coupled systems which really are more demanding than, say, scalar equations. Finally, we deal with nonlinear boundary conditions (note that the boundary operator $\mathcal{B}(u)u$ and the "boundary source function" $g$ depend nonlinearly on $u$). We do not know of other results and techniques which were able to cover this general situation.

This paper relies on [5], where the abstract setting is introduced and the existence and attractivity of center manifolds is proven. We will briefly introduce this abstract setting and extend results to parameter-dependent equations. This is actually a simple task, since we just extend the spaces by the parameter space. In a later section, we prove that the invariant manifolds "carry" the dynamical behavior of small solutions. Section 4 then is devoted to Hopf bifurcation and the computation of the stability algorithm is given in Section 5. In the last part, we apply the results to quasilinear reaction–diffusion systems, using results of [5, Sections 7, 8].

*Notations*

Let $E$ and $F$ be two Banach spaces over the same field $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then we denote by $\mathcal{L}(E, F)$ the vector space of all bounded linear operators from $E$ to $F$ and we equip this space with the uniform operator norm. If two Banach spaces $E, F$ coincide, except for equivalent norms, we express this by writing $E \doteq F$. If $E$ is a subspace of $F$, $E \hookrightarrow F$ means that the natural injection is continuous, that is, $E$ is continuously embedded in $F$. $E \overset{d}{\hookrightarrow} F$ then stands for dense embedding, i.e. $E \subset F$ is densely and continuously embedded.

## 2. INVARIANT MANIFOLDS FOR PARAMETER-DEPENDENT EQUATIONS

In this section we briefly introduce an abstract setting which enables us to deal with parameter-dependent, quasilinear, autonomous evolution equations. We prove the existence of (parameter-dependent) invariant manifolds and show that these attract small solutions at an exponential rate. Our approach uses results on maximal regularity of the Da Prato–Grisvard type and relies on [5]. We refer to [5, Section 2] for a short presentation of the subject, cf. also [9, 22–24]. We refrain from giving more details and assume that the reader is familiar with some of the results and the notation of [5].

Let $X_0$ and $X_1$ be two Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$. Let $F$ be a finite dimensional space and assume that

$$\Lambda \subset F \text{ is open and } 0 \in \Lambda.$$

We then consider the parameter-dependent, quasilinear evolution equation

$$\dot{u} + A(\lambda, u)u = F(\lambda, u), \qquad t > 0, \qquad \lambda \in \Lambda. \tag{2.1$_\lambda$}$$

We assume that

$$(A, F) \in C^k(\Lambda \times U_\beta, \mathcal{L}(X_1, X_0) \times X_0), \qquad k \in \mathbb{N}^*, \tag{2.2}$$

where $U_\beta$ denotes an open subset of the continuous interpolation space $X_\beta := (X_0, X_1)_\beta$, with $0 < \beta < 1$, such that $0 \in U_\beta$.

The next assumption on maximal regularity is crucial for our results. In fact, we can then use (a modified version of) the Ljapunov–Perron method and obtain invariant manifolds as fixed points of an integral equation. For comments in this direction, see the remarks at the end of [5, Section 5]. We impose that

$$A(\lambda, v) \in \mathfrak{M}_\alpha(X_1, X_0), \qquad (\lambda, v) \in \Lambda \times U_\beta, \tag{2.3}$$

where $0 < \beta < \alpha < 1$. For the definition of $\mathfrak{M}_\alpha(X_1, X_0)$, see (2.16) in [5]. In particular, $A(\lambda, v)$ is the negative generator of an analytic semigroup on $X_0$ and $\mathrm{dom}(A(\lambda, v)) \doteq X_1$, in the sense of equivalent norms, for each $\lambda$ and each $v$. We will show in the last section that the quasilinear reaction–diffusion system $(1.1)_\lambda$ can be shaped as an abstract evolution equation of the type $(2.1)_\lambda$ in carefully chosen spaces $X_0$ and $X_1$. The abstract counterpart of (1.15) will then be reflected by

$$F(\lambda, 0) = 0, \qquad \lambda \in \Lambda. \tag{2.4}$$

If

$$L := A(0,0) - \partial_2 F(0,0) \tag{2.5}$$

denotes the linearization of $(2.1)_0$ with respect to the second variable, the condition (1.18) will be recovered as

$$\sigma(-L) = \sigma_c \cup \sigma_s, \qquad \sigma_c \subset i\mathbb{R}, \qquad \sigma_s \subset [\operatorname{Re} z < 0], \tag{2.6}$$

where $\sigma(-L)$ denotes the spectrum of the closed operator $-L$. In an abstract context we assume that

$$\sigma_c \text{ consists of finitely many eigenvalues with finite multiplicity.} \tag{2.7}$$

(Note that the spectrum of $-L$ corresponds to the eigenvalues of the eigenvalue problem $(1.16)_0$. Condition (2.7) is automatically satisfied for the quasilinear reaction–diffusion system described in the Introduction.) Finally, let $X^c$ denote the finite dimensional subspace, obtained by the spectral projection, $\pi^c$, to the spectral set $\sigma_c$ and set $\pi^s := \mathrm{id}_{X_0} - \pi^c$. Again, the space $X^c$ is the abstract counterpart of (1.19). If

$$g(\lambda, u) := (A(0,0) - A(\lambda, u))u + F(\lambda, u) - \partial_2 F(0,0)u, \qquad (\lambda, u) \in \Lambda \times X_1, \tag{2.8}$$

the quasilinear equation $(2.1)_\lambda$ can be restated as

$$\dot{u} + Lu = g(\lambda, u), \qquad t > 0, \lambda \in \Lambda. \tag*{$(2.9)_\lambda$}$$

(Note that we need maximal regularity to justify the last step.) Finally, let $g_\rho \in C^k(\Lambda \times X^c \times U_1^s, X_0)$ denote the modified function

$$g_\rho := g \circ (\mathrm{id}_\Lambda, r_\rho), \qquad 0 < \rho \leq \rho_0, \tag{2.10}$$

where $U_1^s$ is a neighborhood of 0 in $U \cap X_1^s$ and $X_1^s$ is a (direct topological) complement of $X^c$ in $X_1$. Moreover, $r_\rho$ is defined by $r_\rho(x, y) := \chi(\rho^{-1}x)x + y$ for $(x, y) \in X^c \times X_1^s$, where $\chi \in \mathfrak{D}(\mathbb{B}_{X^c}(0, 2))$ denotes a smooth cutoff function for the closed ball $\bar{\mathbb{B}}_{X^c}(0, 1)$ in $X^c$. We then infer that

$$g_\rho(\lambda, 0) = 0 \qquad \text{for } \lambda \in \Lambda, \ \partial g_\rho(0,0) = 0. \tag{2.11}$$

We are now ready to state the following theorem.

THEOREM 2.1 (existence of locally invariant manifolds). There exists $\rho_k \in (0, \rho_0]$ and for each $\rho \in (0, \rho_k]$ there exists a neighborhood of 0 in $\Lambda$, denoted again by $\Lambda = \Lambda(\rho)$, and a uniquely determined mapping

$$\sigma = \sigma_\rho = \sigma_{k,\rho} \in BC^k(\Lambda \times X^c, X_1^s) \tag{2.12}$$

such that

$$\sigma(\lambda, 0) = 0 \qquad \text{for } \lambda \in \Lambda, \ \partial\sigma(0,0) = 0 \tag{2.13}$$

and

$$\|\sigma(\lambda, x) - \sigma(\lambda, x')\|_{X_1^s} \leq b\|x - x'\|, \qquad \lambda \in \Lambda, \qquad x, x' \in X^c \tag{2.14}$$

for a suitable constant $b$,

$$\mathfrak{M}^c(\lambda) := \mathfrak{M}^c_{k,\rho}(\lambda) := \operatorname{graph}(\sigma(\lambda, \cdot)) \tag{2.15}$$

is a locally invariant $C^k$-manifold for small solutions of the parameter-dependent quasilinear equation $(2.1)_\lambda$.

Let

$$z(\cdot) := z(\cdot, x, \lambda) := z(\cdot, x, \lambda, \sigma, \rho) \tag{2.16}$$

be the (global) solution to the *reduced* ordinary differential equation

$$\dot{z}(t) + L_c z(t) = \pi^c g_\rho(\lambda, z(t), \sigma(\lambda, z(t))), \qquad t \in \mathbb{R}, \qquad z(0) = x. \tag{2.17}$$

Then $\sigma$ satisfies the (fixed point) equation

$$\sigma(\lambda, x) = \int_{-\infty}^{0} e^{\tau L_s} \pi^s g_\rho(\lambda, z(\tau, x, \lambda), \sigma(\lambda, z(\tau, x, \lambda)))\, d\tau. \tag{2.18}$$

Finally, $\mathfrak{M}^c(\lambda)$ contains all small global solutions of $(2.1)_\lambda$ for $\lambda \in \Lambda$.

*Proof.* The parameter-dependent situation can be reduced to the setting in [5] by extending the spaces by the "parameter space" $F$. Note that this is a standard trick in this context. More precisely, set

$$\mathbf{X}_0 := F \times X_0, \qquad \mathbf{X}_1 := F \times X_1$$

and

$$\mathbf{L} := \operatorname{diag}[0, L], \qquad \mathbf{g}_\rho = (0, g_\rho),$$

where diag[0, $L$] denotes the diagonal matrix with entries 0 and $L$ on the diagonal. It follows for the extended mapping $\mathbf{g}_\rho$ that

$$\mathbf{g}_\rho \in C^k(\Lambda \times X^c \times U_1^s, \mathbf{X}_0), \qquad \mathbf{g}_\rho(0,0,0) = 0, \qquad \partial \mathbf{g}_\rho(0,0,0) = 0.$$

It is not difficult to see that $\mathbf{L}$ satisfies the assumptions of [5, Section 4]. (Note that in the parameter-dependent case, similar conditions to [5, Section 4] can be imposed, yielding the required maximal regularity results.) It is plain that the spectrum of $-\mathbf{L}$ consists of

$$\sigma(-\mathbf{L}) = \sigma(-L) \cup \{0\} \qquad \text{and} \qquad \mathbf{X}^c = F \times X^c, \tag{2.19}$$

where $\mathbf{X}^c$ denotes the eigenspace according to the spectral projection of the spectral set $\sigma_c \cup \{0\}$. More precisely, let $\Pi^c$ be the spectral projection for $\sigma_c \cup \{0\}$ and set $\Pi^s := \mathrm{id}_{\mathbf{X}_0} - \Pi^c$. Then,

$$\Pi^c = \operatorname{diag}[\mathrm{id}_F, \pi^c] \qquad \text{and} \qquad \Pi^s = \operatorname{diag}[0, \pi^s], \tag{2.20}$$

and, moreover,

$$\mathbf{L}_c = \operatorname{diag}[0, L_c], \qquad \mathbf{L}_s = L_s, \tag{2.21}$$

where we identify $\{0\} \times X_1^s$ with $X_1^s$ and analogously $\{0\} \times X_0^s$ with $X_0^s$. The parameter-dependent equation (2.17) is equivalent to the extended ordinary differential equation

$$\dot{\mathbf{z}}(t) + \mathbf{L}_c \mathbf{z}(t) = \Pi^c \mathbf{g}_\rho(\mathbf{z}(t), \sigma(\mathbf{z}(t))), \qquad \mathbf{z}(0) = \mathbf{x}, \tag{2.22}$$

where $\mathbf{z} = (\lambda, z)$ and $\mathbf{x} = (\lambda, x)$. Then let $\mathbf{z}(\cdot, \mathbf{x}) := \mathbf{z}(\cdot, \mathbf{x}, \sigma, \rho)$ denote the (global) solution to (2.22) and set

$$\mathcal{S}_{k,\rho} := \{\sigma \colon \Lambda(\rho) \times X^c \to X_1^s;\ \sigma(\lambda, 0) = 0,\ \|\sigma(\mathbf{x})\| \le \rho,$$
$$\|\partial^j \sigma(\mathbf{x})\| \le b_j,\ j = 1, \ldots, k-1,\ [\partial^{(k-1)}\sigma]_{1-} \le b_k\},$$

endowed with the sup-norm. Hereby, $\Lambda(\rho)$ denotes a 0-neighborhood in $\Lambda$ such that diam $\Lambda(\rho) \to 0$ for $\rho \to 0$. For a given $\sigma \in \mathcal{S}_{k,\rho}$, let $G$ be defined by

$$G(\sigma)(\mathbf{x}) := \int_{-\infty}^{0} e^{\tau \mathbf{L}_s} \Pi^s \mathbf{g}_\rho(\mathbf{z}(\tau, \mathbf{x}), \sigma(\mathbf{z}(\tau, \mathbf{x})))\, d\tau. \tag{2.23}$$

Now, the same arguments as in [5, theorem 4.1] show that the mapping $G$ has a uniquely determined fixed point $\sigma = \sigma_{k,\rho} \in \mathcal{S}_{k,\rho}$ with

$$\sigma \in BC^k(\Lambda(\rho) \times X^c, X_1^s), \qquad \partial\sigma(0, 0) = 0,$$

provided that $\rho$ is sufficiently small. Note that the right-hand side in (2.23) is just another writing of (2.18). Since we know that graph($\sigma$) is a locally invariant manifold for the equation

$$\dot{\mathbf{u}} + \mathbf{L}\mathbf{u} = \mathbf{g}_\rho(\mathbf{u}), \qquad \mathbf{u} = (\lambda, u), \qquad \mathbf{u}(0) = (\lambda, u_0), \tag{2.24}$$

and since this equation is equivalent to

$$\dot{u} + Lu = g_\rho(\lambda, u), \qquad u(0) = u_0, \tag*{(2.25)$_\lambda$}$$

we can conclude that $\mathcal{M}^c(\lambda)$ is locally invariant for $(2.25)_\lambda$ and, hence, for $(2.9)_\lambda$ (for sufficiently small solutions and $\lambda \in \Lambda(\rho)$). Note that $(2.9)_\lambda$ and $(2.1)_\lambda$ have the same solutions. (We need maximal regularity to handle the equation $(2.9)_\lambda$.) This proves the assertions in theorem 2.1. ■

*Remark* 2.2. Theorem 2.1 ensures the existence of a finite dimensional $C^k$-center manifold, $\mathcal{M}^c$ = graph($\sigma$), for the extended equation (2.24). Note that the second part of (2.13) implies that the space $F \times X^c$ is tangential to $\mathcal{M}^c$ at 0. $\mathcal{M}^c(\lambda)$ = graph($\sigma(\lambda, \cdot)$) is an invariant $C^k$-manifold for the quasilinear equation $(2.1)_\lambda$, but it is no longer tangential to $X^c$ for $\lambda \neq 0$. Nevertheless, it is sometimes convenient to call $\mathcal{M}^c(\lambda)$ a (parameter-dependent) center manifold too.

It is known that the (parameter-dependent) quasilinear equation $(2.1)_\lambda$ generates a smooth semiflow on the continuous interpolation space $X_\alpha$, where $\alpha \in (1, \beta)$ is the same constant as in (2.3). A proof, using maximal regularity results, is given in [25], see also [24, Section 7]. (It is clear that the proofs can be adjusted to the parameter-dependent case. One can again use the same idea as in the proof of theorem 2.1.) On the other hand, the quasilinear equation $(2.1)_\lambda$ regularizes, which means that solutions immediately become more regular than the initial values are. We, thus, have the following situation: if $u(\cdot, u_0, \lambda)$ denotes a solution to $(2.1)_\lambda$ with $u(0, u_0, \lambda) = u_0$ and with an initial value $u_0 \in X_\alpha$, then $u(t, u_0, \lambda) \in X_1$ for each $t \in (0, t^+(u_0, \lambda))$, the maximal interval of existence. We, thus, have two topologies where we can measure the solutions. One is the weaker norm of the space $X_\alpha$, where the initial values are taken from and where the equation governs a semiflow. The other one is the stronger norm of $X_1$, where solutions exist for each positive time. In [5, theorem 5.8], we have used this smoothing (or regularizing) property for solutions of parabolic quasilinear equations, and have shown that center manifolds exist in $X_1$ and attract solutions with initial data lying in the weaker space $X_\alpha$ at an exponential rate. This is the best possible result we can expect. Once having established this (rather technically involved) result, we are awarded with the ability to leave the spaces of maximal regularity and formulate similar statements in any space lying between $X_1$ and $X_\alpha$. For comments as to why we must first invoke maximal regularity, see the remarks at the end of Section 5 in [5].

For the sake of completeness, we state the result for parameter-dependent equations. Let

$$u = u(\cdot, u_0, \lambda) \tag{2.26}$$

denote the solution to the parameter-dependent, quasilinear equation $(2.1)_\lambda$ and let $[0, t^+(u_0, \lambda))$ be its maximal interval of existence. Then the following result holds.

THEOREM 2.3. There exists $\omega > 0$ such that

$$\|\pi^s u(t, u_0, \lambda) - \sigma(\lambda, \pi^c u(t, u_0, \lambda))\|_{X_1} \le \frac{N_\alpha}{t^{1-\alpha}} e^{-\omega t} \|\pi^s u_0 - \sigma(\lambda, \pi^c u_0)\|_{X_\alpha} \tag{2.27}$$

for $t \in (0, t^+(u_0, \lambda))$ and $(\lambda, u_0) \in \Lambda \times U_\alpha$ sufficiently small, as long as $\pi^c u(\cdot, u_0, \lambda)$ is contained in a small neighborhood of 0 in $X^c$.

*Proof.* Let $\mathbf{X}_\gamma := (\mathbf{X}_0, \mathbf{X}_1)_\gamma$ be a continuous interpolation space, $\gamma \in (0, 1)$. Hereby, $\mathbf{X}_1$ and $\mathbf{X}_0$ are the extended spaces defined in the proof of theorem 2.1. It then follows that $\mathbf{X}_\gamma = F \times X_\gamma$, cf. [26, proposition 3.4]. Now, the results in [5, Section 5] together with (2.20) provide the proof. ■

## 3. STABILITY

We shall show that the dynamical behavior of small solutions of the parameter-dependent equation

$$\dot{u}(t) + Lu(t) = g(\lambda, u(t)), \qquad t > 0, \qquad u(0) = u_0 \tag{3.1$_\lambda$}$$

is determined by the behavior of small solutions of the reduced ordinary differential equation

$$z(t) + L_c z(t) = \pi^c g(\lambda, z(t), \sigma(\lambda, z(t))), \qquad t \in \mathbb{R}, \qquad z(0) = \pi^c u_0. \tag{3.2$_\lambda$}$$

Hence, the invariant manifolds $\mathfrak{M}^c(\lambda)$ provide a reduction to a finite dimensional differential equation. Loosely expressed, the invariant manifolds $\mathfrak{M}^c(\lambda)$ carry the dynamics of small solutions to $(3.1)_\lambda$. We assume for the remainder of this section that $\Lambda$ is a (sufficiently small) neighborhood of 0 and fix $\lambda \in \Lambda$.

We briefly recall the definition of *stability* and *asymptotic stability*. Let $X$ then be a metric space and $\varphi: \mathfrak{D} \subset \mathbb{R}^+ \times X \to X$ be a (local) semiflow. A subset $M \subset X$ is called *stable*, if there exists for each neighborhood $U$ of $M$ another neighborhood, $V$, of $M$ such that $t^+(x) = \infty$ and $\gamma^+(x) \subset U$ for each $x \in V$. $M$ is asymptotically stable if it is stable and there is a neighborhood $W$ of $M$ such that

$$\varphi^t(x) \to M \qquad \text{for } t \to \infty,\ x \in W.$$

Finally, $M$ is uniformly asymptotically stable if $M$ is asymptotically stable and $\varphi^t \to M$ for $t \to \infty$ uniformly in $x \in W$.

LEMMA 3.1. Let $z(\cdot) := z(\cdot, x_0, \lambda) := z(\cdot, x_0, \lambda, \sigma, \rho)$ be the global solution of the *reduced* differential equation

$$\dot{z}(t) + L_c z(t) = \pi^c g_\rho(\lambda, z(t), \sigma(\lambda, z(t))), \qquad t \in \mathbb{R}, \qquad z(0) = x_0 \tag{3.3$_{\lambda,\rho}$}$$

and let $x(\cdot) := x(\cdot, x_0, \lambda, \rho)$ denote the solution of

$$\dot{x}(t) + L_c x(t) = \pi^c g_\rho(\lambda, x(t), y(t)), \qquad x(0) = x_0 := \pi^c u_0, \tag{3.4$_{\lambda,\rho}$}$$

where $x(t) = \pi^c u(t)$, $y(t) = \pi^s u(t)$ and $u$ solves the equation

$$\dot{u}(t) + Lu(t) = g_\rho(\lambda, u(t)), \qquad t \in (0, t^+(u_0, \lambda)), \qquad u(0) = u_0. \tag{3.5$_{\lambda,\rho}$}$$

Then there exist positive constants $k_1$ and $k_2$ such that

$$\|z(t) - x(t)\| \leq k_1 \, e^{k_2 t} \|y_0 - \sigma(\lambda, x_0)\|_\alpha, \qquad t \in (0, t^+(u_0, \lambda)), \tag{3.6}$$

and for $u_0$ belonging to a sufficiently small neighborhood, $\mathfrak{U}_\alpha = \mathfrak{U}_\alpha(\rho)$, of 0 in $U_\alpha$.

*Proof.* Let $\eta > 0$ be fixed. Then there exists $M \geq 1$ such that

$$\|e^{-tL_c}\|_{\mathcal{L}(X^c)} \leq M e^{t\eta}, \qquad t \geq 0. \tag{3.7}$$

$x(\cdot)$ and $z(\cdot)$ are given by the variation of constants formula, and we obtain

$$z(t) - x(t) = \int_0^t e^{-(t-\tau)L_c} \pi^c [g_\rho(\lambda, z(\tau), \sigma(\lambda, z(\tau))) - g_\rho(\lambda, x(\tau), y(\tau))] \, d\tau.$$

An analogous argument to [5, proposition 5.3(i)] and (3.7) yield

$$\|z(t) - x(t)\| \leq cML_\alpha(\rho) \int_0^t e^{(t-\tau)\eta} \|y(\tau) - \sigma(\lambda, x(\tau))\|_{X_1} \, d\tau$$
$$+ cML_\alpha(\rho) \int_0^t e^{(t-\tau)\eta} \|z(\tau) - x(\tau)\| \, d\tau.$$

Gronwall's lemma then yields (multiply both sides of the inequality with $e^{-t\eta}$ and then use Gronwall's lemma)

$$\|z(t) - x(t)\| \leq cML_\alpha(\rho) \, e^{k_2 t} \int_0^t e^{-\tau\eta} \|y(\tau) - \sigma(\lambda, x(\tau))\|_{X_1} \, d\tau, \qquad t \in (0, t^+(u_0, \lambda))$$

where $k_2 := cML_\alpha(\rho) + \eta$. We now invoke theorem 2.3 and get

$$\int_0^t e^{-\tau\eta} \|y(\tau) - \sigma(\lambda, x(\tau))\|_1 \, d\tau \leq N_\alpha \|y_0 - \sigma(\lambda, x_0)\|\alpha \int_0^t e^{-(\eta+\omega)\tau} \tau^{\alpha-1} \, d\tau$$
$$\leq N_\alpha \frac{\Gamma(\alpha)}{(\eta + \omega)^\alpha} \|y_0 - \sigma(\lambda, x_0)\|_\alpha.$$

Now the assertion follows for $k_1 := cLM_\alpha(\rho) N_\alpha \Gamma(\alpha)(\eta + \omega)^{-\alpha}$. ∎

We can assume that the neighborhood $\mathfrak{U}_\alpha$ is given by

$$\mathfrak{U}_\alpha := \bar{\mathbb{B}}_{X^c}(0, r_c) \times \bar{\mathbb{B}}_{X_\alpha^s}(0, r_s). \tag{3.8}$$

Hence, theorem 2.3 holds true for all initial values in this set. We assume that $r$ is fixed such that

$$r \leq \min\left(\frac{1}{2} r_c, \frac{1}{4b} r_s\right). \tag{3.9}$$

PROPOSITION 3.2. Let $\gamma \subset \bar{\mathbb{B}}_{X^c}(0, r)$ be a compact subset of $X^c$ and suppose that $\gamma$ is stable for the reduced equation $(3.3)_{\lambda,\rho}$. Then there exists a neighborhood $\mathcal{W}_\alpha$ of $\Gamma := \{(x, \sigma(\lambda, x)); x \in \gamma\}$ in $X_\alpha$ such that:

(a) each solution $u(\cdot, u_0) := u(\cdot, u_0, \lambda, \rho)$ to the equation $(3.5)_{\lambda,\rho}$ exists globally for initial values $u_0 \in \mathcal{W}_\alpha$;

(b) there exists $T \geq 1$ and $M \geq 1$ such that

$$d_{X_1}(u(t, u_0), \Gamma) \leq M(d_{X^c}(z(t, x_0), \gamma) + \mathrm{e}^{-t \ln 2/T} \|y_0 - \sigma(\lambda, x_0)\|_\alpha)$$

for $t \geq T$ and $u_0 \in \mathcal{W}_\alpha$. Here, $d_{X_1}(u(t, u_0), \Gamma)$ denotes the distance in $X_1$ between $u(t, u_0)$ and the set $\Gamma$ and $d_{X^c}$ stands for the distance of the indicated sets in $X^c$.

*Proof.* Let $T \geq 1$ be fixed such that the relations given in (3.13) are satisfied. Then we can choose a neighborhood $\mathcal{O}_\alpha := \mathcal{O}_\alpha(\rho)$ of $\Gamma$ in $X_\alpha$ such that each solution $u(\cdot, u_0)$ with $u_0 \in \mathcal{O}_\alpha$ exists on the interval $[0, T]$. (Note that solutions $u(\cdot, u_0)$ of the modified equation $(3.5)_{\lambda,\rho}$ with initial values in $\mathfrak{M}^c(\lambda)$ exist globally, as is shown in [5, theorem 4.1]. The same statement also holds true for the parameter-dependent case.) We can assume that $\mathcal{O}_\alpha$ is given as

$$\mathcal{O}_\alpha = \left\{(x, y) \in X^c \times X^s_\alpha;\ x \in \bar{\mathbb{B}}_{X^c}(\gamma, \varepsilon),\ \|y - \sigma(\lambda, x)\|_\alpha \leq \frac{1}{2k_1} \mathrm{e}^{-k_2 T} \varepsilon\right\} \tag{3.10}$$

for some $\varepsilon \leq r$. Here, $k_1$ and $k_2$ are given by lemma 3.1. Since $\gamma$ is stable, according to our assumption, we conclude that there exists $\delta = \delta(\varepsilon)$ such that

$$z(t, x_0) \in \bar{\mathbb{B}}_{X^c}(\gamma, \varepsilon/2) \text{ for } t \geq 0 \qquad \text{and} \qquad x_0 \in \bar{\mathbb{B}}_{X^c}(\gamma, \delta). \tag{3.11}$$

Now, we define

$$\mathcal{W}_\alpha := \mathcal{O}_\alpha \cap (\bar{\mathbb{B}}_{X^c}(\gamma, \delta) \times X^s_\alpha). \tag{3.12}$$

Assume that $T \geq 1$ has been fixed such that

$$\frac{1}{2k_1} \mathrm{e}^{-k_2 T} \leq 2b, \qquad \max(1, \|i\|_{\mathcal{L}(X_1, X_\alpha)}) N_\alpha \, \mathrm{e}^{-\omega T} \leq 1/4, \tag{3.13}$$

where $b$ is defined in (2.14) and $N_\alpha$ and $\omega$ have the same meaning as in theorem 2.3. Let $q \geq 1$ be sufficiently large such that

$$\frac{k_1}{q} \mathrm{e}^{2k_2 T} \leq 1/4. \tag{3.14}$$

Let $u_0 = (x_0, y_0) \in \mathcal{O}_\alpha$ be arbitrarily given. We then infer from (3.9), (3.10), (3.13), from (2.13)–(2.14), from the assumption $\gamma \subset \bar{\mathbb{B}}_{X^c}(0, r)$, and from $\varepsilon \leq r$ that

$$\|y_0\|_\alpha \leq \|y_0 - \sigma(\lambda, x_0)\|_\alpha + \|\sigma(\lambda, x_0)\|_\alpha \leq \frac{1}{2k_1} \mathrm{e}^{-k_2 T} \varepsilon + b\|x_0\| \leq r_s,$$

where we use

$$\|x_0\| \leq d_{X^c}(x_0, \gamma) + d_{X^c}(0, \gamma) \leq \varepsilon + r \leq 2r \leq \min\left(r_c, \frac{1}{2b} r_s\right).$$

Thus, we have shown that

$$\mathcal{O}_\alpha \subset \bar{\mathbb{B}}_{X^c}(0, r_c) \times \bar{\mathbb{B}}_{X^s_\alpha}(0, r_s). \tag{3.15}$$

Hence, theorem 2.3 holds for all initial values $(x, y) \in \mathcal{O}_\alpha$. Let $u_0 = (x_0, y_0) \in \mathcal{W}_\alpha$ be given and set $u_1 := u(T, u_0)$. We shall show that $u_1 \in \mathcal{O}_\alpha$. Note that $u_1 = (x(T, x_0), y(T, y_0))$. We then conclude with lemma 3.1 and (3.10), (3.11) that

$$\begin{aligned} d(x(T, x_0), \gamma) &\leq d(z(T, x_0), \gamma) + \|z(T, x_0) - x(T, x_0)\| \\ &\leq d(z(T, x_0), \gamma) + k_1 \mathrm{e}^{k_2 T}\|y_0 - \sigma(\lambda, x_0)\| \leq \varepsilon. \end{aligned}$$

On the other hand, theorem 2.3, the second part of (3.13), and $T \geq 1$ give

$$\begin{aligned} \|y(T, y_0) - \sigma(\lambda, x(T, x_0))\|_\alpha &\leq \|i\|_{\mathcal{L}(X_1, X_\alpha)} N_\alpha \mathrm{e}^{-\omega T}\|y_0 - \sigma(\lambda, x_0)\|_\alpha \\ &\leq 1/4\|y_0 - \sigma(\lambda, x_0)\|_\alpha. \end{aligned}$$

Summarizing, we see that $u_1 \in \mathcal{O}_\alpha$. But then, $u_2 := u(T, u_1)$ exists as well and it follows that

$$\begin{aligned} d(x(T, x_1), \gamma) &\leq d(z(T, x_1), \gamma) + \|z(T, x_1) - x(T, x_1)\| \\ &\leq d(z(T, x_1), \gamma) + k_1 \mathrm{e}^{k_2 T}\|y_1 - \sigma(\lambda, x_1)\|. \end{aligned}$$

Observe that $z(T, x_1) = z(2T, x_0)$. Thus, (3.11) gives $z(T_1, x_1) \in \mathbb{B}_{X^c}(\gamma, \varepsilon/2)$ and we obtain

$$d(z(T, x_1), \gamma) + k_1 \mathrm{e}^{k_2 T}\|y_1 - \sigma(\lambda, x_1)\|_\alpha \leq \varepsilon.$$

Now we can conclude as before that $u_2 \in \mathcal{O}_\alpha$. By repeating these steps we obtain the result that solutions with initial values in $\mathcal{W}_\alpha$ indeed exist for each $t \geq 0$. Using theorem 2.3, lemma 3.1, (2.14) and the fact that $q \geq 1$, a simple computation shows

$$d_{X_1}(u(t, u_0), \Gamma) \leq (1 + b)\left[d_{X^c}(z(t, x_0), \gamma) + \left(\frac{k_1}{q}\mathrm{e}^{k_2 t} + \frac{N_\alpha}{t^{1-\alpha}}\mathrm{e}^{-\omega t}\right)q\|y_0 - \sigma(\lambda, x_0)\|_\alpha\right], \tag{3.16}$$

for each $t \geq 0$. By setting $a(t, u_0) := (1/(1 + b))d_{X_1}(u(t, u_0), \Gamma) - d_{X^c}(z(t, x_0), \gamma)$, (3.16) can be restated as

$$a(t, u_0) \leq \left[\frac{k_1}{q}\mathrm{e}^{k_2 t} + \frac{N_\alpha}{t^{1-\alpha}}\mathrm{e}^{-\omega t}\right]q\|y_0 - \sigma(\lambda, x_0)\|_\alpha. \tag{3.17}$$

Observe that

$$a(t, u_0) \leq (1/2)q\|y_0 - \sigma(\lambda, x_0)\|_\alpha, \qquad t \in [T, 2T], \tag{3.18}$$

due to (3.13) and (3.14). Now let $t \in [2T, 3T]$. Then $a(t, u_0) = a(s, u(T, u_0))$ with $s \in [T, 2T]$. Since $u(T, u_0)$ belongs to $\mathcal{O}_\alpha$ we obtain, as in (3.16) and (3.18),

$$a(t, u_0) \leq (1/2)q\|y(T, y_0) - \sigma(\lambda, x(T, x_0))\|_\alpha, \qquad t \in [2T, 3T]. \tag{3.19}$$

Since

$$\|y(T, y_0) - \sigma(\lambda, x(T, x_0))\|_\alpha \leq \|i\|N_\alpha \mathrm{e}^{-\omega T}\|y_0 - \sigma(\lambda, x_0)\|_\alpha \leq (1/4)\|y_0 - \sigma(\lambda, x_0)\|_\alpha,$$

thanks to theorem 2.3 and (3.13), we get $a(t, u_0) \leq (1/2)^2 q\|y_0 - \sigma(\lambda, x_0)\|_\alpha$ for $t \in [2T, 3T]$. An induction argument then shows that

$$a(t, u_0) \leq (1/2)^n q\|y_0 - \sigma(\lambda, x_0)\|_\alpha, \qquad t \in [nT, (n + 1)T], \qquad n \geq 1. \tag{3.20}$$

We can now infer from (3.20) that

$$a(t, u_0) \le 2q\, e^{-t \ln 2/T} \|y_0 - \sigma(\lambda, x_0)\|_\alpha, \qquad t \ge T.$$

This completes the proof of proposition 3.2. ■

We can now state a stability result for solutions to the equation $(3.1)_\lambda$, provided we know that there exists a stable set for the reduced equation $(3.2)_\lambda$. Proposition 3.2 shows that we can measure the distance of $u(\cdot, u_0)$ in the stronger topology of the space $X_1$, even if the initial values are taken in the weaker space $X_\alpha$. (Since quasilinear parabolic equations regularize, solutions immediately belong to $X_1$ for $t > 0$.) This makes it possible to state results in any Banach space lying between $X_1$ and $X_\alpha$. So let us assume that $X$ is a Banach space with $X_1 \hookrightarrow X \hookrightarrow X_\alpha$. Moreover, assume that the quasilinear equation $(2.1)_\lambda$ governs a (continuous) semiflow on an open subset $U$ of $X$, i.e. the mapping

$$\bigcup_{u_0 \in U} [0, t^+(u_0, \lambda)) \times \{u_0\} \to U, \qquad (t, u_0) \to u(t, u_0, \lambda) \tag{3.21}$$

is continuous, where $u(\cdot, u_0, \lambda)$ denotes the solution to the equation $(3.1)_\lambda$, (which is equivalent to the quasilinear equation $(2.1)_\lambda$). Then the result reads as the following theorem.

Theorem 3.3. Let $\gamma$ be a compact subset of $X^c$ with $\gamma \subset \bar{\mathbb{B}}_{X^c}(0, r)$ for a sufficiently small $r > 0$.

(a) Assume that $\gamma$ is stable [asymptotically stable resp. uniformly asymptotically stable] for (the flow of) the *reduced* ordinary differential equation $(3.2)_\lambda$. Then the set

$$\Gamma := \{x, \sigma(\lambda, x)); x \in \gamma\} \subset \mathcal{M}^c(\lambda) \tag{3.22}$$

is stable [asymptotically resp. uniformly asymptotically stable] in $X$ for (the flow of) the equation $(2.1)_\lambda$.

(b) If $\gamma$ is unstable, $\Gamma$ is unstable in $X$ as well.

*Proof.* (a) Assume that $r$ satisfies (3.9) and $r \le \rho$ for a fixed $\rho$. We first claim that the solutions of $(3.2)_\lambda$ and $(3.3)_{\lambda,\rho}$ (resp. $(3.1)_\lambda$, and $(3.5)_{\lambda,\rho}$) coincide for small initial values. Owing to the definition of the modified function $g_\rho$, it suffices to show that all solutions of $(3.3)_{\lambda,\rho}$ (resp. $(3.4)_{\lambda,\rho}$) are contained in $\bar{\mathbb{B}}_{X^c}(0, \rho)$. The first part of the claim immediately follows from (3.11), since we assumed that $\varepsilon \le r \le \rho$. The second part can be derived from (3.11), lemma 3.1, and proposition 3.2b. (We can decrease $\varepsilon$, if necessary.) Thanks to proposition 3.2, each solution $u(\cdot, u_0)$ of $(3.1)_\lambda$ with $u_0 \in \mathcal{W}_\alpha$ exists globally and

$$d_X(u(t, u_0), \Gamma) \le M'(d_{X^c}(z(t, x_0), \gamma) + e^{-t \ln 2/T} \|y_0 - \sigma(\lambda, x_0)\|_X) \tag{3.23}$$

for $t \ge T$. (We have assumed that the norm of $X$ is weaker than the norm of $X_1$, but stronger than the one of $X_\alpha$.) This gives us information on the behavior of $u(t, u_0)$ for $t \ge T$. If $t$ belongs to the compact interval $[\tau, T]$, where $\tau$ will be defined below, we use (3.16) and note that

$$d_X(u(t, u_0), \Gamma) \le c\left[ d_{X^c}(z(t, x_0), \gamma) + \left(\frac{k_1}{q} e^{k_2 t} + \frac{N_\alpha}{t^{1-\alpha}} e^{-\omega t}\right) q \|y_0 - \sigma(\lambda, x_0)\|_X \right], \tag{3.24}$$

for an appropriate constant $c$. We are now ready to prove the stability of $\Gamma$. Let $U_X$ then be an arbitrary neighborhood of $\Gamma$ in $X$. It follows from (3.21) and the compactness of $\Gamma$ that there exists $\tau > 0$ and a neighborhood $V$ of $\Gamma$ such that

$$u(t, x, \lambda) \in U_X \qquad \text{for } (t, x) \in [0, \tau] \times (V \cap \mathcal{W}_\alpha). \tag{3.25}$$

Observe that we can infer from (3.23)–(3.25) (by decreasing $\varepsilon$ if necessary, see the definition of $\mathcal{W}_\alpha$ and $\mathcal{O}_\alpha$ in (3.12), respectively (3.10)) that

$$u(t, u_0, \lambda) \in U_X \qquad \text{for } t \in \mathbb{R}^+,\ u_0 \in V \times \mathcal{W}_\alpha,$$

showing the stability of $\Gamma$. The remaining assertions of (a) are immediate consequences of the definitions and (3.23). Note that (b) is obviously true. ■

*Remarks* 3.4. Note that proposition 3.2 contains a stronger result than we actually quoted in theorem 3.3. Indeed, proposition 3.2 shows that the set $\Gamma$ attracts solutions in the topology of $X_1$, even if the initial values belong to the weaker space $X_\alpha$. Note that this result again takes the regularizing property of quasilinear parabolic equations into consideration. Thus, we could sharpen theorem 3.3 by stating that $\Gamma$ is stable [asymptotically resp. uniformly asymptotically stable] in the stronger norm of $X_1$. (We then slightly modify the definition of stability in an obvious way.)

Our proof follows [7, theorem 6.1.4], where a related result for semilinear equations is given. The same proof has also been used by [10, theorem 3.4] in the context of fully nonlinear equations. Our contribution for quasilinear equations differs in some points from theirs. First, we get the stronger result mentioned above. This enables us to state results for any Banach space lying in between $X_1$ and $X_\alpha$. Hence, we can leave the spaces of maximal regularity. Second, we can weaken the assumptions in [7, 10] to a certain extent, allowing $\gamma$ to be stable and then still getting the stability of $\Gamma$. (Where [7, 10] require $\gamma$ to be uniformly asymptotically stable.)

## 4. HOPF BIFURCATION

Many interesting problems in physics, chemistry and biology are governed by quasilinear parabolic equations depending on a parameter, i.e. by an equation of the type

$$\dot{u}(t) + A(\lambda, u(t))u(t) = F(\lambda, u(t)), \qquad t > 0. \tag{4.1$_\lambda$}$$

We assume that

$$\Lambda = (-\lambda_0, \lambda_0) \qquad \text{for some } \lambda_0 > 0. \tag{4.2}$$

As discussed in the Introduction, the behavior of solutions may change abruptly if some eigenvalues of the linearized equation (say the linearization at 0) crosses the imaginary axis. Using our results of Sections 2 and 3 we obtain the existence of finite dimensional manifolds $\mathfrak{M}^c(\lambda)$, locally invariant under the flow of $(4.1)_\lambda$ and containing all the local recurrence. This provides a reduction to an ordinary differential equation on a finite dimensional space. Then we may use bifurcation theorems for ordinary equations and finally return to the original equation. We focus our attention on Hopf bifurcation. Assuming that a pair of simple eigenvalues of the linearized equation crosses the imaginary axis with nonzero speed (and there are no other eigenvalues on the imaginary axis), we show that a family of periodic solutions emerge from the equilibrium. We will be particularly interested in the case in which bifurcation to *stable* periodic orbits occurs. Thus, we will assume that the remaining part of the spectrum remains in the left half-plane.

We assume that all the appearing spaces are real. (We mean vector spaces over $\mathbb{R}$, the field of reals.) We will then use complexification wherever needed, e.g. in connection with spectral theory. We impose the same assumptions as in Section 2. In particular

$$(A, F) \in C^k(\Lambda \times U_\beta, \mathcal{L}(X_1, X_0) \times X_0), \qquad k \geq 2, \tag{4.3}$$

$$F(\lambda, 0) = 0, \qquad \lambda \in \Lambda, \tag{4.4}$$

such that 0 is an equilibrium for $(4.1)_\lambda$, independent of $\lambda \in \Lambda$. Suppose, as in Section 2, that the spectrum of $-(A(0,0) - \partial_2 F(0,0))$ admits a decomposition with a part $\sigma_s \subset [\operatorname{Re} z < 0]$ and $\sigma_c \subset i\mathbb{R}$, where

$$\sigma_c = \{\pm i\omega_0\}, \qquad \omega_0 > 0, \tag{4.5}$$

$$i\omega_0 \text{ is a simple eigenvalue of } -(A(0,0) - \partial_2 F(0,0)). \tag{4.6}$$

Let

$$\mu(\cdot) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{C}) \tag{4.7}$$

be the unique (local) continuation of the eigenvalue $i\omega_0$ of $-(A(0,0) - \partial_2 F(0,0))$ along $\Lambda \times \{0\}$, i.e. $\mu(\lambda)$ is (the unique) eigenvalue of

$$-(A(\lambda, 0) - \partial_2 F(\lambda, 0)) := -L(\lambda) \in \mathcal{L}(X_1, X_0). \tag{4.8}$$

(For the existence, cf. [20, lemma 2.1; 27].) Suppose that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\operatorname{Re}\mu(\lambda))|_{\lambda=0} \neq 0, \tag{4.9}$$

i.e. $\mu(\lambda)$ crosses the imaginary axis with nonzero speed at $\lambda = 0$. Then we have the following theorem.

THEOREM 4.1. The quasilinear equation $(4.1)_\lambda$ has in a neighborhood of $(0,0) \in \Lambda \times X_1$ a unique one-parameter family $\{\Gamma(s); 0 < s < \varepsilon\}$ of nontrivial periodic orbits which tend towards the equilibrium 0 as $s \to 0$. More precisely, there exists $\varepsilon > 0$ and a mapping

$$(\lambda(\cdot), T(\cdot), u(\cdot)) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times X_1)$$

satisfying

$$(\lambda(0), T(0), u(0)) = (0, 2\pi/\omega_0, 0),$$

such that

$$\Gamma(s) := \Gamma(u(s))$$

is a nontrivial orbit of $(4.1)_{\lambda(s)}$ of period $T(s)$ passing through $u(s) \in X_1$ for each $0 < s < \varepsilon$. If $0 < s_1 < s_2 < \varepsilon$, then $\Gamma(s_1) \neq \Gamma(s_2)$.

The family $\{\Gamma(s); 0 < s < \varepsilon\}$ contains every nontrivial periodic orbit of $(4.1)_\lambda$ lying in a suitable neighborhood of $(0, T(0), 0) \in \Lambda \times \mathbb{R} \times X_1$.

*Proof.* Owing to theorem 2.1, small periodic solutions of $(4.1)_\lambda$ are contained in $\mathfrak{M}^c(\lambda)$. Hence, it suffices to look for periodic solutions of the reduced differential equation

$$\dot{z}(t) = -L_c z(t) + \pi^c g_\rho(\lambda, z(t), \sigma(\lambda, z(t))) =: h_\rho(\lambda, z(t)), \qquad t \in \mathbb{R}, \tag{4.10$_\lambda$}$$

where $L := L(0)$. Henceforth, we will suppress $\rho$ and write $h(\lambda, z)$ instead. Note that

$$h \in C^k(\Lambda \times X^c, X^c), \qquad h(\lambda, 0) = 0, \qquad \lambda \in \Lambda, \tag{4.11}$$

thanks to (2.11)–(2.13) and the fact that $g_\rho \in C^k(\Lambda \times X^c \times U_1^s, X_0)$. Thus, we may differentiate $h(\lambda, \cdot)$ with respect to the second variable and we obtain, by invoking (4.8) and the definition of $h$,

$$\partial_2 h(\lambda, 0) = -\pi^c L(\lambda)[\mathrm{id}_{X^c} + \partial_2 \sigma(\lambda, 0)]. \tag{4.12}$$

In particular,

$$\partial_2 h(0, 0) = -\pi^c L = -L_c. \tag{4.13}$$

It follows from (4.5), (4.6) that the (two dimensional space) $X^c$ is spanned by

$$(X^c)_{\mathbb{C}} = \ker((-L_c)_{\mathbb{C}} - i\omega_0) \oplus \ker((-L_c)_{\mathbb{C}} + i\omega_0). \tag{4.14}$$

Here, $(X^c)_{\mathbb{C}}$ stands for the complexification of $X^c$ and $(L_c)_{\mathbb{C}}$ denotes the complexification of the linear operator $L_c \in \mathcal{L}(X^c)$. Let

$$z_0 := \xi + i\eta \in \ker((-L_c)_{\mathbb{C}} - i\omega_0), \qquad \xi, \eta \in \mathbb{R}, \tag{4.15}$$

be a fixed eigenvector to the eigenvalue $i\omega_0$ of $(-L_c)_{\mathbb{C}}$. Moreover, let

$$\kappa(\cdot) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{C}) \tag{4.16}$$

be the unique (local) continuation of the eigenvalue $i\omega_0$ of $\partial_2 h(0, 0)$ along $\Lambda \times \{0\}$. Next we shall show that (4.9) implies $\operatorname{Re} \dot\kappa(0) \neq 0$. Hence, the transversality condition for equation $(4.1)_\lambda$ will be recovered for the reduced equation $(4.10)_\lambda$. To prove this claim we consider the following two eigenvalue problems

$$[\partial_2 h(\lambda, 0)]_{\mathbb{C}} z(\lambda) = \kappa(\lambda) z(\lambda), \tag{4.17}$$

$$[-L(\lambda)]_{\mathbb{C}} v(\lambda) = \mu(\lambda) v(\lambda), \tag{4.18}$$

where $\lambda \in (-\varepsilon, \varepsilon)$ for $\varepsilon$ sufficiently small. Note that the corresponding eigenvectors, $z(\cdot)$ and $v(\cdot)$, depend $(k - 1)$-"continuously differentiable" on $\lambda$. Moreover,

$$v(0) = z(0) = z_0 \in (X^c)_{\mathbb{C}}. \tag{4.19}$$

Let $\ker([(-L_c)_{\mathbb{C}}]' - i\omega_0)$ be the kernel of the dual of $([(-L_c)_{\mathbb{C}}] - i\omega_0)$ and let $w := u + iv \in \ker([(-L_c)_{\mathbb{C}}]' - i\omega_0)$ be a linear form such that

$$\langle u, \xi \rangle = -\langle v, \eta \rangle = 1, \qquad \langle u, \eta \rangle = \langle v, \xi \rangle = 0. \tag{4.20}$$

Note that [12, lemma 26.23], for example, ensures its existence. We now differentiate (4.17) with respect to $\lambda$ and evaluate at $\lambda = 0$. We infer from (4.12), (4.13) and (2.13) that

$$\dot\kappa(0) z(0) = -[\pi^c \dot L(0)\, \mathrm{id}_{X^c} + \pi^c L \partial_\lambda \partial_2 \sigma(0, 0)]_{\mathbb{C}} z(0) + ([-L_c]_{\mathbb{C}} - \kappa(0)) \dot z(0). \tag{4.21}$$

Noting that $\partial_\lambda \partial_2 \sigma(0, 0) \in \mathcal{L}(X^c, X_1^s)$ and that $L$ leaves the space $X_1^s$ invariant, we get

$$\dot\kappa(0) z_0 = [-\pi^c \dot L(0)]_{\mathbb{C}} z_0 + ([-L_c]_{\mathbb{C}} - i\omega_0) \dot z(0), \tag{4.22}$$

where $z(0) = z_0$ according to (4.19). Differentiating (4.18) with respect to $\lambda$ and using (4.19) yields

$$\dot{\mu}(0)z_0 = [-\dot{L}(0)]_{\mathbb{C}}z_0 + ([-L]_{\mathbb{C}} - i\omega_0)\dot{v}(0). \tag{4.23}$$

We now apply the linear form $w$ to both sides of (4.22) and get

$$2\dot{\kappa}(0) = \langle w, [-\pi^c\dot{L}(0)]_{\mathbb{C}}z_0\rangle. \tag{4.24}$$

Note that this follows from (4.20) and $([(-L_c)_{\mathbb{C}}]' - i\omega_0)w = 0$. On the other hand, we may apply $[(\pi^c)_{\mathbb{C}}]'w$ to (4.23). It follows that

$$2\dot{\mu}(0) = \langle w, [-\pi^c\dot{L}(0)]_{\mathbb{C}}z_0\rangle. \tag{4.25}$$

For this, observe that

$$\langle [(\pi^c)_{\mathbb{C}}]'w, ([-L]_{\mathbb{C}} - i\omega_0)\dot{v}(0)\rangle = \langle w, (\pi^c)_{\mathbb{C}}([-L]_{\mathbb{C}} - i\omega_0)\dot{v}(0)\rangle.$$

Since there exists a (unique) decomposition of $\dot{v}(0) = \dot{v}_c(0) + \dot{v}_s(0) \in X^c \oplus X_1^s$, the right-hand side reduces to

$$\langle w, (\pi^c)_{\mathbb{C}}([-L]_{\mathbb{C}} - i\omega_0)\dot{v}_c(0)\rangle = \langle w, ([-L_c]_{\mathbb{C}} - i\omega_0)\dot{v}_c(0)\rangle = 0,$$

where we use that $([-L]_{\mathbb{C}} - i\omega_0)$ is reduced by the decomposition $X_1 = X^c \oplus X_1^s$ and finally, that $w$ belongs to the kernel of the dual operator. Therefore, we have

$$\dot{\kappa}(0) = \dot{\mu}(0), \tag{4.26}$$

which is now a transversality condition for the ordinary differential equation

$$\dot{z}(t) = h(\lambda, z(t)) \tag*{$(4.27)_\lambda$}$$

in the two dimensional space $X^c$. Hence, [12, theorem 26.25] ensures the existence of a one-parameter family $\{\gamma(s); 0 < s < \varepsilon\}$ of nontrivial periodic orbits in a neighborhood of $(0, 0) \in \Lambda \times X^c$. More precisely, there exists $\varepsilon > 0$ and

$$(\lambda(\cdot), T(\cdot), x(\cdot)) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times X^c)$$

satisfying the properties of [12, theorem 27.11]. Set $u(\cdot) := \sigma(\lambda(\cdot), x(\cdot))$. Then

$$(\lambda(\cdot), T(\cdot), u(\cdot)) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times X_1).$$

Finally, if

$$\Gamma(s) := (\sigma(\lambda(s), \gamma(s))), \tag{4.28}$$

the first part of theorem 4.1 immediately follows. We are left to prove that each nontrivial periodic orbit in a neighborhood of $(0, 2\pi/\omega_0, 0)$ belongs to $\{\Gamma(s); 0 < s < \varepsilon\}$. Indeed, if $\Gamma$ denotes a periodic orbit of $(4.1)_\lambda$, lying in a small neighborhood of 0 in $X_1$, $\Gamma$ has to be contained in $\mathfrak{M}^c(\lambda)$. Obviously, the projection of $\Gamma$ on $X^c$ is a periodic orbit to the reduced equation $(4.27)_\lambda$ and, hence, belongs to the family $\{\gamma(s); 0 < s < \varepsilon\}$. According to the definition in (4.28), $\Gamma$ is, thus, part of $\{\Gamma(s); 0 < s < \varepsilon\}$. ■

*Remarks* 4.2. (a) Assumption (4.5) can be relaxed in several directions. First, we could allow the spectrum to have some eigenvalues in $[\operatorname{Re} z > 0]$. Then there exist invariant manifolds $\mathfrak{M}^{cu}(\lambda)$ "belonging" to $\sigma_{cu} := \sigma_c \cup (\sigma(-[A(0,0) - \partial_2 F(0,0)]) \cap [\operatorname{Re} z > 0])$, cf. [5, remark 4.2b]. For additional generalizations of conditions (4.5) and (4.9) we refer to [17, 18, 28].

(b) Our formulation of theorem 4.1 is taken from [12, 19]. In the latter paper, the existence of bifurcating periodic solutions for a quasilinear equation is proven by using the Ljapunov–Schmidt reduction.

(c) We were not able to locate statement (4.26) in the literature, although it is certainly known.

## 5. STABILITY CONDITIONS. AN ALGORITHM

In this section we formulate conditions which guarantee that the bifurcating periodic solutions of theorem 4.1 are stable. These conditions will lead to an algorithm which involves (in principle) known information and is (in principle) numerically computable. The algorithm works for ordinary differential equations, for semilinear evolution equations and quasilinear parabolic evolution equations. Although stability conditions have been proven before for the first two classes of equations, it is hoped that our presentation will give some new insights even in this context. In any case, we do not know of similar results which were able to cover the equations presented in the Introduction. It is hoped that this section reveals some new insights even for ordinary differential equations in finite dimensional spaces. However, we are foremost interested in quasilinear reaction–diffusion systems. For the reader's convenience, we include the following results for two dimensional differential equations, which stand at the beginning of our analysis.

PROPOSITION 5.1. Let $\kappa(\cdot)$ be the unique local continuation of the eigenvalue $i\omega_0$ of $\partial_2 h(0, 0)$ along $\Lambda \times \{0\}$. Suppose that

$$\operatorname{Re} \dot{\kappa}(0) > 0,$$

where $h \in C^k(\Lambda \times X^c, X^c)$ denotes the map in (4.27). Let

$$(\lambda(\cdot), T(\cdot), x(\cdot)) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times X^c)$$

be the functions defined in the proof of theorem 4.1. Assume that

$$s\dot{\lambda}(s) > 0 \qquad \text{for } s > 0 \text{ (``supercritical bifurcation'')}.$$

Then each orbit $\gamma(s)$ is asymptotically stable.

If

$$s\dot{\lambda}(s) < 0 \qquad \text{for } s > 0 \text{ (``subcritical bifurcation'')},$$

then each of the orbits $\gamma(s)$, $0 < s < \varepsilon$, is unstable.

*Proof.* The assertions follow from [12, theorem 27.11] and $\sigma(\partial_2 h(0, 0))\backslash\{\pm i\omega_0\} = \emptyset$, since $\dim X^c = 2$ in our context. ■

Thanks to the results of Section 3, the recurrence of all solutions is contained in the locally invariant manifolds $\mathfrak{M}^c(\lambda)$. Hence, it suffices to state stability results for bifurcating solutions of the reduced equation $(4.10)_\lambda$. The next result, which is taken from [12, theorem 27.14], now describes conditions, whether supercritical or subcritical bifurcation occurs. We include it for the reader's convenience. Assume that

$$h \in C^k(\Lambda \times X^c, X^c) \qquad \text{with } k \geq 3.$$

We recall that

$$(X^c)_{\mathbb{C}} = \ker([\partial_2 h(0,0)]_{\mathbb{C}} - i\omega_0) \oplus \ker([\partial_2 h(0,0)]_{\mathbb{C}} + i\omega_0). \tag{5.1}$$

PROPOSITION 5.2. Suppose that we have chosen a basis $\{n_1, n_2\}$ on the two dimensional space $X^c$ such that the linear mapping $\partial_2 h(0,0)$ can be represented by

$$\begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}$$

with respect to this basis. Moreover, let

$$\varphi(x, y) := (\varphi^1(x, y), \varphi^2(x, y)) := h(0, (x, y)),$$

where $(x, y)$, respectively $(\varphi^1, \varphi^2)$, are the coordinates with respect to the basis $\{n_1, n_2\}$. Let

$$\begin{aligned}\delta := \omega_0[\partial_1^3\varphi^1 + \partial_1\partial_2^2\varphi^1 + \partial_1^2\partial_2\varphi^2 + \partial_2^3\varphi^2] - \partial_1^2\varphi^1\partial_1\partial_2\varphi^1 + \partial_1^2\varphi^1\partial_1^2\varphi^2 \\ - \partial_1\partial_2\varphi^1\partial_2^2\varphi^1 - \partial_2^2\varphi^1\partial_2^2\varphi^2 + \partial_1^2\varphi^2\partial_1\partial_2\varphi^2 + \partial_1\partial_2\varphi^2\partial_2^2\varphi^2.\end{aligned}$$

Then if $\delta < 0$, the occurring bifurcation is supercritical and the periodic orbits which emerge from the trivial solution are asymptotically stable. If $\delta > 0$, we have subcritical bifurcation.
Here, we use the notation $\partial_k\partial_j\varphi^i$ for $\partial_k\partial_j\varphi^i(0, 0)$ etc.

Let us now introduce some simplifying notation. We set

$$\phi_\lambda(u) := -A(\lambda, u)u + F(\lambda, u), \qquad \text{for } \lambda \in \Lambda,\ u \in X_1. \tag{5.2}$$

We can assume that $\Lambda$ has been decreased such that all occurring formulas are defined for $\lambda \in \Lambda$. Moreover, we suppress $\lambda$ in our notation whenever $\lambda = 0$. Note that $X_1$ and $X_0$ admit a decomposition, $X_1 = X^c \oplus X_1^s$ and $X_0 = X^c \oplus X_0^s$, into a stable subspace and the space $X^c$ (belonging to the eigenvalues $\{\pm i\omega_0\}$). We can identify an element $z \in X^c$ with its coordinates, $(x, y)$, in the fixed basis $\{n_1, n_2\}$. Moreover, we identify the direct sum with the product of its factors. Hence, we write

$$\phi(x, y, v) = (\phi^1(x, y, v), \phi^2(x, y, v), \phi^3(x, y, v)) := \phi_0(x, y, v),$$

where $((x, y), v) \in X^c \times X_1^s$. Moreover, $(\phi^1, \phi^2)$ denotes the coordinates of the part of $\phi$ in $X^c$, expressed in the basis $\{n_1, n_2\}$, and $\phi^3 \in X_0^s$ is the part of $\phi$ in $X_0^s$. We set

$$\begin{aligned} h(\lambda, s) &= \pi^c\phi_\lambda(z + \sigma(\lambda, z)), \\ \varphi^i(x, y) &= \phi^i(x, y, \sigma(x, y)), \qquad i \in \{1, 2\}, \end{aligned} \tag{5.3}$$

where we now write

$$\sigma(x, y) := \sigma(0, (x, y)), \tag{5.4}$$

slightly abusing notation. Remark that $\phi$ "stands for the full equation", while $\varphi$ denotes the equation reduced through the locally invariant manifolds, see (5.3). We show that derivatives of $\varphi$ can be expressed by the derivatives of $\phi$ and $\sigma$. Note that we only have to deal with $\lambda = 0$. We thus omit any reference to the parameter.

LEMMA 5.3. Let $\varphi$ be given by (5.3). Then

$$\partial_k\partial_j\varphi^i(0,0) = \partial_k\partial_j\phi^i(0,0,0), \qquad i,j,k\in\{1,2\},$$

$$\partial_l\partial_k\partial_j\varphi^i(0,0) = \partial_l\partial_k\partial_j\phi^i(0,0,0) + \partial_3\partial_j\phi^i(0,0,0)\partial_l\partial_k\sigma(0,0) + \partial_k\partial_3\phi^i(0,0,0)\partial_l\partial_j\sigma(0,0) + \partial_l\partial_3\phi^i(0,0,0)\partial_k\partial_j\sigma(0,0), \qquad i,j,k,l\in\{1,2\}.$$

*Proof.* Pick any $i,j,k,l\in\{1,2\}$. It follows from the chain rule that the derivative of (5.3) with respect to the $j$th coordinate is given by

$$\partial_j\varphi^i(x,y) = \partial_j\phi^i(x,y,\sigma(x,y)) + \partial_3\phi^i(x,y,\sigma(x,y))\partial_j\sigma(x,y).$$

Taking the derivative along the $k$th coordinate we now obtain

$$\begin{aligned}\partial_k\partial_j\varphi^i(x,y) &= \partial_k\partial_j\phi^i(x,y,\sigma(x,y)) + \partial_3\partial_j\phi^i(x,y,\sigma(x,y))\partial_k\sigma(x,y)\\ &\quad + \partial_k\partial_3\phi^i(x,y,\sigma(x,y))\partial_j\sigma(x,y) + \partial_3^2\phi^i(x,y,\sigma(x,y))\partial_k\sigma(x,y)\partial_j\sigma(x,y)\\ &\quad + \partial_3\phi^i(x,y,\sigma(x,y))\partial_k\partial_j\sigma(x,y).\end{aligned} \tag{5.5}$$

By evaluating (5.5) at $(x,y)=(0,0)$ we infer from the second part of (2.13) that the first assertion in lemma 5.3 holds true. We proceed by taking the derivative of (5.5) with respect to the $l$th coordinate and then evaluate at the point $(x,y)=(0,0)$. We now note that those terms containing first derivatives of $\sigma$ vanish, thanks to (2.13). Hence, we only have to take into consideration those terms which include derivatives of $\sigma$ of second and third orders. Moreover, we have to pay attention to $\partial_l\partial_k\partial_j\phi^i(0,0,0)$, which is found by taking the derivative $\partial_l$ of the very first term in (5.5). Observe that third order derivatives of $\sigma$ can only occur from the last term of (5.5), namely as

$$\partial_3\phi^i(0,0,0)\partial_l\partial_k\partial_j\sigma(0,0). \tag{5.6}$$

Note that $\partial_3\phi^i\partial_l\partial_k\partial_j\sigma(0,0) = \pi^i\partial\phi\partial_l\partial_k\partial_j\sigma(0,0) = \pi^i[-L\partial_l\partial_k\partial_j\sigma(0,0)]$, where $\pi^i$ denotes the projection to the one dimensional space span $\{n_i\}$. Since $\partial_l\partial_k\partial_j\sigma(0,0)\in X_1^s$, due to (2.12), and $-L$ is decomposed by $X_1^s$, we see that $-L\partial_l\partial_k\partial_j\sigma(0,0)\in X_0^s$ and, hence, lies in a complemented space of span $\{n_i\}$. Therefore, (5.6) does not provide a contribution. Finally, derivatives of second order of $\sigma$ (without additional first order derivatives) can only stem from the terms at the second, third and fifth place of the right-hand side of (5.5). Summarizing, we have proven lemma 5.3. ■

PROPOSITION 5.4.

$$\partial_k\partial_j\sigma(0,0) = \int_{-\infty}^0 e^{\tau L_s}\pi^s\partial_2^2 g(0,0)[e^{-\tau L_c}n_k, e^{-\tau L_c}n_j]\,d\tau.$$

More precisely, we have for $c(\tau) := \cos(\omega_0\tau)$ and $s(\tau) := \sin(\omega_0\tau)$

$$\partial_1^2\sigma(0,0) = \int_{-\infty}^0 e^{\tau L_s}[\partial_1^2\phi^3 c(\tau)^2 - 2\partial_1\partial_2\phi^3 c(\tau)s(\tau) + \partial_2^2\phi^3 s(\tau)^2]\,d\tau,$$

$$\partial_1\partial_2\sigma(0,0) = \int_{-\infty}^0 e^{\tau L_s}[\partial_1^2\phi^3 c(\tau)s(\tau) + \partial_1\partial_2\phi^3(c(\tau)^2 - s(\tau)^2) - \partial_2^2\phi^3 c(\tau)s(\tau)]\,d\tau,$$

$$\partial_2^2\sigma(0,0) = \int_{-\infty}^0 e^{\tau L_s}[\partial_1^2\phi^3 s(\tau)^2 + 2\partial_1\partial_2\phi^3 c(\tau)s(\tau) + \partial_2^2\phi^3 c(\tau)^2]\,d\tau,$$

where $\partial_k\partial_j\phi^i := \partial_k\partial_j\phi^i(0,0,0)$ and $g = g(\lambda,u)$ for $(\lambda,u)\in\Lambda\times X_1$.

*Proof.* We will again suppress $\lambda = 0$ in our notation and simply write $\sigma(x, y) = \sigma(0, (x, y))$ and

$$g((x, y), v) := g(0, ((x, y), v)), \tag{5.7}$$

where $((x, y), v) \in X^c \times X_1^s$. Note that the functions $g$ and $g_\rho$ coincide for small values of $(x, y)$. Taking derivatives at 0 only requires information within an arbitrary small neighborhood of 0. Hence, we can always use the "original" function $g$ instead of the modified $g_\rho$ which was actually needed for the fixed point argument of Section 2. Note that $g(\cdot, \cdot) \in C^k(X^c \times X_1^s, X_0^s)$, where we again identify $X^c \times X_1^s$ with $X^c \oplus X_1^s$. Theorem 2.1 shows that $\sigma$ satisfies the (fixed point) equation

$$\sigma(x, y) = \int_{-\infty}^{0} e^{\tau L_s} \pi^s g(z(\tau, (x, y)), \sigma(z(\tau, (x, y)))) \, d\tau. \tag{5.8}$$

Let $h, k \in X^c$ be given and let $\mathbf{x} := (x, y) \in X^c$. Then we may differentiate (5.8) and we first get

$$\partial\sigma(\mathbf{x})h = \int_{-\infty}^{0} e^{\tau L_s} \pi^s \partial g(z(\tau, \mathbf{x}), \sigma(z(\tau, \mathbf{x})))[\partial_2 z(\tau, \mathbf{x})h + \partial\sigma(z(\tau, \mathbf{x}))\partial_2 z(\tau, \mathbf{x})h] \, d\tau,$$

and also

$$\begin{aligned}\partial^2\sigma(\mathbf{x})[h, k] = \int_{-\infty}^{0} & e^{\tau L_s} \pi^s \{\partial^2 g(z(\tau, \mathbf{x}), \sigma(z(\tau, \mathbf{x}))) \\ & \times [\partial_2 z(\tau, \mathbf{x})h + \partial\sigma(z(\tau, \mathbf{x}))\partial_2 z(\tau, \mathbf{x})h, \partial_2 z(\tau, \mathbf{x})k + \partial\sigma(z(\tau, \mathbf{x}))\partial_2 z(\tau, \mathbf{x})k] \\ & + \partial g(z(\tau, \mathbf{x}), \sigma(z(\tau, \mathbf{x})))[\text{derivatives of } \sigma(z(\tau, \cdot)) \text{ and } z(\tau, \cdot)]\} \, d\tau.\end{aligned}$$

Recall that $z(\cdot, \mathbf{x})$ is the solution of the (ordinary) differential equation

$$\dot{z}(t) = -L_c z(t) + g(z(t), \sigma(z(t))), \qquad t \in \mathbb{R}, \qquad z(0) = \mathbf{x} \tag{5.9}$$

on $X^c$. Owing to (2.11) (which actually is a restatement of the very same property of $g$) and (2.13), it follows that $z(t, 0) \equiv 0$ is the solution of (5.9) with initial value $\mathbf{x} = (0, 0)$. (2.13) then tells us that $\sigma(z(\tau, 0)) \equiv 0$. We can then infer from (2.11) that

$$\partial g(z(\tau, 0), \sigma(z(\tau, 0))) = 0, \qquad \tau \in \mathbb{R}.$$

Hence,

$$\partial^2\sigma(0, 0)[h, k] = \int_{-\infty}^{0} e^{\tau L_s} \pi^s \partial^2 g(0, 0)[\partial_2 z(\tau, 0)h, \partial_2 z(\tau, 0)k] \, d\tau. \tag{5.10}$$

We will now give a representation for the derivative $\partial_2 z(\tau, 0)$ of the solution to (5.9). Using a well-known result in the theory of ordinary differential equations, cf. [12, theorem 9.2], and the fact that $\partial g(0, 0) = 0$, we obtain that $\partial_2 z(\cdot, 0)$ solves the linearized problem

$$\dot{v}(t) = -L_c v(t), \qquad t \in \mathbb{R}, \qquad v(0) = \mathrm{id}_{X^c},$$

in $\mathcal{L}(X^c)$. It then follows that

$$\partial_2 z(\tau, 0)h = e^{-\tau L_c} h \qquad \text{for } \tau \in \mathbb{R} \tag{5.11}$$

and we can write

$$\partial^2\sigma(0,0)[h,k] = \int_{-\infty}^{0} e^{\tau L_s}\pi^s\partial^2 g(0,0)[e^{-\tau L_c}h, e^{-\tau L_c}k]\,d\tau. \tag{5.12}$$

The first assertion of proposition 5.4 is now an immediate consequence of the well-known relation

$$\partial_k\partial_j\sigma(0,0) = \partial^2\sigma(0,0)[n_k, n_j],$$

where $\partial_j$ and $\partial_k$ denote the partial derivatives with respect to the $j$th and $k$th coordinates in the basis $\{n_1, n_2\}$. (Note again that $\sigma(x,y) := \sigma(0,(x,y))$ in slight abuse of the notation of Section 2.) Using the representation of $\partial_2 h(0,0) = -L_c$ with respect to the basis $\{n_1, n_2\}$, cf. proposition 5.2, we obtain

$$e^{-\tau L_c} = \begin{bmatrix} \cos(\omega_0\tau) & \sin(\omega_0\tau) \\ -\sin(\omega_0\tau) & \cos(\omega_0\tau) \end{bmatrix}, \qquad \tau\in\mathbb{R}. \tag{5.13}$$

If $c(\tau) := \cos(\omega_0\tau)$ and $s(\tau) := \sin(\omega_0\tau)$, we then get

$$e^{-\tau L_c}n_1 = (c(\tau), -s(\tau)), \qquad e^{-\tau L_c}n_2 = (s(\tau), c(\tau)), \qquad \tau\in\mathbb{R}. \tag{5.14}$$

Let

$$g((x,y),v) = (g^1((x,y),v), g^2((x,y),v), g^3((x,y),v)), \qquad ((x,y),v)\in X^c\times X_1^s$$

denote the components of $g := g(0,\cdot)$ in $X^c\times X_0^s$. (Note that we drop $\lambda = 0$ from our notation and collect $(x,y)\in X^c$). It is obvious that

$$\pi^s\partial^2 g(0,0) = \partial^2\pi^s g(0,0) = \partial^2 g^3(0,0), \tag{5.15}$$

where $\partial^2$ stands for the second order Fréchet derivative of $g\in C^k(X^c\times X_1^s, X_0^s)$. Note that

$$g((x,y),v) = \phi((x,y),v) + \operatorname{diag}(L_c, L_s)((x,y),v), \qquad ((x,y),v)\in X^c\times X_1^s,$$

due to the definition of $g$ and $\phi$, cf. (2.8) and (5.2), thus, showing that the second derivatives of $g$ and $\phi$ coincide. With (5.15) we can restate (5.12) as

$$\partial^2\sigma(0,0)[h,k] = \int_{-\infty}^{0} e^{\tau L_s}\partial^2\phi((0,0),0)[e^{-\tau L_c}h, e^{-\tau L_c}k]\,d\tau. \tag{5.16}$$

We are left to give a representation of $\partial^2\phi^3(0,0,0)[e^{-\tau L_c}n_1, e^{-\tau L_c}n_2]$. Observe that $\partial^2\phi^3(0,0,0)$ denotes the "total" (Fréchet) derivative of $\phi^3\in C^k(X^c\times X_1^s, X_0^s)$. (We do not hesitate to change notation and write $\phi^3(0,0,0)$ for $\phi^3((0,0),0)$.) Note first that the group $e^{-\tau L_c}$ leaves the space $X^c$ invariant. Thus, there will be no contribution from the "partial" derivative with respect to the "third" variable $v\in X_1^s$. We only have to take

$$\partial_{X^c}^2\phi^3(0,0,0)[e^{-\tau L_c}n_1, e^{-\tau L_c}n_2]$$

into consideration, and, thus, end up with derivatives with respect to $x$ and $y$. An easy computation, using (5.14), shows

$$\begin{aligned}\partial_{X^c}^2\phi^3(0,0,0)&[c(\tau)n_1 - s(\tau)n_2, s(\tau)n_1 + c(\tau)n_2] \\ &= \partial_1^2\phi^3 c(\tau)s(\tau) + \partial_1\partial_2\phi^3(c(\tau)^2 - s(\tau)^2) - \partial_2^2\phi^3 c(\tau)s(\tau),\end{aligned}$$

where we simplify notation by $\partial_1\partial_2\phi^3 := \partial_1\partial_2\phi^3\phi^3(0,0,0)$. The remaining two formulas of proposition 5.4 follow by adjusting the last step to this situation. ■

*Remarks* 5.5. (a) Let

$$f(t) := \partial_1^2\phi^3 c(t)s(t) + \partial_1\partial_2\phi^3(c(t)^2 - s(t)^2) - \partial_2^2\phi^3 c(t)s(t) \tag{5.17}$$

and let $v(\cdot) \in BC(\mathbb{R}, X_1^s) \cap BC^1(\mathbb{R}, X_0^s)$ be the (unique) bounded solution of the Cauchy problem

$$\dot{v}(t) + L_s v(t) = f(t), \qquad t \in \mathbb{R}.$$

Then we obtain

$$\partial_1\partial_2\sigma(0, 0) = v(0),$$

due to the representation of $\partial_1\partial_2\sigma(0, 0)$ in proposition 5.4 and [5, theorem 2.4]. This shows that $\partial_1\partial_2\sigma(0, 0)$ is the value of the (uniquely determined) bounded solution $v(\cdot)$, evaluated at zero. Analogous statements hold true for the remaining terms of proposition 5.4. Note that the "formulas" of proposition 5.4 involve derivatives $\partial_j\partial_k\phi^3$ and the knowledge, how the semigroup $e^{-tL}$ acts on the stable subspace $X_0^s$. These are (in principle) known "quantities". Moreover, the integrals appearing in proposition 5.4 are (in principle) numerically computable. We have, thus, derived an algorithm which is able to provide computable conditions for stability. The algorithm comes out of our proofs and statements. We just have to follow the procedure stated in proposition 5.2 and then use the expressions in lemma 5.3 and proposition 5.4. An application to a "concrete" quasilinear reaction–diffusion system will be given elsewhere.

(b) The situation simplifies if

$$\partial_k\partial_j\phi^3(0, 0, 0) = 0, \qquad j, k \in \{1, 2\}.$$

Proposition 5.4 then shows that $\partial_k\partial_j\sigma(0, 0) = 0$ and we can infer from lemma 5.3 that

$$\begin{aligned}\delta = \omega_0[\partial_1^3\phi^1 + \partial_1\partial_2^2\phi^1 + \partial_1^2\partial_2\phi^2 + \partial_2^3\phi^2] - \partial_1^2\phi^1\partial_1\partial_2\phi^1 + \partial_1^2\phi^1\partial_1^2\phi^2 \\ - \partial_1\partial_2\phi^1\partial_2^2\phi^1 - \partial_2^2\phi^1\partial_2^2\phi^2 + \partial_1^2\phi^2\partial_1\partial_2\phi^2 + \partial_1\partial_2\phi^2\partial_2^2\phi^2.\end{aligned} \tag{5.18}$$

If $\partial_k\partial_j\phi^i(0, 0, 0) = 0$ for each combination of $i \in \{1, 2, 3\}$, $j, k \in \{1, 2\}$, $\delta$ has the simple form

$$\delta := \omega_0[\partial_1^3\phi^1 + \partial_1\partial_2^2\phi^1 + \partial_1^2\partial_2\phi^2 + \partial_2^3\phi^2], \tag{5.19}$$

where $\partial_l\partial_k\partial_j\phi^i = \partial_l\partial_k\partial_j\phi^i(0, 0, 0)$.

(c) The stability condition of proposition 5.2, which always holds for a two dimensional ordinary differential equation, is taken from [12, theorem 27.14]. It corresponds to [6, p. 126], except for a different scaling factor. The derivation in [12] simplifies the calculations in [6, pp. 111–126] a great deal, see the remark on p. 125 of [6]. Our conditions in proposition 5.4, though, are different from the formula (4A.6) of [6, p. 134], which seems to work best only in three dimensional spaces. We would also like to draw attention to the references quoted in [28].

We can now combine the results of the last sections and state our main result on Hopf bifurcation for (abstract) quasilinear evolution equations. Assume that the mapping in (3.21) generates a semiflow on an open subset $U$ of $X$, where $X$ can be any space lying between $X_1$ and $X_\alpha$ (including both of them). Then we have the following theorem.

THEOREM 5.6 (Hopf bifurcation for quasilinear equations). Let

$$\mu(\lambda) \in \sigma(-[A(\lambda, 0) - \partial_2 F(\lambda, 0)])$$

be the (unique) local continuation of the eigenvalue $i\omega_0$ of $-[A(0,0) - \partial_2 F(0,0)]$ along $\Lambda \times \{0\}$. Assume that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\operatorname{Re}\mu(\lambda))|_{\lambda=0} > 0, \tag{5.20}$$

$$\sigma(-[A(\lambda, 0) - \partial_2 f(\lambda, 0)])\backslash\{\pm i\omega_0\} \subset [\operatorname{Re} z < 0] \tag{5.21}$$

for sufficiently small values of $\lambda$.

Then the quasilinear equation $(4.1)_\lambda$ has in a neighborhood of $(0,0) \in \Lambda \times X$ a unique one-parameter family $\{\Gamma(s); 0 < s < \varepsilon\}$ of nontrivial periodic orbits which tend towards 0 as $s \to 0$. More precisely, there exists $\varepsilon > 0$ and a mapping

$$(\lambda(\cdot), T(\cdot), u(\cdot)) \in C^{k-1}((-\varepsilon, \varepsilon), \mathbb{R} \times \mathbb{R} \times X)$$

satisfying

$$(\lambda(0), T(0), u(0)) = (0, 2\pi/\omega_0, 0),$$

such that

$$\Gamma(s) := \Gamma(u(s))$$

is a nontrivial orbit of $(4.1)_{\lambda(s)}$ of period $T(s)$ passing through $u(s) \in X$ for each $0 < s < \varepsilon$. If $0 < s_1 < s_2 < \varepsilon$, then $\Gamma(s_1) \neq \Gamma(s_2)$.

The family $\{\Gamma(s); 0 < s < \varepsilon\}$ contains every nontrivial periodic orbit of $(4.1)_\lambda$ lying in a suitable neighborhood of $(0, T(0), 0) \in \Lambda \times \mathbb{R} \times X$.

If

$$s\dot{\lambda}(s) > 0 \qquad \text{for } s > 0 \text{ (supercritical bifurcation)},$$

then each orbit $\gamma(s)$ is asymptotically stable in $X$.

If

$$s\dot{\lambda}(s) < 0 \qquad \text{for } s > 0 \text{ (subcritical bifurcation)},$$

then every orbit $\gamma(s)$, $0 < s < \varepsilon$, is unstable in $X$.

Finally, the stability algorithm derived in proposition 5.2, lemma 5.3 and proposition 5.4 applies.

*Proof.* Note that (5.20) and (4.26) imply $[\operatorname{Re}\dot{\kappa}(0)] > 0$, where $\kappa(\cdot)$ denotes the continuation of the eigenvalue $i\omega_0$ of $\partial_2 h(0,0)$. The results then follow from theorem 4.1, proposition 5.1 and theorem 3.3 (cf. also the last part of the proof of theorem 4.1). Moreover, proposition 5.2, lemma 5.3 and proposition 5.4 apply, giving the quoted stability algorithm. ∎

## 6. PROOF OF THEOREMS 1.1 AND 1.2

We will briefly explain how the quasilinear reaction-diffusion system $(1.1)_\lambda$ fits into the abstract framework of Sections 2–5. This part relies on the work carried out in [5, Sections 7–8]. We shall show that there exist Banach spaces $X_1$ and $X_0$, satisfying all the assumptions of Sections 2–3. Our approach to handling the nonlinear boundary conditions uses, implicitly, the extrapolation setting, cf. [1, 2, 29]. It is shown in [5] that we can find suitable defined extrapolation spaces such that the device of maximal regularity can be used.

Let $(\mathcal{A}(\lambda, u), \mathcal{B}(\lambda, u))$ be a (formal) boundary value system, where $\mathcal{A}(\lambda, u)$ is defined in (1.3) and $\mathcal{B}(\lambda, u)$ stands for the boundary operator introduced in (1.5). Note that we may identify

$$(\mathcal{A}(\lambda, u), \mathcal{B}(\lambda, u)) \leftrightarrow ((a_{jk}(\lambda, \cdot, u)), (a_j(\lambda, \cdot, u)), a_0(\lambda, \cdot, u), b_0(\lambda, \cdot, u)), \tag{6.1}$$

where $\lambda \in \Lambda$ and $u$ belongs to an appropriate function space, say $u \in C(\bar{\Omega}, G)$ to fix ideas. Assume for the moment that $\lambda$ and $u$ are fixed. Then let $X := \mathcal{L}(\mathbb{R}^N)$ and set

$$\mathbb{E}_p^\rho(\Omega) := C^\rho(\bar{\Omega}, X)^{n^2} \times C^\rho(\bar{\Omega}, X)^n \times L_p(\Omega, X) \times L_p(\partial\Omega, X) \tag{6.2}$$

with a generic element $e := ((a_{jk}), (a_j), a_0, b_0)$, where $\rho \in (0, 1)$. In general, we write

$$e(\lambda, u) := ((a_{jk}(\lambda, \cdot, u)), (a_j(\lambda, \cdot, u)), a_0(\lambda, \cdot, u), b_0(\lambda, \cdot, u)), \tag{6.3}$$

where $(\lambda, u) \in \Lambda \times C(\bar{\Omega}, G)$. Note that (6.2) defines a topology on the set of linear (formal) boundary value systems. Let $\mathcal{E}_p^\rho(\Omega)$ denote the (open) subset of $\mathbb{E}_p^\rho(\Omega)$, given by

$$\begin{gathered}((a_{jk}(\lambda, \cdot, \eta)), \delta(a_{jk}(\lambda, \cdot, \eta)\nu^j) + (1 - \delta)),\\ ((a_{jk}(\lambda, x_0, \eta)), \delta(a_{jk}(\lambda, x_0, \eta)\nu^j) + (1 - \delta)),\end{gathered} \tag{6.4}$$

define normally elliptic boundary value systems for (each) $(\lambda, \eta) \in \Lambda \times G$ and each $x_0 \in \Omega$. For details, see [1–3], especially for the proof that $\mathcal{E}_p^\rho(\Omega)$ is open in $\mathbb{E}_p^\rho(\Omega)$.

In the following, all of the indicated function spaces are assumed to consist of $\mathbb{R}^N$-valued functions. For $s \in (0, 2)$ and $p \in (1, \infty)$, let $b_{p,\infty}^s(\Omega)$ be the *little Nikol'skii* spaces, defined by

$$b_{p,\infty}^s(\Omega) := (L_p(\Omega), H_p^2(\Omega))_{s/2,\infty}^0, \tag{6.5}$$

where $(\cdot, \cdot)_{\theta,\infty}^0$ denotes the continuous interpolation method, see [3, 29] and [5, 24]. Moreover, let us define the *Besov spaces*

$$B_{p,1}^s(\Omega) := (L_p(\Omega), H_p^2(\Omega))_{s/2,1}, \tag{6.6}$$

where $(\cdot, \cdot)_{\theta,1}$ stands for the real interpolation method, cf. [30, 31]. It should be noted that (6.5) and (6.6) differ slightly from the definitions given in [5, Section 6]. However, using reiteration properties of the real and the continuous interpolation method, it follows that the spaces coincide (except for equivalent norms). Finally, we set

$$b_{p,\infty,\mathcal{B}}^s(\Omega) := \{u \in b_{p,\infty}^s(\Omega); (1 - \delta)\gamma u = 0\}, \qquad s \in (1/p, 1 + 1/p). \tag{6.7}$$

(The restriction $s < 1 + 1/p$ avoids conflicts with the notation in some of the quoted papers.) Note that the function $\delta$, the boundary characterization map, appears in (1.5). The space $B_{p,1,\mathcal{B}}^s(\Omega)$ is defined analogously, where we also take $s \in (1/p, 1 + 1/p)$. We can now set

$$b_{p,\infty,\mathcal{B}}^{s-2}(\Omega) := \mathrm{cl}(L_p(\Omega)) \qquad \text{in } (B_{p',1,\mathcal{B}}^{2-s}(\Omega))',\ 1/p < s < 1 + 1/p, \tag{6.8}$$

where the duality pairing is induced by the standard $L_{p'} \times L_p$ pairing. Note that the Besov space $B_{p',1,\mathcal{B}}^{2-s}(\Omega)$ is densely embedded in $L_{p'}(\Omega)$. Therefore,

$$L_p(\Omega) \hookrightarrow (B_{p',1,\mathcal{B}}^{2-s}(\Omega))',$$

(by identifying, as usual, the dual of $L_{p'}(\Omega)$ with $L_p(\Omega)$). Hence, (6.8) is well defined. Note that the space $B_{p',1}^{2-s}(\Omega)$ (and, thus, $B_{p',1,\mathcal{B}}^{2-s}(\Omega)$) is not reflexive. An obvious density argument connects the definition (6.8) with [5, lemma 7.6]. In general, i.e. if $\delta$ is not the zero matrix, $b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)$ is no longer a space of distributions. Assume from now on that

$$p \in (n,\infty), \qquad 1/p < s < 1 + 1/p, \qquad \rho = \rho(s) > |s-1|. \tag{6.9}$$

It has been proved in [5, theorem 7.11] that there exists for each $e \in \mathcal{E}^{\rho}_{p}(\Omega)$ an operator

$$A(e) \in \mathfrak{M}_{\alpha}(b^{s}_{p,\infty,\mathcal{B}}(\Omega), b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)), \tag{6.10}$$

(with $0 < \alpha \leq 1$ arbitrary), such that the mapping

$$\mathcal{E}^{\rho}_{p}(\Omega) \to \mathcal{L}(b^{s}_{p,\infty,\mathcal{B}}(\Omega), b^{s-2}_{p,\infty,\mathcal{B}}(\Omega)), \qquad [e \mapsto A(e)], \tag{6.11}$$

is analytic. Moreover,

$$\langle w, A(e)u\rangle = a(e)(w,u), \qquad e \in \mathcal{E}^{\rho}_{p}(\Omega), \qquad (w,u) \in (b^{s-2}_{p,\infty,\mathcal{B}}(\Omega))' \times b^{s}_{p,\infty,\mathcal{B}}(\Omega), \tag{6.12}$$

where

$$a(e)(e,u) := \int_{\Omega} \{\langle \partial_j w, a_{jk}\partial_k u\rangle + \langle w, a_j\partial_j u + a_0 u\rangle\}\,dx + \int_{\partial\Omega} \langle \gamma w, b_0 \gamma u\rangle\, d\sigma$$

denotes the Dirichlet form associated with $e \in \mathbb{E}^{\rho}_{p}(\Omega)$, and $\langle \xi, \eta\rangle$, $\xi, \eta \in \mathbb{R}^N$, stands for the duality pairing in $\mathbb{R}^N$. Note that $(b^{s-2}_{p,\infty,\mathcal{B}}(\Omega))' \doteq B^{2-s}_{p,1,\mathcal{B}}(\Omega)$ and that the Dirichlet form $a(e)$ is (i.e can be extended to be) continuous and bilinear on each of the spaces $B^{2-s}_{p,1,\mathcal{B}}(\Omega) \times b^{s}_{p,\infty,\mathcal{B}}(\Omega)$, cf. [5, corollary 7.3]. Hence, (6.12) is meaningful. Observe that (6.10) is a statement on maximal regularity. We now fix

$$n/p < r < 1 < s_0 < 1 + 1/p, \qquad s_0 - 1 < \rho < (r - n/p) \wedge 1/p \tag{6.13}$$

and

$$\alpha = 1 - (s_0 - 1)/2, \qquad \beta \in (1 - (s_0 - r)/2, \alpha). \tag{6.14}$$

Then let

$$X_1 := b^{s_0}_{p,\infty,\mathcal{B}}(\Omega) \qquad \text{and} \qquad X_0 := b^{s_0-2}_{p,\infty,\mathcal{B}}(\Omega). \tag{6.15}$$

It can be shown that

$$X_{\alpha} \doteq b^{1}_{p,\infty,\mathcal{B}}(\Omega), \qquad X_{\beta} \doteq b^{s_0-2(\beta-1)}_{p,\infty,\mathcal{B}}(\Omega),$$

see [5, proposition 7.13]. Moreover,

$$X_{\beta} \hookrightarrow H^{r}_{p,\mathcal{B}}(\Omega) \hookrightarrow C^{r-n/p}(\bar{\Omega}), \tag{6.16}$$

thanks to (6.13), (6.14), Sobolev's embedding theorem and [5, (7.22)]. Thus,

$$U_{\beta} := \{u \in X_{\beta};\, u(\bar{\Omega}) \subset G\} \tag{6.17}$$

is an open subset of $X_{\beta}$. Finally, we define

$$A(\lambda,u) := A(e(\lambda,u)), \qquad F(\lambda,u) := f(\lambda,\cdot,u) + \gamma' g(\lambda,\cdot,u), \qquad (\lambda,u) \in \Lambda \times U_{\beta} \tag{6.18}$$

where $\gamma'$ denotes the dual of the trace operator

$$\gamma \in \mathcal{L}(H^{\sigma}_{p',\mathcal{B}}(\Omega), B^{\sigma-1/p'}_{p',p'}(\partial\Omega)), \qquad 1 - 1/p < \sigma < 2 - s_0.$$

Note that the mapping

$$\begin{aligned} &\Lambda \times U_\beta \to \mathcal{E}_p^\rho(\Omega), \\ &(\lambda, u) \mapsto e(\lambda, u) := ((a_{jk}(\lambda, \cdot, u)), (a_j(\lambda, \cdot, u)), a_0(\lambda, \cdot, u), b_0(\lambda, \cdot, u)) \end{aligned} \tag{6.19}$$

is $C^\infty$, as can be seen from an analogous argument in [5, proposition 8.1]. An inspection of [5, lemma 8.4] shows that

$$[(\lambda, u) \mapsto F(\lambda, u)] \in C^\infty(\Lambda \times U_\beta, X_0). \tag{6.20}$$

We can now infer from (6.10), (6.11) and (6.19), (6.20) that the assumptions (2.2) and (2.3) of Section 2 are satisfied. Moreover, (1.15) and (6.18) imply that (2.4) holds as well. It should be noted that solutions of the quasilinear problem

$$\dot{u}(t) + A(\lambda, u(t))u(t) = F(\lambda, u(t)), \qquad t > 0, \lambda \in \Lambda \tag*{$(6.21)_\lambda$}$$

are, in particular, weak solutions of the quasilinear reaction–diffusion system $(1.1)_\lambda$. Using the smoothing property of the "parabolic semiflow", we obtain the result that solutions of $(6.21)_\lambda$ are in fact classical solutions of $(1.1)_\lambda$, cf. [3, 19, 29]. Finally, it is not difficult to see that the spectrum of $-L := -(A(0,0) - \partial_2 F(0,0))$ corresponds to the eigenvalues of the eigenvalue problem $(1.16)_0$. Thus, condition (1.18) can indeed be recovered in (2.6). Moreover, the space $X^c$, according to the spectral projection $\pi^c$, can be described by (1.19). Hence, theorem 1.1 follows from theorem 2.1 and theorem 2.3 (see also remark 2.2).

It is now clear that the assumptions of theorem 1.2 are recovered in (4.2)–(4.6). Moreover, $X := H^1_{p,\mathcal{B}}(\Omega)$ satisfies the assumption in (3.21), due to (1.14) and $X_1 \hookrightarrow x \hookrightarrow X_\alpha$, see [5, Sections 6 and 8]. Theorem 1.2 now follows from theorem 5.6. ■

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