Hamiltonicity of graphs on surfaces

Kenta Ozeki

(National Institute of Informatics, Japan)

Partially Joint Work with

J. Fujisawa (Keio univ.), K. Kawarabayashi (NII),
A. Nakamoto (Yokohama N. Univ.), K. Ota (Keio Univ.),
P. Vrána (West Bohemia Univ.)
Hamilton cycles

Hamilton cycle: Connection to TSP or other topics

Hamiltonicity of graphs on a surface

Tait (1884):

\[ \exists \text{Hamilton cycles in } \forall \text{ cubic map} \quad \Downarrow \quad \exists \text{4-coloring in } \forall \text{ map} \]

\[ \quad \text{False} \quad \text{True (4 Color Thm)} \]
**Toughness of a graph**

Toughness (type) condition: Necessary condition!!

\[ (*) \forall S \subseteq V(G): \text{cutset}, \ (# \text{ of components in } G - S) \leq a|S| + b \]

\forall \text{ graph } G \text{ satisfies } (*) \text{ with }

\[ (a, b) = (1, 1) \text{ if } G : \exists \text{ Hamilton path} \]

\[ = (1, 0) \quad \exists \text{ Hamilton cycle} \]

\[ = (1, -1) \quad \text{Hamilton connected} \]

\[ (# \text{ comp.s in } G - S) \leq |S| \]
## Toughness of a graph

**Toughness (type) condition:**

\[(\ast) \forall S \subset V(G): \text{cutset}, (\# \text{ of components in } G - S) \leq a|S| + b\]

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, -t))</td>
<td>∃ H-cycle if we delete (\forall t) vertices</td>
</tr>
<tr>
<td>((1, -t + 1))</td>
<td>(t)-leaf connected</td>
</tr>
<tr>
<td>((1, -1))</td>
<td>Hamilton-connected</td>
</tr>
<tr>
<td>((1, 0))</td>
<td>∃ Hamilton cycles</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>∃ Hamilton path</td>
</tr>
<tr>
<td>((1, t - 1))</td>
<td>∃ Special tree with (\leq t) leaves</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>((a, b))</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>((k, 0))</td>
<td>∃ Special closed walk passing (\forall) vrt. (\leq k) times</td>
</tr>
<tr>
<td>((k, 0))</td>
<td>∃ vertex cover with (\leq k) cycles</td>
</tr>
<tr>
<td>((k, 0))</td>
<td>(k)-prism Hamiltonian</td>
</tr>
<tr>
<td>((k - 1, 1))</td>
<td>∃ Special tree with max. deg. (\leq k)</td>
</tr>
<tr>
<td>((k - 1, t + 1))</td>
<td>∃ Special tree with te (\leq t) from (k)</td>
</tr>
</tbody>
</table>

Related to ∃ 1-factor, matching extension etc.
Toughness of a graph

Toughness (type) condition:

\[(*) \forall S \subset V(G): \text{cutset}, \ (# \text{ of components in } G - S) \leq a|S| + b\]

We expect that a graph satisfying \((*)\) has a good property

Ex. \(\exists H\)-cycle, \(\exists H\)-path, … \(\Rightarrow \) ???

**Conjecture (Chvátal `73)**

\[\exists t : \text{integer s.t. } \forall G \text{ satisfying } (*)\]

\[\text{with } (a, b) = \left(\frac{1}{t}, 0\right) \text{ has an } H\text{-cycle}\]

But how about graphs on surfaces??
### Graphs on surfaces

| Surface          | Euler Characteristic | Graph on a surface, $|V(G)| - |E(G)| + |F(G)| \geq \chi$ |
|------------------|----------------------|--------------------------------------------------|
| Plane            | $\chi = 2$          |                                                  |
| Proj. Plane      | $\chi = 1$          |                                                  |
| Torus            | $\chi = 0$          |                                                  |
| Klein bottle     | $\chi = 0$          |                                                  |
| Others           | $\chi < 0$          |                                                  |
Graphs on surfaces

\[ \forall F^2: \text{a surface } \quad 3 \leq k \leq 5 \]
\[ \forall G: k\text{-conn. graph on } F^2 \]
\[ \forall S: \text{a vertex set of } G \]

\[ \# \text{ comp.s in } G - S \leq \frac{2}{k - 2} |S| + \frac{-2\chi(F^2)}{k - 2} \]

\[ |E(R)| \geq kt \]
\[ |E(R)| \leq 2(|S| + t) - 2\chi \]

holds for \( (a, b) = \left( \frac{2}{k - 2}, \frac{-2\chi}{k - 2} \right) \)
Graphs on surfaces

∀ $F^2$: a surface $3 \leq k \leq 5$

∃ $G$: $k$-conn. triangulation

∃ $S$: a vertex set of $G$

s.t.

$\# \text{ comp. s in } G - S$

$= |F(H)| = \frac{2}{k - 2}|S| + \frac{-2\chi(F^2)}{k - 2}$

$G$: $k$-conn. graph on $F^2$

$H$: Tri/quadl/pent-anguration of $F^2$

$G$: Face sub. of $H$

$S = V(H)$

$(\# \text{ comp. s in } G - S) \leq a|S| + b$

holds for $(a, b) = \left(\frac{2}{k - 2}, \frac{-2\chi}{k - 2}\right)$
Graphs on surfaces

- Graphs on surfaces, connectivity and toughness

\[ G : k\text{-conn. graph on } F^2 \]

\[ (\# \text{ comp.s in } G - S) \leq a|S| + b \text{ holds for } (a, b) = \left( \frac{2}{k-2}, \frac{-2\chi}{k-2} \right) \]

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement: \[ \forall k\text{-conn. graph on a surface } F^2 \text{ is } [...] \]

We have to think about toughness type necessary condition

What else do we need?
Graphs on surfaces

Nothing???

There are NO known counterexample to the following type statement,
EXCEPT for the toughness reason

Properties concerning Hamiltonicity (and toughness)

Statement: \( \forall k\text{-conn. graph on a surface } F^2 \text{ is } [...] \)

We have to think about toughness type necessary condition
What else do we need?

I’ll show several results and conjectures as “evidences” for it
Graphs on surfaces

Nothing???
There are NO known counterexample to the following type statement, EXCEPT for the toughness reason

Properties concerning Hamiltonicity (and toughness)

Statement: $\forall k$-conn. graph on a surface $F^2$ is $[...]$

Ex. $\exists$ Hamilton cycle $\Rightarrow (a, b) = (1, 0)$

I’ll show several results and conjectures as “evidences” for it
Graphs on surfaces, connectivity and toughness

- \( G : k \)-conn. graph on \( F^2 \)

\[ (\# \text{ comp. s in } G - S) \leq a|S| + b \]

\( k_c = 4 \)  
Necessary condition

\[ b \geq -\chi \Rightarrow \text{OK.} \]
\[ b < -\chi \Rightarrow \text{Not satisfied} \]

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement: \( \forall k \)-conn. graph on a surface \( F^2 \) is [...] 

Ex. \( \exists \) Hamilton cycle \( \rightarrow (a, b) = (1, 0) \rightarrow k_c = 4 \)

Graphs on a surface with \( \chi \geq 0 \) satisfies the necessary condition
### 4-conn. graphs on surfaces \((\alpha = 1)\)

<table>
<thead>
<tr>
<th></th>
<th>Plane (\chi = 2)</th>
<th>Proj. plane (\chi = 1)</th>
<th>Torus (\chi = 0)</th>
<th>K-bottle (\chi = 0)</th>
<th>(N_3) (\chi = -1)</th>
<th>Others (\chi &lt; -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(k = 4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(b \geq -\chi \Rightarrow) OK.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>H-cycle</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(b &lt; -\chi \Rightarrow) Not satisfied</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>H-conn</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = -1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(\chi\) is the Euler characteristic of the surface.
4-conn. graphs on surfaces \((a = 1)\)

<table>
<thead>
<tr>
<th></th>
<th>Plane (\chi = 2)</th>
<th>Proj. plane (\chi = 1)</th>
<th>Torus (\chi = 0)</th>
<th>K-bottle (\chi = 0)</th>
<th>(N_3) (\chi = -1)</th>
<th>Others (\chi &lt; -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-path</td>
<td>(b = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>Thomas, Yu &amp; Zang ('05)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H-cycle</td>
<td>(b = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Tutte ('56)</td>
<td>○</td>
<td>Thomas &amp; Yu ('94)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H-conn</td>
<td>(b = -1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Thomassen ('83)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## 4-connected graphs on surfaces ($\alpha = 1$)

<table>
<thead>
<tr>
<th></th>
<th>Plane $\chi = 2$</th>
<th>Proj. plane $\chi = 1$</th>
<th>Torus $\chi = 0$</th>
<th>K-bottle $\chi = 0$</th>
<th>$N_3$ $\chi = -1$</th>
<th>Others $\chi &lt; -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-path</td>
<td></td>
<td></td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
<td>$\bigcirc$</td>
</tr>
<tr>
<td>$b = 1$</td>
<td></td>
<td></td>
<td>Thomas, Yu &amp; Zang (’05)</td>
<td>K.K. &amp; Oz (’12+)</td>
<td></td>
<td>$\times$</td>
</tr>
<tr>
<td>H-cycle</td>
<td>$\bigcirc$</td>
<td></td>
<td>Tutte (’56)</td>
<td>Thomas &amp; Yu (’94)</td>
<td>Grunbaum (’70) Nash-Williams (’73)</td>
<td>$\times$</td>
</tr>
<tr>
<td>$b = 0$</td>
<td></td>
<td></td>
<td></td>
<td>$\times$</td>
<td></td>
<td>$\times$</td>
</tr>
<tr>
<td>H-conn</td>
<td>$\bigcirc$</td>
<td></td>
<td>Thomassen (’83)</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$b = -1$</td>
<td></td>
<td></td>
<td>K.K. &amp; Oz (’12+)</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
</tbody>
</table>

2nd June 2012

Cycles in Graphs
4-conn. graphs on surfaces \((a = 1)\)

Conjecture (Grunbaum ’70, Nash-Williams ’73)

\[ \forall G : \text{4-conn graph on the torus} \implies \exists \text{H-cycle} \]

Why is this conjecture difficult?

- A surface (and graphs on it) is more complicated
- To find an H-cycle \((b = 0)\)
  
  we usually think the property for \(b = -1\)

Ex. H-conn, \(\exists \text{H-cycle through a given edge} \ldots\)
### 4-conn. graphs on surfaces ($\alpha = 1$)

<table>
<thead>
<tr>
<th></th>
<th>Plane ($\chi = 2$)</th>
<th>Proj. plane ($\chi = 1$)</th>
<th>Torus ($\chi = 0$)</th>
<th>K-bottle ($\chi = 0$)</th>
<th>$N_3$ ($\chi = -1$)</th>
<th>Others ($\chi &lt; -1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = 1$</td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>?</td>
<td>![ x ]</td>
</tr>
<tr>
<td><strong>H-cycle</strong></td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>![ ? ]</td>
<td>![ ? ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
</tr>
<tr>
<td>$b = 0$</td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>![ ? ]</td>
<td>![ ? ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
</tr>
<tr>
<td><strong>H-conn</strong></td>
<td></td>
<td></td>
<td>![ x ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
</tr>
<tr>
<td>$b = -1$</td>
<td>![ circle ]</td>
<td>![ circle ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
<td>![ x ]</td>
</tr>
</tbody>
</table>

- Thomas, Yu & Zang ('05)
- K.K. & Oz ('12+)
- Grunbaum ('70)
- Nash-Williams ('73)
- Thomassen ('83)
- K.K. & Oz ('12+)
4-conn. graphs on surfaces \((\alpha = 1)\)

**Conjecture (Grunbaum ’70, Nash-Williams ’73)**

\[ \forall G : 4\text{-conn graph on the torus} \implies \exists \text{ H-cycle} \]

**Theorem (Fujisawa, Nakamoto, Oz ’12+)**

\[ \forall G : 4\text{-conn graph on the torus} \implies (\text{# of comp.s in } G - S) \leq |S'| \]

\[ \exists S : \text{the equality in above holds} \implies \exists \text{ H-cycle} \]
4-conn. graphs on surfaces \((a = 1)\)

**Conjecture (Grunbaum ’70, Nash-Williams ’73)**

\[ \forall G : \text{4-conn graph on the torus} \Rightarrow \exists \text{ H-cycle} \]

**Theorem (Kawarabayashi, Oz ’12+)**

\[ \forall G : \text{4-conn triangulation on the torus} \Rightarrow \exists \text{ H-cycle} \]
### 4-conn. graphs on surfaces \((\alpha = 1)\)

<table>
<thead>
<tr>
<th>(b = 1)</th>
<th>(b = 0)</th>
<th>(b = -1)</th>
<th>(b = -2)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td><strong>H-cycle</strong></td>
<td><strong>H-conn</strong></td>
<td></td>
</tr>
<tr>
<td>Plane (\chi = 2)</td>
<td>Proj. plane (\chi = 1)</td>
<td>Torus (\chi = 0)</td>
<td>K-bottle (\chi = 0)</td>
</tr>
<tr>
<td>(k = 4)</td>
<td>(b \geq -\chi \implies \text{OK.})</td>
<td>(b &lt; -\chi \implies \text{Not satisfied})</td>
<td>(N_3) (\chi = -1)</td>
</tr>
</tbody>
</table>

**Necessary condition**

- \(b \geq -\chi \implies \text{OK.}\)
- \(b < -\chi \implies \text{Not satisfied}\)

**References**
- Tutte ('56)
- Thomas & Yu ('94)
- Nash-Williams('73)
- Thomassen ('83)
- K.K. & Oz ('12+)

**Remark:**
- \(c_2 > 0\) for \(b = 1\)
- \(c_2 = 0\) for \(b = 0\)
- \(c_2 < 0\) for \(b = -1\)
- \(c_2 = 0\) for \(b = -2\)
4-conn. graphs on surfaces \((a = 1)\)

2-H : \(\forall x, y \in V(G), \exists \text{H-cycle in } G - \{x, y\}\)

1-H-conn : \(\forall x \in V(G), G - x \text{ is H-conn}\)

2-e-H-conn : \(\forall e, f \in \binom{V}{2}, \exists \text{H-cycle through } e, f \text{ in } G + \{e, f\}\)

Necessary condition

(# of comp.s in \(G - S\)) \(\leq |S| - 2\)

\(G : 2\text{-e-H-conn} \iff G : 2\text{-H}\)

\(\downarrow \quad \times\)

\(G : 1\text{-H-conn}\)

• \(\forall 4\text{-conn. plane graph is 2-H}\) (Tomas & Yu, `94)
• \(\forall 4\text{-conn. plane graph is 1-H-conn.}\) (Sanders, `97)
• \(\forall 4\text{-conn. plane graph is 2-e-H-conn.}\) (Oz & Vrána `12+)
### 4-conn. graphs on surfaces \((a = 1)\)

<table>
<thead>
<tr>
<th></th>
<th>Plane (\chi = 2)</th>
<th>Proj. plane (\chi = 1)</th>
<th>Torus (\chi = 0)</th>
<th>K-bottle (\chi = 0)</th>
<th>(N_3) (\chi = -1)</th>
<th>Others (\chi &lt; -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-path (b = 1)</td>
<td>(k = 4)</td>
<td>Necessary condition (b \geq -\chi \Rightarrow) OK.</td>
<td></td>
<td>(\ast)</td>
<td>?</td>
<td>(\ast)</td>
</tr>
<tr>
<td>H-cycle (b = 0)</td>
<td>Tutte ('56)</td>
<td>Thomas &amp; Yu ('94)</td>
<td>Nash-Williams ('73)</td>
<td></td>
<td></td>
<td>(\ast)</td>
</tr>
<tr>
<td>H-conn (b = -1)</td>
<td>Thomassen ('83)</td>
<td>K.K. &amp; Oz ('12+)</td>
<td></td>
<td></td>
<td>(\ast)</td>
<td>(\ast)</td>
</tr>
<tr>
<td>(b = -2)</td>
<td>Oz, V ('12+)</td>
<td></td>
<td></td>
<td></td>
<td>(\ast)</td>
<td>(\ast)</td>
</tr>
</tbody>
</table>

2nd June 2012

Cycles in Graphs
Spanning tree with (# of leaves) ≤ t

(# of comp.s in $G - S$) ≤ $a|S| + b$

$(a, b) = (1, t - 1)$

$k = 4$

Problems (Oz `12+)

∀ $F^2$: a surface  $\chi = \chi(F^2) < 0$

∀ $G$: 4-conn. graph on $F^2$

⇒ ∃ Spanning tree with (# of leaves) ≤ $-\chi + 1$

Necessary condition

$\Rightarrow b \geq -\chi \Rightarrow$ OK.

$\Rightarrow b < -\chi \Rightarrow$ Not satisfied
Representativity

$G : \text{graphs on a surface on } F^2$

Rep : sufficiently large $\iff$ Locally planar

Representativity of $G$ (rep of $G$):

$$= \min \left\{ |G \cap \gamma| : \gamma \text{ is a non-contractible curve on } F^2 \right\}$$

$\exists$ Graphs $K_{n,m} (n \neq m)$ s.t. connectivity : large
rep : small $\not\exists$ H-cycle

2nd June 2012 Cycles in Graphs 24
4-conn. graphs on surfaces \( (a = 1) \)

- Graphs on surfaces, connectivity and toughness

\[ G : k \text{-conn. graph on } F^2 \]

\[ \text{(number of components in } G - S ) \leq a|S| + b \]

This bound is best possible \( \text{O}(\chi) \) is needed (AHL `96)

\[ (a, b) = \left( \frac{2}{k-2}, \frac{-2\chi}{k-2} \right) \]

This implies \( |S| : \text{large} \)

\[ \text{Rep : sufficiently large } \implies \text{It holds for } a > \frac{2}{k-2}, \quad b \geq -ka + 2 \]

A necessary condition is OK if \( k = 4, a > 1 \), and rep: large

Ex. \( \exists \text{Sp. tree with max. deg. } \leq 3 \implies (a, b) = (2, 1) \)

This holds for \( \forall \)4-conn. graph on a surface with rep : large (Ellingham & Gao `94)
4-conn. graphs on surfaces \( (\alpha = 1) \)

**Conjecture (Mohar ’95)**

\[ \forall F^2 : \text{a surface} \quad \chi = \chi(F^2) < 0 \quad \exists r(F^2) : \text{constant} \]

s.t. \( \forall G : 4\text{-connected} \text{ graph on } F^2 \text{ with rep } > r(F^2) \)

\[ \implies \exists \text{Spanning tree with max. deg } \leq 3 \]

s.t. (\# of leaves) = \( O(-\chi) \)

**Problem (Oz `12+)**

\[ \forall F^2 : \text{a surface} \quad \chi = \chi(F^2) < 0 \]

\[ \forall G : 4\text{-conn. graph on } F^2, \]

\[ \implies \exists \text{Spanning tree with (\# of leaves) } \leq -\chi + 1 \]
4-conn. graphs on surfaces \((\alpha = 1)\)

Conjecture (Mohar ’95)

\[
\forall F^2 : \text{a surface} \quad \chi = \chi(F^2) < 0 \quad \exists r(F^2) : \text{constant}
\]

\[\text{s.t. } \forall G : \text{4-connected graph on } F^2 \text{ with rep } > r(F^2)\]

\[\Rightarrow \exists \text{Spanning tree with max. } \text{deg} \leq 3\]

\[\text{s.t. (} \# \text{ of leaves} \text{) } = O(-\chi)\]

Theorem (Kawarabayashi, Oz ’12+)

The above conjecture is true if \(G\) is a triangulation
### 4-conn. graphs on surfaces \((a = 1)\)

<table>
<thead>
<tr>
<th></th>
<th>Plane (\chi = 2)</th>
<th>Proj. plane (\chi = 1)</th>
<th>Torus (\chi = 0)</th>
<th>K-bottle (\chi = 0)</th>
<th>(N_3) (\chi = -1)</th>
<th>Others (\chi &lt; -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(k' = 4)</td>
</tr>
<tr>
<td>(b = 1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(b \geq -\chi \Rightarrow) OK.</td>
</tr>
<tr>
<td><strong>H-cycle</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(\times)</td>
</tr>
<tr>
<td>(b = 0)</td>
<td>Tutte (‘56)</td>
<td>Thomas &amp; Yu (‘94)</td>
<td>Nash-Williams (‘73)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>H-conn</strong></td>
<td>Thomassen (‘83)</td>
<td>K.K. &amp; Oz (‘12+)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = -1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = -2)</td>
<td>(\bigcirc) Oz,V (‘12+)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = -3)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2nd June 2012  
Cycles in Graphs  
28
5-conn. graphs on surfaces \((a = \frac{2}{3})\)

- Graphs on surfaces, connectivity and toughness

\[
G : k\text{-conn. graph on } F^2
\]

\[
(# \text{ comp. s in } G - S) \leq a|S| + b
\]

This bound is best possible

This implies \(|S| : \text{large}\)

Rep : sufficiently large \(\Rightarrow\) It holds for \(a > \frac{2}{k - 2}, b \geq -ka + 2\)

A necessary condition is OK if \(k = 5, a > \frac{2}{3}\), and rep: large

Ex. Hamiltonicity of graphs \(\Rightarrow (a, b) = (1, 0)\)

We do not need any assumption on rep for large \(\chi\)
### 5-conn. graphs on surfaces (\(a = \frac{2}{3}\))

<table>
<thead>
<tr>
<th></th>
<th>Plane (\chi = 2)</th>
<th>Proj. plane (\chi = 1)</th>
<th>Torus (\chi = 0)</th>
<th>K-bottle (\chi = 0)</th>
<th>(N_3) (\chi = -1)</th>
<th>Others (\chi &lt; -1)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = 1)</td>
<td>○</td>
<td>○</td>
<td>○ Thomas, Yu &amp; Zang (‘05)</td>
<td>○ K.K. &amp; Oz (’12+)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>H-cycle</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = 0)</td>
<td>○ Tutte (‘56)</td>
<td>○ Thomas &amp; Yu (‘94)</td>
<td>○ Thomas &amp; Yu (‘97)</td>
<td>○ K.K. &amp; Mohar (‘12+)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td><strong>H-conn</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b = -2)</td>
<td>○</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2nd June 2012

Cycles in Graphs
5-connected graphs on surfaces \((a = \frac{2}{3})\)

**Conjecture (Thomassen `94)**

\[
\forall F^2 : \text{a surface} \quad \chi = \chi(F^2) < 0 \quad \exists r(F^2) : \text{constant}
\]

s.t. \(\forall G : 5\)-connected graphs on \(F^2\),

\[
\text{rep} > r(F^2) \implies \exists \text{Hamilton cycles}
\]

**Locally planar**

\[
G : 5\text{-connected graphs on } F^2 \text{ with rep : large enough}
\]

\[
\exists \text{Hamilton cycle if } G \text{ is a triangulation (Yu `95)}
\]

\[
\exists \text{Hamilton cycle (Kawarabayashi, Oz `11+) if } \geq 3 \text{ triangles are incident with } \forall \text{vertex}
\]
3-conn. graphs on surfaces $(a = 2)$

$G$: 3-conn. graph on $F^2$

$(\# \text{ comp.s in } G - S) \leq 2|S| - 2\chi$

<table>
<thead>
<tr>
<th>$(k - 1, 1)$</th>
<th>(\exists) Sp. tree with max. deg. (\leq k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k - 1, t + 1)$</td>
<td>(\exists T : \text{Sp. tree with } te(T, k) \leq t)</td>
</tr>
</tbody>
</table>

\(\exists\) Sp. tree with bounded total excess
3-conn. graphs on surfaces \((\alpha = 2)\)

\[
te(T, k) := \sum_{x \in V(T)} \max\{d_T(x) - k, 0\}
\]

Total excess of a sp. tree \(T\) from \(k\)

\[
te(T, k) = 0
\]

\(\Updownarrow\)

\(T: \text{sp. tree with max. deg.} \leq k\)

\[\exists T: \text{Sp. tree with } te(T, k) \leq t\]

\(\Downarrow\)

\((\text{# comp.s in } G - S)\)

\[\leq (k - 1)|S| + t - 1\]

\[te(T, 2) = 6\]
3-conn. graphs on surfaces \( (\alpha = 2) \)

\[ G : \text{3-conn. graph on } F^2 \]

\[(\# \text{ comp. s in } G - S) \leq 2|S| - 2\chi \]

\[
\begin{array}{c|c}
(k - 1, 1) & \exists \text{Sp. tree with max. deg. } \leq k \\
(k - 1, t + 1) & \exists T : \text{Sp. tree with } te(T, k) \leq t
\end{array}
\]

\( \chi \geq 0 \) \quad \exists \text{Sp. tree with max. deg. } \leq 3 \quad \text{(Barnette `66, `92)}

\( \chi \leq -1 \) \quad \exists T : \text{Sp. tree with } te(T, 3) \leq -2\chi - 1 \quad \text{(Oz `12+)}

\[(\# \text{ comp. s in } G - S) \leq 3|S| + 1 \]

\( \exists \text{Sp. tree with max. deg. } \leq 4 \quad \text{(Yu `96)} \)

\( \exists \text{Sp. tree with max. deg. } \leq 4 \text{ and } te(T, 3) \leq -2\chi - 1 \quad \text{(Kawarabayashi, Nakamoto, Ota `97)} \)

\[(\# \text{ comp. s in } G - S) \leq \left\lfloor \frac{-2\chi + 5}{3} \right\rfloor |S| + 1 \quad \therefore |S| \geq 3 \]

\( \exists \text{Sp. tree with max. deg. } \leq \left\lfloor \frac{-2\chi + 8}{3} \right\rfloor \quad \text{(Sanders, Zhao `01, Ota, Oz `12)} \)
Graphs on surfaces

- Graphs on surfaces, connectivity and toughness

\[ G: k\text{-conn. graph on } F^2 \]

\[ (\text{# comp. s in } G - S ) \leq a|S| + b \]

holds for \( (a, b) = \left( \frac{2}{k-2}, \frac{-2\chi}{k-2} \right) \)

This bound is best possible

- Properties concerning Hamiltonicity (and toughness)

Statement: \( \forall k\text{-conn. graph on a surface } F^2 \text{ is } […] \)

We have to think about toughness type necessary condition

What else do we need?
### 4-conn. graphs on surfaces ($a = 1$)

<table>
<thead>
<tr>
<th></th>
<th>Plane $\chi = 2$</th>
<th>Proj. plane $\chi = 1$</th>
<th>Torus $\chi = 0$</th>
<th>K-bottle $\chi = 0$</th>
<th>$N_3$ $\chi = -1$</th>
<th>Others $\chi &lt; -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>H-path</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = 1$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>H-cycle</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = 0$</td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
</tr>
<tr>
<td><strong>H-conn</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b = -1$</td>
<td><img src="image13" alt="Diagram" /></td>
<td><img src="image14" alt="Diagram" /></td>
<td><img src="image15" alt="Diagram" /></td>
<td><img src="image16" alt="Diagram" /></td>
<td><img src="image17" alt="Diagram" /></td>
<td><img src="image18" alt="Diagram" /></td>
</tr>
<tr>
<td>$b = -2$</td>
<td><img src="image19" alt="Diagram" /></td>
<td><img src="image20" alt="Diagram" /></td>
<td><img src="image21" alt="Diagram" /></td>
<td><img src="image22" alt="Diagram" /></td>
<td><img src="image23" alt="Diagram" /></td>
<td><img src="image24" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Necessary condition**

- $k = 4$
- $b \geq -\chi \Rightarrow$ OK.
- $b < -\chi \Rightarrow$ Not satisfied

*2nd June 2012 Cycles in Graphs*
### 5-conn. graphs on surfaces \( (\alpha = \frac{2}{3}) \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>Plane ( \chi = 2 )</th>
<th>Proj. plane ( \chi = 1 )</th>
<th>Torus ( \chi = 0 )</th>
<th>K-bottle ( \chi = 0 )</th>
<th>( N_3 ) ( \chi = -1 )</th>
<th>Others ( \chi &lt; -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b = 1 )</td>
<td>H-path</td>
<td></td>
<td></td>
<td></td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( b = 0 )</td>
<td>H-cycle</td>
<td></td>
<td>Thomas, Yu &amp; Zang (‘05)</td>
<td>K.K. &amp; Oz (’12+)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( b = -1 )</td>
<td>H-conn</td>
<td>Tutte (‘56)</td>
<td>Thomas &amp; Yu (‘94)</td>
<td>Thomas &amp; Yu (‘97)</td>
<td>K.K. &amp; Mohar (‘12+)</td>
<td>?</td>
</tr>
<tr>
<td>( b = -2 )</td>
<td></td>
<td>Thomassen (‘83)</td>
<td>K.K. &amp; Oz (’12+)</td>
<td>K.K. &amp; Oz (’12+)</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

2nd June 2012

Cycles in Graphs
3-conn. graphs on surfaces (\(a = 2\))

\[ G : \text{3-conn. graph on } F^2 \]

| \((k - 1, 1)\) | \(\exists \text{Sp. tree with max. deg. } \leq k\) |
| \((k - 1, t + 1)\) | \(\exists T: \text{Sp. tree with } te(T, k) \leq t\) |

\((\# \text{ comp.s in } G - S) \leq 2|S| - 2\chi\)

\(\chi \geq 0\) \(\exists \text{Sp. tree with max. deg. } \leq 3\) (Barnette `66, `92)

\(\chi \leq -1\) \(\exists T: \text{Sp. tree with } te(T, 3) \leq -2\chi - 1\) (Oz `12+)

\((\# \text{ comp.s in } G - S) \leq 3|S| + 1\)

If \(\text{rep}\) : sufficiently large

\(\exists \text{Sp. tree with max. deg. } \leq 4\) (Yu `96)

\(\exists \text{Sp. tree with max. deg. } \leq 4\) and \(\text{te}(T, 3) \leq -2\chi - 1\)

(Kawarabayashi, Nakamoto, Ota `97)

\((\# \text{ comp.s in } G - S) \leq \left\lceil \frac{-2\chi + 5}{3} \right\rceil |S| + 1\) \(\because |S| \geq 3\)

\(\exists \text{Sp. tree with max. deg. } \leq \left\lceil \frac{-2\chi + 8}{3} \right\rceil\) (Sanders, Zhao `01, Ota, Oz `12)
$K_{3,t}$-minor free graphs

$t \geq 3$

$G : 3$-conn. $K_{3,t}$-minor-free graphs

$(\# \text{ comp. s in } G - S) \leq a|S| + b$

These bounds are best possible

holds for $(a, b) = (t - 1, -2t + 2)$ if $t$: odd

(Chen, Egawa, Kawarabayashi, Mohar, Ota, `11)

holds for $(a, b) = (t - 2, -2t + 6)$ if $t$: even

(95% true, Ota, Oz, `12)

$\exists$ Sp. tree with max. deg. $\leq t$ if $t$: odd

$\exists$ Sp. tree with max. deg. $\leq t - 1$ if $t$: even

$(k - 1, 1) \exists$ Sp. tree with max. deg. $\leq k$
$K_{3,t}$-minor free graphs

$t \geq 3$

$(k - 1, 1) \exists$ Sp. tree with max. deg. $\leq k$

$G : 4$-conn. $K_{3,t}$-minor-free graphs

$(\# \text{ comp. s in } G - S) \leq a|S| + b$

Is there good toughness bound? (I expect $a = 1$.)

$\rightarrow \exists H$-cycle or $\exists$ sp. tree with bounded # of leaves?

How about $4$-conn. $K_{4,t}$-minor-free graphs??

Toughness bound holds for $(a, b) = (3t - 3, -9t + 9)$

(Chen, Egawa, Kawarabayashi, Mohar, Ota, `11)
Thank you for your attention