Long Monochromatic Cycles
in
Edge-Colored Complete Graphs

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The **circumference** $c(G)$ of a graph $G$ is the length of a longest cycle in $G$.

**Theorem 1** (Faudree, Lesniak & Schiermeyer 2009)

Let $G$ be a graph of order $n \geq 6$. Then

$$\max\{c(G), c(\overline{G})\} \geq \left\lceil \frac{2n}{3} \right\rceil$$

and this bound is sharp.

**Comment:** The proof of Theorem 1 depended heavily on the ramsey number for two even cycles.
Notation (Fujita 2011): Let \( f(r, n) \) be the maximum integer \( k \) such that every \( r \)-edge-coloring of \( K_n \) contains a monochromatic cycle of length at least \( k \). (For \( i \in \{1, 2\} \), we regard \( K_i \) as a cycle of length \( i \).)

Thus, Faudree et al. proved \( f(2, n) = \left\lceil \frac{2n}{3} \right\rceil \) for \( n \geq 6 \).
Comments:

1. To show that $f(r, n) \geq k$ we must show that every $r$-edge-coloring of $K_n$ contains a monochromatic cycle of length at least $k$.

2. To show that $f(r, n) \leq k$ we must exhibit one $r$-edge-coloring of $K_n$ in which the longest monochromatic cycle has length (exactly) $k$. 
Using affine planes of index $r - 1$ and order $(r - 1)^2$ Gyárfas exhibited $r$-edge-colorings of $K_n$ in which the longest monochromatic cycle had length $\left\lceil \frac{n}{r-1} \right\rceil$. In fact, $f(r, n) \leq \left\lceil \frac{n}{r-1} \right\rceil$ for infinitely many $r$ and, for each such $r$, infinitely many $n$. 

Conjecture 2 (Faudree, Lesniak & Schiermeyer 2009)

For $r \geq 3$, $f(r, n) \geq \left\lceil \frac{n}{r-1} \right\rceil$. 
Using affine planes of index $r - 1$ and order $(r - 1)^2$ Gyárfas exhibited $r$-edge-colorings of $K_n$ in which the longest monochromatic cycle had length $\left\lfloor \frac{n}{r-1} \right\rfloor$. In fact, $f(r, n) \leq \left\lfloor \frac{n}{r-1} \right\rfloor$ for infinitely many $r$ and, for each such $r$, infinitely many $n$.

**Conjecture 2 (Faudree, Lesniak & Schiermeyer 2009)**

For $r \geq 3$, $f(r, n) \geq \left\lceil \frac{n}{r-1} \right\rceil$. 
However, Conjecture 2 is not true as stated. For example, $K_{2r}$ can be factored into $r$ hamiltonian paths. Giving each path a different color shows that $f(r, 2r) \leq 2$, whereas the conjecture would give $f(r, 2r) \geq \left\lceil \frac{2r}{r-1} \right\rceil = 3$. 
Theorem 3 (Fujita 2011)

For $1 \leq r \leq n$, $f(r, n) \geq \left\lceil \frac{n}{r} \right\rceil$.

Proof.

Arbitrarily color the edges of $K_n$ with $r$ colors. Let $G_i$ be the maximal monochromatic spanning subgraph of $G$ with color $i$ for $1 \leq i \leq r$. Then some $G_{i_0}$ contains at least $\frac{n^2}{r}$ edges. So,

$$|E(G_{i_0})| \geq \frac{\binom{n}{2}}{r} = \left(\frac{n}{r}\right)\left(\frac{n-1}{2}\right) \geq \frac{\left\lceil \frac{n}{r} \right\rceil - 1}{2} (n-1).$$

Thus $G_{i_0}$ contains a cycle of length at least $\left\lceil \frac{n}{r} \right\rceil$.

(Erdös & Gallai 1959)
Fujita determined $f(r, n)$ for $n \leq 2r + 1$ and proposed:

**Problem 4 (Fujita 2011)**

For $r \geq 2$ and $n \geq 2r + 2$ determine $f(r, n)$.

Rest of the talk:

1. Determine $f(r, 2r + 2)$.
2. Determine $f(r, sr + c)$ for $r$ sufficiently large with respect to $s$ and $c$.
3. Open questions.
Theorem 5 (Ray-Chaudhury & Wilson 1971)

For any \( t \geq 1 \), the edge set of \( K_{6t+3} \) can be partitioned into \( 3t + 1 \) parts, where each part forms a graph isomorphic to \( 2t + 1 \) disjoint triangles.

Theorem 6 (Fujita, Lesniak & Toth 2012+)

For \( r \geq 3 \), \( f(r, 2r + 2) = 3 \). For \( r = 1, 2 \), \( f(r, 2r + 2) = 4 \).

Outline of proof for \( r \geq 4 \).

By Theorem 3, \( f(r, 2r + 2) \geq \left\lceil \frac{2r+2}{r} \right\rceil = 3 \).

We show \( f(r, 2r + 2) \leq 3 \) by exhibiting an \( r \)-edge-coloring of \( K_{2r+2} \) in which the longest monochromatic cycle is a triangle.
Case 1: \( r = 3k+1 \).

Then \( n = 2r + 2 = 6k + 4 \). Begin with a coloring of the edges of \( K_{6k+3} \) on the vertices \( v_1, v_2, \ldots, v_{6k+3} \) with colors \( c_1, c_2, \ldots, c_{3k+1} \) according to Theorem 5.
Case 1 (continued).

\[ c_{3k+1} : \]

\[ \begin{array}{c}
V_2 & V_3 & V_5 & V_6 & \cdots & V_{6k+2} & V_{6k+3} \\
V_1 & & & & & & \\
V_{6k+4} & & & & & & \\
\end{array} \]
Case 1 (continued).

\[ c_{3k+1} : \]

\[ c_{1} : \]

\[ \text{etc.} \]

\[ c_{2k+1} : \]

\[ 2k + 1 \leq 3k \]
Case 2: \( r = 3k + 2 \).

Then \( n = 2r + 2 = 6k + 6 \). Begin with a coloring of the edges of \( K_{6k+3} \) on the vertices \( v_1, v_2, \ldots, v_{6k+3} \) with colors \( c_1, c_2, \ldots, c_{3k+1} \) according to Theorem 5.

\[ K_{6k+3}: \]

\[ c_1: \]

\[ c_2: \]

\[ c_{3k+1}: \]

\[ 2k+1 \]
Case 2: \( r=3k+2 \).

Then \( n = 2r + 2 = 6k + 6 \). Begin with a coloring of the edges of \( K_{6k+3} \) on the vertices \( v_1, v_2, \ldots, v_{6k+3} \) with colors \( c_1, c_2, \ldots, c_{3k+1} \) according to Theorem 5.

\[ K_{6k+3}: \]

\[ c_1: \]

\[ c_2: \]

\[ c_{3k+1}: \]

One unused color \( c_{3k+2} \)
Case 3: $r=3k$.

Then $n = 2r + 2 = 6k + 2$. Begin with a coloring of the edges of $K_{6k+3}$ on the vertices $v_1, v_2, \ldots, v_{6k+3}$ with colors $c_1, c_2, \ldots, c_{3k+1}$ according to Theorem 5.

In this case we used one extra color and have one extra vertex. We recolor to remove color $c_{3k+1}$ so that the longest monochromatic cycle is a triangle. Thus

$$f(r, 2r+3) \leq 3.$$
Case 3: $r=3k$.

Then $n = 2r + 2 = 6k + 2$. Begin with a coloring of the edges of $K_{6k+3}$ on the vertices $v_1, v_2, \ldots, v_{6k+3}$ with colors $c_1, c_2, \ldots, c_{3k+1}$ according to Theorem 5.

In this case we used one extra color and have one extra vertex. We recolor to remove color $c_{3k+1}$ so that the longest monochromatic cycle is a triangle. Thus $f(r, 2r + 2) \leq f(r, 2r + 3) \leq 3$. 

$\square$
We’ve determined \( f(r, n) \) for \( n \leq 2r + 2 \).

**Problem 7**

* Determine \( f(r, sr + c) \) for integers \( s, c \geq 2 \).

**Comment:** \( f(r, sr + c) \geq \left\lceil \frac{sr+c}{r} \right\rceil \geq s + 1 \) by Theorem 3.

We’ll show that \( f(r, sr + c) = s + 1 \) for \( r \) sufficiently large with respect to \( s \) and \( c \).
Theorem 8 (Chang 2000)

Let $q \geq 3$. Then for $t$ sufficiently large, the edge set of $K_{q(q-1)t+q}$ can be partitioned into $qt + 1$ parts, where each part is isomorphic to $(q-1)t + 1$ disjoint copies of $K_q$.

Comment: Chang’s result was stated in terms of resolvable balanced incomplete block designs. Thank you, Wal Wallis.

Comment: $q = 3$ in Theorem 8 is our previous Theorem 5.
Theorem 9 (Fujita, Lesniak & Toth 2012+)

For any pair of integers \( s, c \geq 2 \) there is an \( R \) such that \( f(r, sr + c) = s + 1 \) for all \( r \geq R \).

**Comment:** For \( s = c = 2 \), \( R = 3 \) (Theorem 8).

**Outline of Proof.**

1. We show \( f(r, sr + c) \leq s + 1 \) with an \( r \)-edge-coloring of \( K_{sr+c} \) in which the longest monochromatic cycle has length \( s + 1 \).
2. Since \( f(r, n) \) is monotone increasing in \( n \), we may assume that \( sr + c = (s + 1)st + (s + 1) \) for some \( t \).
Outline of Proof (continued).

3. Color the edges of $K_{(s+1)t+(s+1)} = K_{sr+c}$ with $(s + 1)t + 1 = r + \frac{c-1}{s}$ colors using Chang’s Theorem for $q = s + 1$.

\[ c_1 : \quad K_{S+1} \quad K_{S+1} \quad \cdots \quad K_{S+1} \]

\[ \vdots \quad \vdots \quad \cdots \quad \vdots \]

\[ c_{(s+1)t+1} : \quad K_{S+1} \quad K_{S+1} \quad \cdots \quad K_{S+1} \quad s \text{t+1} \]
Outline of Proof (continued).

4 For \( r \) sufficiently large with respect to \( s \) and \( c \) we can recolor to reduce the number of colors by \( \frac{c-1}{s} \) without creating monochromatic cycles of length greater than \( s + 1 \).

\[ \square \]

Comment: To remove one color class, we need at most

\[
\log \frac{s(s+1)+1}{s(s+1)} \left( \frac{s}{s+1} r + \frac{c}{s+1} \right)
\]

other classes. Thus we can avoid \( \frac{c-1}{s} \) color class with the remaining \( r \) classes if

\[
\left( \frac{c - 1}{2} \right) \log \frac{s(s+1)+1}{s(s+1)} \left( \frac{s}{s+1} r + \frac{c}{s+1} \right) \leq r,
\]

which is true for sufficiently large \( r \) compared with \( s \) and \( c \).
Some open questions:

1. Determine $f(r, n)$ for $n$ “large” with respect to $r$.

Comment: We know $f(r, n) \geq \left\lceil \frac{n}{r} \right\rceil$ for all $1 \leq r \leq n$.

Comment: We know $f(r, n) \leq \left\lceil \frac{n}{r-1} \right\rceil$ for infinitely many $r$ and, for each such $r$, infinitely many $n$. 
2. The proof that $f(2, n) = \lceil \frac{2n}{3} \rceil$ for $n \geq 6$ depended heavily on the Ramsey number for two even cycles. Can the Ramsey number for three even cycles be used to determine $f(3, n)$ for $n$ sufficiently large?

**Theorem 10 (Benevides & Skokan 2009)**

*There exists an $n_1$ such that for every even $n \geq n_1$, $r(C_n, C_n, C_n) = 2n.$*