Approximating Weak Chebyshev Subspaces by Chebyshev Subspaces

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Abstract. We examine to what extent finite-dimensional spaces defined on locally compact subsets of the line and possessing various weak Chebyshev properties (involving sign changes, zeros, alternation of best approximations, and peak points) can be uniformly approximated by a sequence of spaces having related properties.

§1. Introduction

Let T be a locally compact Hausdorff space that contains at least $n+1$ points, and let $C_0(T)$ be the space of all real-valued continuous functions f on T which vanish at infinity, i.e., for each $\varepsilon > 0$, $\{t \in T : |f(t)| \geq \varepsilon\}$ is compact. Then $C_0(T)$ is a Banach space with the uniform norm $\|f\| := \sup\{|f(t)| : t \in T\}$. If T is actually compact, then $C_0(T)$ is just the space $C(T)$ of all continuous functions on T.

In this paper we are interested in finite-dimensional linear subspaces of $C_0(T)$ with the following kinds of properties (see below for precise definitions):

1) Chebyshev properties,
2) weak-Chebyshev properties,
3) peak point properties,
4) alternation properties,
5) interpolation properties.

Our aim is to investigate the relationship between these various properties, and also the extent to which such properties are preserved in the limit, as well as the extent to which a space with given properties can be approximated by a sequence of spaces with stronger properties. In this connection, we make the following definition.

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Definition 1.1. *Let* $G = \text{span}\{g_1, g_2, \ldots, g_n\}$ *be an* n*-dimensional subspace of* $C_0(T)$*. Then we say that* G *is* approximable by subspaces having property P *provided that for each* $\varepsilon > 0$*, there exists an* n*-dimensional subspace* $G_\varepsilon = \text{span}\{g_{\varepsilon,1}, g_{\varepsilon,2}, \ldots, g_{\varepsilon,n}\}$ *of* $C_0(T)$ *having property* P *such that* $\|g_{\varepsilon,i} - g_i\| < \varepsilon$ *for each* $i = 1, 2, \ldots, n$*.*

We shall focus on the following questions:

Question 1.2. *If a subspace* G *of* $C_0(T)$ *has a certain property P, can* G *be approximated by subspaces with some stronger property Q?*

Question 1.3. *If the subspace* G *is approximable by subspaces having property P, what properties must* G *have?*

As just one example of what we have in mind, we recall the following result of Jones and Karlovitz, which was the starting point for this work.

Theorem 1.4 [6]**.** *A finite-dimensional subspace of* $C[a, b]$ *has the "weak Chebyshev" property (see property W-2 in Section 4) if and only if it is approximable by Chebyshev subspaces (see property C-2 in Section 3).*

The remainder of the paper is organized as follows. In Section 2 of the paper we describe an alternative formulation of the concept of approximability. In Sections 3–5 we introduce the various properties of interest in this paper, and explore their interconnection (see Figures 1–3). In particular, these sections deal with the three cases: 1) T is an arbitrary locally compact set, 2) T is a locally compact subset of $\mathbb{R}$, and 3) T is an interval. Sections 6 and 7 contain the main results of the paper where we answer Questions 1.2 and 1.3. Our results are summarized in Tables 1–6. Section 8 is devoted to a collection of examples which are useful for tracing the connection between various properties as well as for answering Questions 1.2 and 1.3. In Section 9 we examine spaces which are limits of weak Chebyshev spaces, and in Section 10 we discuss properties of spaces of extended functions. Finally, in Section 11 we establish a generalization of the Jones-Karlovitz theorem (see Corollary 11.2).

§2. An Equivalent Formulation of Approximability

In this section we present an interesting reformulation of the concept of approximability. First we need a simple lemma concerning linearly independent sets in a normed linear space.

Lemma 2.1. *Let* $\{g_1, g_2, \ldots, g_n\}$ *be a linearly independent set of* n *vectors in a normed linear space. Then there exists an* $\varepsilon > 0$ *such that if* $\{h_1, h_2, \ldots, h_n\}$ *is any set of* n *vectors satisfying* $\|h_i - g_i\| < \varepsilon$ *for* $i = 1, 2, \ldots, n$*, then* $\{h_1, h_2, \ldots, h_n\}$ *is also linearly independent.*

Proof: Since all norms are equivalent on a finite-dimensional space, there exists a constant $c > 0$ such that

$$c\sum_{i=1}^{n}|\alpha_i| \le \Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\|, \quad \text{for all } n\text{-tuples } (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ of real numbers.} \tag{2.1}$$

Now let $\varepsilon = c$, and choose h_i such that $\|h_i - g_i\| < \varepsilon$ for $i = 1, 2, \ldots, n$. If $h_1, h_2, \ldots, h_n$ were not linearly independent, there would exist a nonzero n-tuple $(\alpha_1, \ldots, \alpha_n)$ of real numbers such that $\sum_{i=1}^{n}\alpha_i h_i = 0$. Then

$$c\sum_{i=1}^{n}|\alpha_i| \le \Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\| = \Big\|\sum_{i=1}^{n}\alpha_i (g_i - h_i)\Big\| \le \sum_{i=1}^{n}|\alpha_i|\|g_i - h_i\| < c\sum_{i=1}^{n}|\alpha_i|,$$

which is impossible. □

Before stating the main result of this section, we need to introduce some additional notation. Let G be a finite-dimensional subspace of $C_0(T)$, and suppose $f \in C_0(T)$. Then the set of best approximations to f from G is defined to be the set

$$P_G(f) := \{g \in G : \|f - g\| = d(f, G)\},$$

where $d(f, G) := \inf\{\|f - g\| : g \in G\}$. It is well known that $P_G(f)$ is nonempty whenever G is finite-dimensional.

Proposition 2.2. *Let $G = \operatorname{span}\{g_1, g_2, \ldots, g_n\}$ be an n-dimensional subspace of $C_0(T)$. Then the following statements are equivalent:*

1) *G is approximable by subspaces having property P;*
2) *there exists a sequence of n-dimensional subspaces $G_k = \operatorname{span}\{g_{k,1}, g_{k2}, \ldots, g_{k,n}\}$ having property P such that $\lim_{k\to\infty}\|g_{k,i} - g_i\| = 0$ for each $i = 1, 2, \ldots, n$;*
3) *for each $\varepsilon > 0$, there exists an n-dimensional subspace G_ε having property P such that*

$$\sup_{g\in B(G)} d(g, B(G_\varepsilon)) < \varepsilon, \tag{2.2}$$

where $B(Y) := \{y \in Y : \|y\| \le 1\}$ is the unit ball in Y.

Proof: The equivalence 1) ⇔ 2) is clear. We now prove 1) ⇒ 3). Given $\varepsilon > 0$, by (2.1) and property 1), there exists a subspace $G_\varepsilon = \operatorname{span}\{g_{\varepsilon,1}, \ldots, g_{\varepsilon,n}\}$ having property P such that

$$\|g_{\varepsilon,i} - g_i\| < \frac{c\,\varepsilon}{4}, \qquad i = 1, \ldots, n,$$

where c is the constant in (2.1). To verify statement 3), we now show that for every $g \in B(G)$, there exists $g_\varepsilon \in B(G_\varepsilon)$ such that

$$\|g_\varepsilon - g\| \le \frac{\varepsilon}{2} < \varepsilon. \tag{2.3}$$

Let $g = \sum_{i=1}^{n} \alpha_i g_i \in B(G)$, and set $h_\varepsilon := \sum_{i=1}^{n} \alpha_i g_{\varepsilon,i}$. Then $h_\varepsilon \in G_\varepsilon$ and

$$\|g - h_\varepsilon\| \leq \sum_{i=1}^{n} |\alpha_i| \|g_i - g_{\varepsilon,i}\| < \sum_{i=1}^{n} |\alpha_i| \frac{c\,\varepsilon}{4} \leq \|g\| \frac{c\,\varepsilon}{4\,c} \leq \frac{\varepsilon}{4}.$$

We also have $\|h_\varepsilon\| \leq \|h_\varepsilon - g\| + \|g\| < \frac{\varepsilon}{4} + \|g\| \leq 1 + \frac{\varepsilon}{4}$. We consider two cases.

Case 1. $\|h_\varepsilon\| \leq 1$. Take $g_\varepsilon = h_\varepsilon$.

Case 2. $\|h_\varepsilon\| > 1$. Setting $g_\varepsilon = \frac{h_\varepsilon}{\|h_\varepsilon\|}$, we see that

$$\|g_\varepsilon - g\| \leq \|g_\varepsilon - h_\varepsilon\| + \|h_\varepsilon - g\| \leq \|\frac{h_\varepsilon}{\|h_\varepsilon\|} - h_\varepsilon\| + \frac{\varepsilon}{4} = \|h_\varepsilon\| - 1 + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}.$$

In either case, $g_\varepsilon \in B(G_\varepsilon)$ and (2.3) holds.

We now show that 3) $\Rightarrow$ 1). Suppose 3) holds. Then for each $\varepsilon > 0$, there exists an n-dimensional subspace G_ε having property P such that (2.2) holds. We may assume that ε is sufficiently small so that the conclusion of Lemma 2.1 holds. Assume first that $\|g_i\| \leq 1$ for all i. Then by (2.2), there exist $g_{\varepsilon,i} \in B(G_\varepsilon)$ such that $\|g_{\varepsilon,i} - g_i\| < \varepsilon$ for each i. By Lemma 2.1, $\{g_{\varepsilon,1}, \ldots, g_{\varepsilon,n}\}$ is a basis for G_ε, and 2) holds.

In general, if $\|g_i\| > 1$ for some i, let $\rho := \max_i \|g_i\|$. Letting $\bar{g}_i := \frac{g_i}{\|g_i\|}$, we see that $\{\bar{g}_1, \ldots, \bar{g}_n\}$ is a basis for G, and by the first part of the proof, there exists an n-dimensional space $G_\varepsilon = \operatorname{span}\{h_{\varepsilon,1}, \ldots, h_{\varepsilon,n}\}$ having property P such that $\|h_{\varepsilon,i} - \bar{g}_i\| < \frac{\varepsilon}{\rho}$ for all i. Setting $g_{\varepsilon,i} = \|g_i\| h_{\varepsilon,i}$ for each i, we see that $\{g_{\varepsilon,1}, \ldots, g_{\varepsilon,n}\}$ is a basis for G_ε and

$$\|g_{\varepsilon,i} - g_i\| = \big\| \|g_i\| h_{\varepsilon,i} - \|g_i\| \bar{g}_i \big\| = \|g_i\| \|h_{\varepsilon,i} - \bar{g}_i\| < \varepsilon.$$

This proves 1). $\square$

§3. T an Arbitrary Locally Compact Set

Let T be a fixed locally compact Hausdorff space that contains at least $n+1$ points.

Definition 3.1. *Suppose G is an n-dimensional subspace of $C_0(T)$. Then we say that G has the* Chebyshev property *C-1 or C-2, respectively, provided that*

(C-1) *$P_G(f)$ is a singleton for every $f \in C_0(T)$,*

or

(C-2) *each nonzero $g \in G$ has at most $n-1$ distinct zeros in T.*

A space G with property C-1 is usually referred to as a Chebyshev subspace. If G has property C-2, it is usually referred to as a Haar subspace. Basic linear algebra shows that the subspace $G = \operatorname{span}\{g_1, g_2, \ldots, g_n\}$ of $C_0(T)$ is a Haar subspace if and only if

$$D\begin{pmatrix} g_1, & g_2, & \cdots, & g_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} \neq 0$$

for each choice of n distinct points $\{t_1, t_2, \ldots, t_n\}$ in T, where

$$D\begin{pmatrix} g_1, & g_2, & \cdots, & g_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} := \begin{vmatrix} g_1(t_1) & g_2(t_1) & \cdots & g_n(t_1) \\ g_1(t_2) & g_2(t_2) & \cdots & g_n(t_2) \\ \vdots & \vdots & \cdots & \vdots \\ g_1(t_n) & g_2(t_n) & \cdots & g_n(t_n) \end{vmatrix} \tag{3.1}$$

is the determinant of the $n \times n$ matrix whose ijth entry is $g_j(t_i)$. Equivalently, G is a Haar subspace if and only if for any distinct $t_1, t_2, \ldots, t_n$ and any real numbers $z_1, \ldots, z_n$, there exists a unique $g \in G$ such that

$$g(t_i) = z_i, \qquad i = 1, \ldots, n.$$

Haar [5] showed that the Chebyshev and Haar properties are *equivalent* when T is compact, see also [10]. The analogous result for locally compact T was established by Phelps [12], see also [1] and [15, pp. 215–218]. Thus, we have

Fact 3.2. *A finite-dimensional subspace of $C_0(T)$ has property C-1 if and only if it has property C-2.*

We now recall two further properties of finite-dimensional subspaces of $C_0(T)$ that are related to the ability to interpolate given values at given points.

Definition 3.3. [2]. *An n-dimensional subspace G of $C_0(T)$ is said to be* weakly interpolating *(WI) provided that for each set of n distinct points $t_i \in T$ and each set of n signs $\sigma_i \in \{-1, 1\}$, there exist neighborhoods U_i of t_i and a nontrivial $g \in G$ such that*

$$\sigma_i g(t) \geq 0 \quad \text{for all } t \in U_i \text{ and } i = 1, 2, \ldots, n. \tag{3.2}$$

G is said to be interpolating *(I) if strict inequality holds in (3.2).*

So far we have discussed only properties of a subspace G which can be defined wholly in terms of elements of the subspace G without any reference to any elements outside of G. We call such properties intrinsic properties. We now introduce several nonintrinsic properties. Recall that a point $t \in T$ is called a peak point for $f \in C_0(T)$ if $|f(t)| = \|f\|$. The following three peak point properties were introduced and studied in [2].

Definition 3.4. *The n-dimensional subspace G is said to have the* peak point properties *P-1, P-2, or P-3 provided that*

(P-1) *for each $f \in C_0(T)$ that has a unique best approximation $g_0 \in G$, $f - g_0$ has at least $n + 1$ peak points;*

$$\begin{array}{ccccc}
\text{P-1} & \Leftarrow & \text{WI} & \Leftarrow & \text{I} \\
\Uparrow & & & & \Updownarrow \\
\text{P-2} & & & & \text{C-1} \\
\Uparrow & & & & \Updownarrow \\
\text{P-3} & & \Leftarrow & & \text{C-2}
\end{array}$$

Fig. 1. Relationships for general locally compact T.

(P-2) *for each $f \in C_0(T)$, there exists $g_0 \in P_G(f)$ such that $f - g_0$ has at least $n+1$ peak points;*

(P-3) *for each $f \in C_0(T)$ and each $g_0 \in P_G(f)$, $f - g_0$ has at least $n+1$ peak points.*

Theorem 3.5. *Let T be an arbitrary locally compact set. Then the various properties introduced in this section are related as shown in Figure 1. Moreover, with the possible exception of the implication WI $\Rightarrow$ P-1, none of the one-sided implications is reversible.*

Proof: Proofs of all of the direct implications in Figure 1 can be found in [2,3]. It is not known whether P-1 $\Rightarrow$ WI, but all of the other one-sided implications are known to be nonreversible, see [2,3]. $\square$

Concerning the question of whether P-1 $\Rightarrow$ WI, it is known (see [2, Theorems 3.10–3.11]) that under certain additional restrictions on G, the answer is affirmative. To conclude this section, we note the (perhaps surprising) fact that for certain sets T, there are no finite-dimensional Chebyshev subspaces in $C_0(T)$ (see Example 8.3 below).

§4. T a Locally Compact Subset of $\mathbb{R}$

Throughout this section we assume that T is a locally compact subset of the set of real numbers $\mathbb{R}$ (with its usual topology). Equivalently (see, e.g., [3]), we assume $\overline{T} \setminus T$ is closed. We now introduce several other types of Chebyshev and weak Chebyshev properties, and discuss the relationships that hold between them.

Definition 4.1. *An n-dimensional subspace G of $C_0(T)$ is said to have the* Chebyshev property C-3 *provided that for any basis $\{g_1, g_2, \ldots, g_n\}$ of G and every collection of points $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots < s_n$ in T,*

$$D\begin{pmatrix} g_1, & g_2, & \cdots, & g_n \\ t_1, & t_2, & \cdots, & t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1, & g_2, & \cdots, & g_n \\ s_1, & s_2, & \cdots, & s_n \end{pmatrix} > 0. \tag{4.1}$$

It is not hard to show that an n-dimensional subspace G has property C-3 if and only if (4.1) holds for *some* basis of G, see [13, p. 32, Theorem 2.26]. The following result explores the connection between the properties C-1, C-2, and C-3. Recall that an interval in $\mathbb{R}$ is any set of the form $[a,b]$, $[a,b)$, $(a,b]$, or (a,b), where $a \le b$, and $a = -\infty$ or $b = \infty$ is allowed on the open ends. Note that every interval is locally compact.

Theorem 4.2. *Let G be an n-dimensional subspace of $C_0(T)$.*

1) *If G has property C-3, then G has property C-1. The converse does not hold in general.*
2) *If T is an interval, then G has any one of the properties C-1, C-2, or C-3 if and only if it has all of them.*

Proof: Fact 3.2 asserts that C-1 and C-2 are equivalent. To prove 1), suppose G has property C-3. Then it is clear from (4.1) that G has property C-2, and hence also C-1. To see that the converse fails, see either Example 8.4 or Example 8.5 below.

To prove 2), suppose T is an interval and that $G = \operatorname{span}\{g_1, g_2, \ldots, g_n\}$ is an n-dimensional subspace having property C-2 but not property C-3. In this case there must exist points $t_1 < \cdots < t_n$ and $s_1 < \cdots < s_n$ in T such that

$$D\begin{pmatrix} g_1, & \cdots, & g_n \\ t_1, & \cdots, & t_n \end{pmatrix} < 0 < D\begin{pmatrix} g_1, & \cdots, & g_n \\ s_1, & \cdots, & s_n \end{pmatrix}.$$

For each $\lambda \in [0,1]$, let $r_i(\lambda) = \lambda t_i + (1-\lambda)s_i$ for $i = 1, 2, \ldots, n$. Then each $r_i(\lambda) \in T$ since T is an interval, $r_i(0) = s_i$, $r_i(1) = t_i$, and $r_1(\lambda) < \cdots < r_n(\lambda)$ so that $r_1(\lambda), \ldots, r_n(\lambda)$ form a distinct set of points for each λ. Now $r_i(\lambda)$ is a continuous function of λ implies that $D\begin{pmatrix} g_1, & \cdots, & g_n \\ r_1(\lambda), & \cdots, & r_n(\lambda) \end{pmatrix}$ is also a continuous function of λ. By the mean value theorem, it follows that there is some $\lambda \in (0,1)$ such that $D\begin{pmatrix} g_1, & \cdots, & g_n \\ r_1(\lambda), & \cdots, & r_n(\lambda) \end{pmatrix} = 0$, which contradicts our assumption that G has property C-2. □

The following types of "weak Chebyshev" properties were introduced in [3]. They are all intrinsic.

Definition 4.3. *An n-dimensional subspace $G = \operatorname{span}\{g_1, g_2, \ldots, g_n\}$ of $C_0(T)$ is said to have the* Weak-Chebyshev property *W-1, W-1′, W-2, W-2′, W-3, or W-4 provided that:*

(W-1′) for every $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$ with $t_1, \ldots, t_{n-1} \in T$, there exists a nontrivial $g \in G$ such that

$$(-1)^i g(t) \geq 0 \text{ for all } t \in [t_i, t_{i+1}) \cap T, \quad \text{for } i = 0, \ldots, n-1;$$

(W-1) for every $1 \leq m \leq n$ and $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ with $t_1, \ldots, t_{m-1} \in T$, there exists a nontrivial $g \in G$ such that

$$(-1)^i g(t) \geq 0 \text{ for all } t \in [t_i, t_{i+1}) \cap T, \quad \text{for } i = 0, \ldots, m-1;$$

(W-2′) for every $-\infty = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$ with $t_1, \ldots, t_{n-1} \in T$, there exists a nontrivial $g \in G$ such that

$$(-1)^i g(t) \geq 0 \text{ for all } t \in [t_i, t_{i+1}] \cap T, \quad \text{for } i = 0, \ldots, n-1;$$

(W-2) for every $1 \leq m \leq n$ and $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$ with $t_1, \ldots, t_{m-1} \in T$, there exists a nontrivial $g \in G$ such that

$$(-1)^i g(t) \geq 0 \quad \text{for all } t \in [t_i, t_{i+1}] \cap T, \qquad \text{for } i = 0, \ldots, m-1;$$

(W-3) for each choice of points $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots s_n$ in T,

$$D\begin{pmatrix} g_1, & \cdots, & g_n \\ t_1, & \cdots, & t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1, & \cdots, & g_n \\ s_1, & \cdots, & s_n \end{pmatrix} \geq 0;$$

(W-4) each $g \in G$ has at most $n-1$ sign changes, i.e., there do not exist $n+1$ points $t_1 < \cdots < t_{n+1}$ in T such that $g(t_i)g(t_{i+1}) < 0$ for $i = 1, \ldots, n$.

Despite the terminology, it turns out that a Chebyshev subspace is not always a Weak Chebyshev subspace (see Example 8.4 below). Recall that a set of points $t_1 < t_2 < \cdots < t_k$ in T are called alternating peak points for the function $f \in C_0(T)$ if there exists $\sigma = \pm 1$ such that $f(t_i) = \sigma(-1)^i \|f\|$ for $i = 1, \ldots, k$.

Definition 4.4. *An n-dimensional subspace G of $C_0(T)$ is said to have the* alternation property *A-1, A-2, or A-3 provided*

(A-1) *for every $f \in C_0(T)$ that has a unique best approximation $g_0 \in G$, $f - g_0$ has at least $n+1$ alternating peak points;*

(A-2) *for every $f \in C_0(T)$, there exists $g_0 \in P_G(f)$ such that $f - g_0$ has at least $n+1$ alternating peak points;*

(A-3) *for every $f \in C_0(T)$ and every $g_0 \in P_G(f)$, $f - g_0$ has at least $n+1$ alternating peak points.*

Theorem 4.5. *Let T be a locally compact subset of $\mathbb{R}$. Then the various properties introduced above are related as shown in Figure 2. It is an open question whether P-1 $\Rightarrow$ WI, but none of the other one-sided implications can be reversed.*

Proof: The direct implications are established in [2,3]. These papers also establish the nonreversibility of all one-sided implications, except for the following two cases:

1) A-1 $\not\Leftarrow$ A-2: If this implication were true, then it would follow from Figure 2 that property W-1 implies property W-2$'$, which is known not to be the case.

2) A-2 $\not\Leftarrow$ A-3: If this implication were true, then it would follow from Figure 2 that property W-2 implies property C-2, which is known not to be the case. $\square$

$$
\begin{array}{ccccccccccc}
 & & \text{W-1}' & & & & & & & & \\
 & & \Uparrow & & & & & & & & \\
 & & \text{W-1} & \Leftrightarrow & \text{A-1} & \Rightarrow & \text{P-1} & \Leftarrow & \text{WI} & & \\
 & & \Uparrow & & & & & & & & \\
 & & \text{W-2}' & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & \Updownarrow & & & & & & & & \\
\text{WI} & \Leftarrow & \text{W-2} & \Leftrightarrow & \text{A-2} & \Rightarrow & \text{P-2} & \Leftarrow & \text{I} & & \\
 & & \Updownarrow & & & & & & \Updownarrow & & \\
 & & \text{W-3} & & & & & & & & \\
 & & \Updownarrow & & \Uparrow & & \Uparrow & & \text{C-1} & & \\
 & & \text{W-4} & & & & & & & & \\
 & & \Uparrow & & & & & & \Updownarrow & & \\
\text{C-3} & \Rightarrow & \text{C-1} + \text{W-2} & \Leftrightarrow & \text{A-3} & \Rightarrow & \text{P-3} & \Leftrightarrow & \text{C-2} & \Leftarrow & \text{C-3}
\end{array}
$$

Fig. 2. Relationships for T a locally compact subset of $\mathbb{R}$.

§5. T an Interval

Theorem 5.1. *Let T be an interval. Then the various properties introduced above are related as shown in Figure 3. Moreover, with the possible exception of the implication WI $\Rightarrow$ P-1, none of the one-sided implications can be reversed.*

Proof: Except for A-3 $\Leftrightarrow$ P-3, all of the implications indicated by the arrows in Figure 3 are explicitly stated in [2,3]. The implication A-3 $\Rightarrow$ P-3 is trivial. Now suppose G has property P-3. Then G has property C-1 by [3, Theorem 7.1], and by [3, Corollary 3.2(4)(c)] also property W-2. Thus, G has both of the properties C-1 and W-2, and hence must also have property A-3 by [3, Theorem 5.1].

The non-reversibility of the one-sided implications was established in [2,3] except for the cases listed in the proof of Theorem 4.5. But the discussion of these cases given there also applies here. $\square$

§6. Properties of Limit Spaces

In this section we answer Question 1.2 concerning what properties a space G must have if it is the limit of a sequence of a spaces G_k with some specified property. Our results are summarized in Tables 1 – 3, which correspond to the cases where

1) T is an interval,
2) T is a locally compact subset of $\mathbb{R}$,
3) T is an arbitrary locally compact set.

$$
\begin{array}{ccccccccccc}
 & & \text{W-1}' & & & & & & & & \\
 & & \Updownarrow & & & & & & & & \\
 & & \text{W-1} & \Leftrightarrow & \text{A-1} & \Rightarrow & \text{P-1} & \Leftarrow & \text{WI} & & \\
 & & \Updownarrow & & & & & & & & \\
 & & \text{W-2}' & & \Updownarrow & & \Uparrow & & \Uparrow & & \\
 & & \Updownarrow & & & & & & & & \\
\text{WI} & \Leftarrow & \text{W-2} & \Leftrightarrow & \text{A-2} & \Rightarrow & \text{P-2} & \Leftarrow & \text{I} & & \\
 & & \Updownarrow & & & & & & \Updownarrow & & \\
 & & \text{W-3} & & & & & & & & \\
 & & \Updownarrow & & \Uparrow & & \Uparrow & & \text{C-1} & & \\
 & & \text{W-4} & & & & & & & & \\
 & & \Uparrow & & & & & & \Updownarrow & & \\
\text{C-3} & \Leftrightarrow & \text{C-1} + \text{W-2} & \Leftrightarrow & \text{A-3} & \Leftrightarrow & \text{P-3} & \Leftrightarrow & \text{C-2} & \Leftrightarrow & \text{C-3}
\end{array}
$$

Fig. 3. The case where T is an interval in $\mathbb{R}$.

In each of the tables, the properties of G are listed in the first column, and the properties of the G_k are listed across the top. An entry Y (yes) in the (i, j)-th position in the table means that if a sequence of subspaces G_k has the property in the j-th column, then the limit space G has the property in the i-th row. An entry N (no) means that the limit space does not have that property.

To simplify the tables, we have combined groups of equivalent properties. Thus, in Table 1, C stands for C-1 $\equiv$ C-2 $\equiv$ C-3. Similarly, W stands for any of the equivalent properties W-1, W-1$'$, W-2, W-2$'$, W-3, or W-4. In Table 2, C-2 $\equiv$ C-1. Note that we have ordered the properties so that in general (but not always) the properties become weaker as we move down or to the right. The process of justifying the entries in the tables can be greatly simplified by taking account of the following observations:

- An N in a table implies an N in the same position in subsequent tables (since T is increasingly more general).
- A Y in a table implies a Y in the same position in previous tables.
- An N in the (i, j)-th position of a table implies an N in row i and all columns corresponding to properties that are weaker than the property in column j. It also implies an N in column j and all rows corresponding to properties that are stronger than the property in row i.
- A Y in the (i, j)-th position of a table implies a Y in row i and all columns corresponding to properties that are stronger than the property in column j. It also implies a Y in column j and all rows corresponding to properties that are weaker than the property in row i.

	C	W	P-2	WI	P-1
C	N	N	N	N	N
W	Y	Y	N	N	N
P-2	Y	Y		N	N
WI	Y	Y		N	N
P-1	Y	Y		N	N

Table 1. Limit relations for T an interval.

	C-3	C-2	W	W-1	P-2	WI	P-1
C-3	N	N	N	N	N	N	N
C-2	N	N	N	N	N	N	N
W	Y	N	Y	N	N	N	N
W-1	Y	N	Y	Y	N	N	N
P-2	Y		Y	N		N	N
WI	Y		Y	N		N	N
P-1	Y		Y	Y		N	N

Table 2. Limit relations for T an arbitrary locally compact subset of $\mathbb{R}$.

	C-2	WI	P-3	P-2	P-1
C-2	N	N	N	N	N
P-3		N		N	N
WI		N			N
P-2		N			N
P-1		N			N

Table 3. Limit relations for an arbitrary locally compact T.

- If a subspace G has property P but not property Q, then the constant sequence $G_k = G$ will have the property P, but the limit will not have the property Q.

We now give explicit justifications for the key entries in each of the tables. All other entries can be deduced using the above principles. We identify each entry by the pair (i, j) describing its position in the table.

Table 1.

(1,1): Example 8.1 shows that this entry is N. This gives N in all other columns of row 1.

(2,2): Theorem 9.1 shows that this entry is Y. This implies a Y in the $(i, 1)$ and $(i, 2)$ positions for all $i \geq 2$.

$(i,4)$ and $(i,5)$ for $1 \leq i \leq 5$: To see that these entries are N, we can take $G_k = G$ to be as in Example 8.2.

Table 2.

(3,3), (4,4): Theorem 9.1 shows that these entries are Y. This implies that the entries in $(i,3)$ with $i \geq 3$ are also Y. Since C-3 implies W, we know that the entry in (3,1) is also Y. This implies that the entries in $(i,1)$ with $i \geq 3$ are also Y. The Y in (4,4) implies a Y in (7,4).

(3,4). To see that this entry is N, we can take $G_k = G$ to be a space that does not have property W, but which does have the property W-1.

(4,2): We can take $G_k = G$ to be the space in Example 8.5.

(4,5): To see that this entry is N, we can take $G_k = G$ to be the space in Example 8.5.

§7. Approximability of Spaces

In this section we answer Question 1.3 concerning whether a space G with a given property can be approximated by a sequence of a spaces G_k with some other property. Our results are summarized in Tables 4 – 6. In each of the tables the properties of G are listed in the first column, and the properties of the G_k are listed in the first row. An entry Y (yes) in the (i,j)-th position in the table means that a space G with the property in the i-th row can be approximated by spaces G_k with the property in the j-th column. An entry N (no) means that G cannot be approximated by such a sequence.

As in Section 6, to simplify the tables we have combined groups of equivalent properties. Thus, in Table 4, C stands for C-1 $\equiv$ C-2 $\equiv$ C-3. Similarly, W stands for any of the equivalent properties W-1, W-1$'$, W-2, W-2$'$, W-3, or W-4. In Table 5, C-2 $\equiv$ C-1.

The process of justifying the entries in the tables can be greatly simplified by taking account of the following observations:

- All three tables contain Y's on the diagonals since a space can always be approximated by itself.
- An N in a table implies an N in the same position in subsequent tables (since the tables deal with increasingly more general sets T).
- A Y in a table implies a Y in the same position in previous tables.
- An N in the (i,j)-th position of a table implies an N in column j and all other rows corresponding to properties that are weaker than the property in row i. It also implies an N in row i and all other columns corresponding to properties that are stronger than the property in column j.
- A Y in the (i,j)-th position of a table implies a Y in row i for all other columns corresponding to properties that are weaker than the property in column j. It

	C	W	P-2	WI	P-1
C	Y	Y	Y	Y	Y
W	Y	Y	Y	Y	Y
P-2	N	N	Y		Y
WI	N	N		Y	Y
P-1	N	N			Y

Table 4. Approximability for T an interval.

	C-3	C-2	W	W-1	P-2	WI	P-1
C-3	Y	Y	Y	Y	Y	Y	Y
C-2	N	Y	N	N	Y	Y	Y
W			Y	Y	Y	Y	Y
W-1	N		N	Y			Y
P-2	N		N	N	Y		Y
WI	N		N	N		Y	Y
P-1	N		N				Y

Table 5. Approximability for T a locally compact subset of $\mathbb{R}$.

	C-2	WI	P-3	P-2	P-1
C-2	Y	Y	Y	Y	Y
P-3			Y	Y	Y
WI	N	Y			Y
P-2	N			Y	Y
P-1	N				Y

Table 6. Approximability for arbitrary locally compact T.

also implies a Y in column j and all other rows corresponding to properties that are weaker than the property in row i.

- Many entries of N can be deduced from the limit properties of Tables 1–3, since if a subspace G has property P but not property Q, and if the sequence G_k has property Q in the limit, then we cannot approximate G by such G_k.

We now give explicit justifications for the key entries in Tables 4–6. All other entries can be deduced by using the above principles. We identify each entry by the pair (i, j) describing its position in the table. As observed above, we clearly have Y on the diagonals of all three tables. The many Y's in the upper triangular parts of the table follow from the fifth observation in the above list.

Table 4.

(2,1): Theorem 1.4 implies that this entry is Y.

(3,2): Taking G to be a space which has property P2 but not property W, we conclude that this entry must be N, since by Table 1 the limit of spaces with property W must have property W.

(4,2): We take G to be a space with property WI but not property W.

Table 5.

(3,1): This entry remains open, but Theorem 11.1 implies that it would be Y if we allow approximation by functions which are just continuous and bounded (rather than in $C_0(T)$).

(4,3): Taking G to be a space which has property W-1 but not property W, we conclude that the entry must be N, since by Table 1 the limit of spaces with property W must have property W.

Table 6.

(3,1), (4,1), (5,1): To see that these entries are N, we can take any space G with the desired property defined on a uncountable set with the discrete topology, since then (cf. Example 8.3) there are no Chebyshev subspaces.

§8. Examples

In this section we collect several examples that are used throughout the paper. Recall that the characteristic function of a subset $S \subset T$ is the function $\chi_S : T \to \mathbb{R}$ whose value is 1 if $t \in S$ and is 0, otherwise. A delta function at the point $t_0 \in T$, denoted δ_{t_0}, is the characteristic function of the singleton $\{t_0\}$, i.e., $\delta_{t_0} = \chi_{\{t_0\}}$. Clearly, δ_{t_0} is in $C_0(T)$ if and only if t_0 is an isolated point of T.

Example 8.1. *Let $T := [0,1]$ and let G be the one-dimensional space spanned by $g(t) = t$. For each $k \geq 1$, let G_k be the one-dimensional space spanned by $g_k(t) = 1/k + t(k-1)/k$. Then $g_k \to g$, and although each of the subspaces G_k has property C-3, the limit space G does not have any of the equivalent properties C-1, C-2, or C-3.*

Proof: For each $k \geq 1$, it is easy to check from the definition that G_k has property C-3 since g_k is positive on $[0,1]$. Since g has a zero at 0, the subspace G does not have property C-2. By Theorem 4.2, all the C-i properties are equivalent. □

Example 8.2. *Let $T := [-1,1]$ and let G be the one-dimensional space spanned by $g(t) = t$. Then G has none of the properties in Figure 3. However, G is approximable by the sequence of one-dimensional spaces G_k spanned by*

$$g_k(t) = \begin{cases} t + 1/k, & -1 \leq t \leq -1/k, \\ 0, & -1/k \leq t \leq 1/k, \\ t - 1/k, & 1/k \leq t \leq 1, \end{cases}$$

and the G_k have property WI for all $k \geq 1$.

Proof: For the first assertion, it suffices to show that G does not have property P-1. To see this, we observe that the function $f(t) := 1 - t^2$ has a unique best approximation in G, namely $g \equiv 0$, and the difference $f - g$ has just one peak point. It is clear that the G_k approximate G, and it is easy to check from the definition that each G_k has property WI. □

Example 8.3. *Let T be an uncountable set with the discrete topology (i.e., all sets are open). Then there are no finite-dimensional Chebyshev subspaces in $C_0(T)$.*

Proof: Clearly, the only compact sets in T are the finite sets. Thus, given a function $f : T \to \mathbb{R}$, we see that $f \in C_0(T)$ if and only if for each $\varepsilon > 0$, the set $\{t \in T : |f(t)| \geq \varepsilon\}$ is finite. Thus,

$$\operatorname{supp} f := \{t \in T : f(t) \neq 0\} = \bigcup_{k=1}^{\infty} \{t \in T : |f(t)| \geq 1/k\}$$

is countable. Now let $G = \operatorname{span}\{g_1, g_2, \ldots, g_n\}$ be any n-dimensional subspace of $C_0(T)$, and let $S = \bigcup_1^n \operatorname{supp} g_i$. Then S is countable, and $g(t) = 0$ for all $t \in T \setminus S$ and each $g \in G$. Since S is countable, there exists a point $t_0 \in T \setminus S$, and we define f on T by $f(t_0) = 1$ and $f(t) = 0$ for all $t \in T \setminus \{t_0\}$. Then $f \in C_0(T)$, and for each $g \in G$, $|f(t_0) - g(t_0)| = |f(t_0)| = 1 = \|f\|$, and so $\|f - g\| \geq 1 = \|f\|$ for each $g \in G$. This proves that $0 \in P_G(f)$. But for any $g \in G$ with $\|g\| = 1$, we see that $\|f - g\| = 1$ and hence $g \in P_G(f)$. This proves that both 0 and g are best approximations to f from G. Thus G is not Chebyshev. (This result can also be proved by using Fact 3.2 and the observation that for this class of T, every function in $C_0(T)$ has infinitely many zeros). □

Example 8.4. *Let $T = \{1, 2\}$, and let G be the one-dimensional subspace spanned by the function $g(t) = (-1)^t$. Then G has property C-2, but does not have property C-3 or any of the weak Chebyshev properties. In fact, it is not approximable by spaces having property C-3 or any of the weak Chebyshev properties.*

Proof: Clearly, G has property C-2 since any nontrivial $g \in G$ cannot vanish. On the other hand, since g is not of one sign on T, if g_k is any sufficiently close function, it must also have more than one sign. We conclude that G cannot be approximated by spaces G_k with property W-1$'$ or any of the other Weak-Chebyshev properties. □

As shown in Figure 1, a Chebyshev subspace must have properties I, WI, P-1, P-2, and P-3. We now describe a Chebyshev subspace that, aside from these properties that all Chebyshev subspaces must have, *fails* to have any of the other properties that are studied in this paper.

Example 8.5. *Let $T = [0, 1] \cup \{2\}$, $g_1 = \chi_{[0,1]} - \delta_2$, and $G = \operatorname{span}\{g_1\}$. Then*

1) *G has properties C-1 and C-2, but not property C-3,*

2) *G has the properties I, WI, P-1, P-2, and P-3.*
3) *G fails to have any of the properties W-1, W-1′, W-2, W-2′, W-3, or W-4.*
4) *G fails to have any of the properties A-1, A-2, or A-3.*

Proof: Clearly, nontrivial functions in G cannot have any zeros, so G has the Haar property and is thus Chebyshev. On the other hand,

$$D\begin{pmatrix} g_1 \\ 1 \end{pmatrix} \cdot D\begin{pmatrix} g_1 \\ 2 \end{pmatrix} = 1(-1) = -1 < 0,$$

so G does not have property C-3. This proves 1). The properties listed in 2) follow from Figure 1. To prove 3), note that for 1-dimensional subspaces, all of the weak Chebyshev properties are obviously equivalent to the condition that the basis element g_1 have one sign on T. But $g_1(1)g_1(2) < 0$ shows that this fails. Finally, to prove 4), note that the best approximation in G of the function $f = \delta_2$ is $g_0 := -\frac{1}{2}g_1$, and $f - g_0 \geq 0$. Thus G fails to have property A-1. □

Example 8.6. *Let $T = \{1, 2, 3, 4\}$, $g_1 = \delta_1$, $g_2 = \delta_2 - \delta_3$, and $G = \operatorname{span}\{g_1, g_2\}$. Then G has properties W-1 and W-1′, but not any of the equivalent properties W-2, W-2′, W-3, or W-4. Moreover, G cannot be approximated by either C-3 or W-2 subspaces.*

Proof: To see that G has property W-1, we note that for any $t_1 \in T$, the function g_1 itself has the required signs on $[1, t_1)$ and on $[t_1, 4]$. Clearly, G does not have property W-4 since $g := g_1 - g_2$ has 2 sign changes. This implies G does not have the equivalent properties W-2, W-2′, W-3, and W-4. Now suppose G were approximatable by subspaces having property C-3. Then by the first part of Theorem 11.1 it would have property W-4, which we have just shown is not the case. Finally, G is not approximable by subspaces having property W-2, since as noted in Table 2 the limit of a sequence of spaces with property W-2 must also have property W-2 (and G does not). □

§9. Limits of Sequences of Weak Chebyshev Subspaces

Throughout this section we suppose that T is a locally compact subset of $\mathbb{R}$.

Theorem 9.1. *Suppose that G is approximable by a sequence of subspaces G_k, all having property W, where W is one of the properties W-1, W-1′, W-2, W-2′, W-3, or W-4. Then G also has property W.*

Proof: Suppose $G = \operatorname{span}\{g_1, \ldots, g_n\}$ and $G_k = \operatorname{span}\{g_{k,1}, \ldots, g_{k,n}\}$. We first consider the case where the G_k all have property W-3. Then

$$\begin{aligned} &D\begin{pmatrix} g_1, & \cdots, & g_n \\ t_1, & \cdots, & t_n \end{pmatrix} \cdot D\begin{pmatrix} g_1, & \cdots, & g_n \\ s_1, & \cdots, & s_n \end{pmatrix} \\ &\qquad = \lim_{k\to\infty} D\begin{pmatrix} g_{k,1}, & \cdots, & g_{k,n} \\ t_1, & \cdots, & t_n \end{pmatrix} \cdot D\begin{pmatrix} g_{k,1}, & \cdots, & g_{k,n} \\ s_1, & \cdots, & s_n \end{pmatrix} \geq 0, \end{aligned}$$

for all $t_1 < t_2 < \cdots < t_n$ and $s_1 < s_2 < \cdots s_n$ in T, and thus G also has property W-3. This proves the theorem for W-3.

Now suppose that the G_k have property W-4, but that G does not. Then there exist $n+1$ points $t_1 < \cdots < t_{n+1}$ in T and nontrivial $g \in G$ such that

$$g(t_i)g(t_{i+1}) < 0, \qquad i = 1, \ldots, n.$$

Without loss of generality we may assume that $\|g\| = 1$. Let

$$\varepsilon := \min_{1 \le i \le n+1} |g(t_i)|.$$

Then $\varepsilon > 0$, and there exists a subspace G_ε with property W-4 such that

$$\sup_{g \in B(G)} d(g, B(G_\varepsilon)) < \varepsilon.$$

Hence, there exists $g_\varepsilon \in B(G_\varepsilon)$ with $\|g - g_\varepsilon\| < \varepsilon$, and in particular

$$|g(t_i) - g_\varepsilon(t_i)| < \varepsilon, \qquad i = 1, \ldots, n+1.$$

This implies that $g_\varepsilon(t_i)g(t_i) > 0$, and it follows that

$$g_\varepsilon(t_i)g_\varepsilon(t_{i+1}) < 0, \qquad i = 1, \ldots, n,$$

which implies that G_ε could not have property W-4, a contradiction. This proves the theorem for W-4.

We now prove the theorem when W denotes one of the properties W-1$'$, W-1, W-2$'$, or W-2. Such a property can always be expressed in the following form: G has property W iff for each set of points $-\infty = t_0 < t_1 < \cdots < t_{m-1} < t_m = \infty$, there exists a nontrivial $g \in G$ with

$$(-1)^i g(t) \ge 0, \qquad \text{for all } t \in S_i, \quad i = 0, \ldots, m-1,$$

where $S_i = [t_i, t_{i+1}) \cap T$ or $S_i = [t_i, t_{i+1}] \cap T$ and $1 \le m \le n$ or $m = n$.

Now suppose each G_k has property W. Then there exist nontrivial $h_k \in G_k$ such that

$$(-1)^i h_k(t) \ge 0, \qquad \text{for all } t \in S_i, \quad i = 0, \ldots, m-1. \tag{9.1}$$

Without loss of generality, we may assume $\|h_k\| = 1$ for all k. Since all norms are equivalent on a finite-dimensional space, there exist constants $c_k > 0$ and $c > 0$ such that for all $\alpha \in \mathbb{R}^n$ and all k,

$$\Big\|\sum_{i=1}^n \alpha_i g_{k,i}\Big\| \ge c_k \sum_{i=1}^n |\alpha_i|,$$
$$\Big\|\sum_{i=1}^n \alpha_i g_i\Big\| \ge c \sum_{i=1}^n |\alpha_i|.$$

In fact (see A. E. Taylor [16, proof of Theorem 3.12A, p. 95]),

$$c_k = \inf\Big\{\Big\|\sum_{i=1}^{n}\alpha_i g_{k,i}\Big\| : \sum_{i=1}^{n}|\alpha_i| = 1\Big\},$$

$$c = \inf\big\{\big\|\sum_{i=1}^{n}\alpha_i g_i\big\| : \sum_{i=1}^{n}|\alpha_i| = 1\big\}.$$

We now show that $c_k \to c$ as $k \to \infty$. Suppose $\varepsilon > 0$, and choose k sufficiently large so that

$$\|g_{k,i} - g_i\| < \varepsilon, \qquad i = 1, \ldots, n.$$

This implies that for every $\alpha \in \mathbb{R}^n$ with $\sum_{i=1}^{n}|\alpha_i| = 1$,

$$\begin{aligned}\left|\Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\| - \Big\|\sum_{i=1}^{n}\alpha_i g_{k,i}\Big\|\right| &\le \Big\|\sum_{i=1}^{n}\alpha_i g_i - \sum_{i=1}^{n}\alpha_i g_{k,i}\Big\| \\ &\le \sum_{i=1}^{n}|\alpha_i|\|g_i - g_{k,i}\| < \sum_{i=1}^{n}|\alpha_i|\varepsilon = \varepsilon.\end{aligned}$$

Thus,

$$\Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\| - \varepsilon < \Big\|\sum_{i=1}^{n}\alpha_i g_{k,i}\Big\| < \Big\|\sum_{i=1}^{n}\alpha_i g_i\Big\| + \varepsilon$$

for every $\alpha \in \mathbb{R}^n$ with $\sum_{i=1}^{n}|\alpha_i| = 1$. It follows that $c - \varepsilon \le c_k \le c + \varepsilon$, and so $c_k \to c$ as asserted.

For the remainder of the proof we assume that k is sufficiently large so that $c_k > c/2$. We now show that the sequences $(\alpha_{k,i})_{k=1}^{\infty}$ are uniformly bounded for all $i = 1, \ldots, n$. We have

$$1 = \|h_k\| = \Big\|\sum_{i=1}^{n}\alpha_{k,i} g_{k,i}\Big\| \ge c_k \sum_{i=1}^{n}|\alpha_{k,i}| \ge \frac{c}{2}\sum_{i=1}^{n}|\alpha_{k,i}|.$$

Hence,

$$\sum_{i=1}^{n}|\alpha_{k,i}| \le \frac{2}{c} \qquad \text{for all } k,$$

and in particular $|\alpha_{k,i}| \le 2/c$ for all k and i.

To complete the proof, we now assume (by passing to a subsequence if necessary) that

$$\alpha_{k,i} \to \alpha_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Since $g_{k,i}$ converges uniformly to g_i for $i = 1, \ldots, n$, it follows that

$$h_k = \sum_{i=1}^{n}\alpha_{k,i} g_{k,i} \to \sum_{i=1}^{n}\alpha_i g_i =: g.$$

Clearly, $g \in G$, $\|g\| = 1$ (since $\|h_k\| = 1$ for all k), and by (9.1),

$$(-1)^i g(t) \ge 0 \qquad \text{for all } t \in S_i, \quad i = 0, \ldots, m-1.$$

Thus, G also has property W as asserted. □

§10. Properties of Extended Functions

A key tool for our study of the approximability of weak Chebyshev spaces by Chebyshev spaces is the idea of *extending* a function defined on T to an interval, see [2]. Given T, let $\widetilde{T}$ be the smallest closed interval containing T, i.e., $\widetilde{T}$ is the intersection of all closed intervals containing T. Given a function $f \in C_0(T)$, we define its extension $\widetilde{f}$ on $\widetilde{T}$ as follows:

$$\widetilde{f}(t) = \begin{cases} f(t), & \text{if } t \in T, \\ 0, & \text{if } t \in \overline{T} \setminus T, \\ \text{linear}, & \text{in each open subinterval of } \widetilde{T} \setminus \overline{T}. \end{cases}$$

It is shown in [2] that for every function $f \in C_0(T)$, the extension $\widetilde{f}$ belongs to $C_0(\widetilde{T})$.

Given an n-dimensional space $G = \operatorname{span}\{g_1, \ldots, g_n\}$, we write $\widetilde{G}$ for the subspace of all extensions of elements in G. Then it is easy to see that $\widetilde{G} := \operatorname{span}\{\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_n\}$. The following lemma was established in [2].

Lemma 10.1. *A finite-dimensional subspace G of $C_0(T)$ has all of the equivalent properties W-2, W-2′, W-3, and W-4 if and only $\widetilde{G}$ has the same properties in $C_0(\widetilde{T})$.*

We now show that the analog of this lemma fails for the properties W-1 or W-1′.

Proposition 10.2. *Let $T = \{1, 2, 3, 4\}$, $g_1 = \delta_1$, $g_2 = \delta_2 - \delta_3$, and suppose $G = \operatorname{span}\{g_1, g_2\}$. Then G is a 2-dimensional subspace of $C_0(T)$ that has properties W-1 and W-1′, but $\widetilde{G}$ fails to have either property in $C_0(\widetilde{T})$.*

Proof: The fact that G has properties W-1 and W-1′ was shown in Example 8.6. We now show that the extension $\widetilde{G}$ does *not* have property W-1′. Here $\widetilde{T} = [1, 4]$, and $\widetilde{g}_1$ and $\widetilde{g}_2$ are piecewise linear functions on $\widetilde{T}$ with knots at the integers $\{2, 3\}$. For any given $1 < t_1 < 2$, it is impossible to find a $\widetilde{g} \in \widetilde{G}$ that is nonnegative on $[1, t_1]$ and nonpositive on $[t_1, 4]$, and so $\widetilde{G}$ does not have either property W-1 or W-1′. $\square$

§11. A Generalization of the Jones-Karlovitz Theorem

Throughout this section we suppose that T is a locally compact subset of $\mathbb{R}$ and that $C_b(T)$ is the Banach space of all real-valued continuous bounded functions f on T equipped with the supremum norm $\|f\| := \sup_{t \in T} |f(t)|$. Note that $C_0(T) \subset C_b(T)$, but the spaces are not the same in general. For example, consider $T := (0, 1]$. Then the nontrivial constant functions are in $C_b(T)$ but not in $C_0(T)$, since $f \in C_0(T)$ implies that $\lim_{t \to 0+} f(t) = 0$. However, when T is compact, $C_0(T) = C_b(T) = C(T)$.

Suppose G is a subspace of $C_b(T)$ that has one of the properties W-2, W-2′, W-3, or W-4. Since these properties are equivalent in this setting (cf. Figure 2), we follow the convention used in Tables 2 and 5, and simply write W for this property. We call such a subspace a weak Chebyshev subspace.

Theorem 11.1. *Let $G := \operatorname{span}\{g_1, \ldots, g_n\}$ be an n-dimensional weak Chebyshev subspace of $C_0(T)$. Then there exists a sequence of subspaces $H_k := \operatorname{span}\{h_{k,1}, \ldots, h_{k,n}\}$ in $C_b(T)$ such that*

1) *each H_k has property C-3,*
2) $\lim_{k\to\infty} \|h_{k,i} - g_i\| = 0$ *for $i = 1, \ldots, n$.*

In other words, every weak Chebyshev subspace in $C_0(T)$ is approximable by C-3 subspaces in $C_b(T)$.

Proof: Since $\widetilde{T}$ is the smallest closed interval containing T, it must have one of the following forms: $(-\infty, \infty)$, $(-\infty, b]$, $[a, \infty)$, or $[a, b]$, where $a < b$. Now let $\widetilde{G} := \operatorname{span}\{\widetilde{g}_1, \ldots, \widetilde{g}_n\}$ be the space obtained from G by extension as described in Section 10. Since G has property W-3 in $C_0(T)$, it follows that $\widetilde{G}$ has property W-3 in $C_0(\widetilde{T})$. We now further extend each $\widetilde{g}_i$ to all of $\mathbb{R}$ by setting $\widetilde{g}_i(t) = \widetilde{g}_i(a)$ for all $t < a$ and $\widetilde{g}_i(t) = \widetilde{g}_i(b)$ for all $t > b$. Note that the (extended) space $\widetilde{G}$ has property W-3 in $C_b(\mathbb{R})$.

For each $i = 1, \ldots, n$ and each $k \in \mathbb{N}$, let $h_{k,i}$ be the function mapping T to $\mathbb{R}$ defined by

$$h_{k,i}(t) := \int_{-\infty}^{\infty} K_k(t-s)\widetilde{g}_i(s)\, ds, \qquad t \in T, \tag{11.1}$$

where

$$K_k(u) := \frac{k}{\sqrt{\pi}} e^{-k^2u^2}, \qquad u \in \mathbb{R}, \tag{11.2}$$

is the standard Gauss (or Weirstrass) kernel. We recall the following well-known properties:

$$K_k \in C_0(\mathbb{R}) \text{ and } K_k(u) \geq 0 \text{ for all } u \in \mathbb{R}, \tag{11.3}$$

$$\int_{-\infty}^{\infty} K_k(u)\, du = 1, \qquad k = 1, 2, \ldots, \tag{11.4}$$

$$\int_{-\infty}^{\infty} u^2 K_k(u)\, du = \frac{1}{k^2}, \qquad k = 1, 2, \ldots, \tag{11.5}$$

Clearly, for every $t \in T$, (11.4) implies that

$$|h_{k,i}(t)| \leq \int_{-\infty}^{\infty} K_k(t-s)\|\widetilde{g}_i\|\, ds = \|\widetilde{g}_i\|, \tag{11.6}$$

and so $h_{k,i}$ is a well-defined bounded function on T. We now show that it is continuous. Fix any $t_0 \in T$, and let $\varepsilon > 0$. By (11.4), there exits $M > 0$ such that

$$\int_{\mathbb{R}\setminus[-M,M]} K_k(s)\,ds < \frac{\varepsilon}{4\|\widetilde{g}_i\|}. \tag{11.7}$$

By (11.3), K_k is uniformly continuous on $\mathbb{R}$, and thus there exits a $\delta \in (0,1)$ such that

$$|K_k(u) - K_k(v)| < \frac{\varepsilon}{4(M + |t_0| + 1)\|\widetilde{g}_i\|}, \tag{11.8}$$

whenever $|u - v| < \delta$. It follows that for each $t \in T$ with $|t - t_0| < \delta$, we have

$$\begin{aligned} |h_{k,i}(t) - h_{k,i}(t_0)| &= \Big| \int_{\mathbb{R}} [K_k(t-s) - K_k(t_0 - s)]\widetilde{g}_i(s)\,ds \Big| \\ &\le \int_{\mathbb{R}} \big|K_k(t-s) - K_k(t_0 - s)\big| |\widetilde{g}_i(s)|\,ds \\ &= I_1 + I_2, \end{aligned} \tag{11.9}$$

where

$$I_1 := \int_{[-(M+|t_0|+1),M+|t_0|+1]} \big|K_k(t-s) - K_k(t_0 - s)\big| |\widetilde{g}_i(s)|\,ds$$

and

$$I_2 := \int_{\mathbb{R}\setminus[-(M+|t_0|+1),M+|t_0|+1]} \big|K_k(t-s) - K_k(t_0 - s)\big| |\widetilde{g}_i(s)|\,ds.$$

Using (11.8), we obtain

$$I_1 \le \int_{[-(M+|t_0|+1),M+|t_0|+1]} \frac{\varepsilon}{4(M + |t_0| + 1)\|\widetilde{g}_i\|} \|\widetilde{g}_i\|\,ds = \frac{\varepsilon}{2}.$$

Using (11.7), we obtain

$$\begin{aligned} I_2 &\le \|\widetilde{g}_i\| \int_{\mathbb{R}\setminus[-(M+|t_0|+1),M+|t_0|+1]} [K_k(t-s) - K_k(t_0 - s)]\,ds \\ &< \|\widetilde{g}_i\| \Big(\frac{\varepsilon}{4\|\widetilde{g}_i\|} + \frac{\varepsilon}{4\|\widetilde{g}_i\|} \Big) = \frac{\varepsilon}{2}. \end{aligned}$$

Substituting these two estimates into (11.9) gives

$$|h_{k,i}(t) - h_{k,i}(t_0)| < \varepsilon,$$

whenever $t \in T$ and $|t - t_0| < \delta$. This shows that $H_k \subset C_b(T)$.

Since $\widetilde{g}_i \in C_0(\widetilde{T})$, it follows that $\widetilde{g}_i$ is uniformly continuous on $\widetilde{T}$. By the way in which $\widetilde{g}_i$ was extended to $\mathbb{R}$, it follows that $\widetilde{g}_i$ is uniformly continuous on $\mathbb{R}$. Thus, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $s, t \in \mathbb{R}$,

$$|\widetilde{g}_i(s) - \widetilde{g}_i(t)| < \frac{\varepsilon}{2} + \frac{2\|\widetilde{g}_i\|}{\delta^2}(s-t)^2.$$

Using this and equations (11.4)–(11.5), we obtain

$$\begin{aligned} |h_{k,i}(t) - \widetilde{g}_i(t)| &= \left| \int_{\mathbb{R}} K_k(t-s)[\widetilde{g}_i(s) - \widetilde{g}_i(t)]\, ds \right| \\ &\le \int_{\mathbb{R}} K_k(t-s)|\widetilde{g}_i(s) - \widetilde{g}_i(t)|\, ds \\ &\le \int_{\mathbb{R}} K_k(t-s)\left[\frac{\varepsilon}{2} + \frac{2\|\widetilde{g}_i\|}{\delta^2}(s-t)^2\right] ds \\ &= \frac{\varepsilon}{2} + \frac{2\|\widetilde{g}_i\|}{\delta^2}\frac{1}{k^2}. \end{aligned}$$

Since the right-hand side is independent of t, this implies that

$$\|h_{k,i} - \widetilde{g}_i\| \le \frac{\varepsilon}{2} + \frac{2\|\widetilde{g}_i\|}{\delta^2}\frac{1}{k^2}.$$

For k sufficiently large, it follows that $\|h_{k,i} - \widetilde{g}_i\| < \varepsilon$, which proves statement 2) of the theorem.

It remains to verify statement 1) of the theorem. For each set of points $t_1 < \cdots < t_n$ in T, the space of exponentials spanned by $\{K_k(t_1 - \cdot), \ldots, K_k(t_n - \cdot)\}$ has property C-3 in $C_0(\mathbb{R})$ (see, e.g., [11, p. 11]). After interchanging two of these functions if necessary, we may assume that

$$D\begin{pmatrix} K_k(t_1 - \cdot), & \cdots, & K_k(t_n - \cdot) \\ s_1, & \cdots, & s_n \end{pmatrix} > 0 \tag{11.10}$$

for every choice of points $s_1 < \cdots < s_n$ in $\mathbb{R}$. Similarly, since span $\{\widetilde{g}_1, \ldots, \widetilde{g}_n\}$ has property W-3 in $C_b(\mathbb{R})$, we may assume that

$$D\begin{pmatrix} \widetilde{g}_1, & \cdots, & \widetilde{g}_n \\ s_1, & \cdots, & s_n \end{pmatrix} \ge 0$$

for every $s_1 < \cdots < s_n$ in $\mathbb{R}$. By the well-known composition formula for determinants (see e.g. [11, p. 81]),

$$A := \begin{vmatrix} h_{k,1}(t_1) & h_{k,2}(t_1) & \cdots & h_{k,n}(t_1) \\ \cdots & & \cdots & \\ h_{k,1}(t_n) & h_{k,2}(t_n) & \cdots & h_{k,n}(t_n) \end{vmatrix} =$$

$$= \int_{\triangle_n} D\begin{pmatrix} \widetilde{g}_1, & \cdots, & \widetilde{g}_n \\ s_1, & \cdots, & s_n \end{pmatrix} D\begin{pmatrix} K_k(t_1 - \cdot), & \cdots, & K_k(t_n - \cdot) \\ s_1, & \cdots, & s_n \end{pmatrix} ds_1 ds_2 \cdots ds_n,$$

where

$$\triangle_n := \{(s_1, \ldots, s_n) \in \mathbb{R}^n : s_1 < \cdots < s_n\}.$$

Now the first integrand cannot vanish for all $(s_1, \ldots, s_n) \in \triangle_n$ since $\widetilde{g}_1, \ldots, \widetilde{g}_n$ are linearly independent. Thus,

$$D\begin{pmatrix} \widetilde{g}_1, & \cdots, & \widetilde{g}_n \\ s_1, & \cdots, & s_n \end{pmatrix} > 0$$

on some open subset of $\triangle_n$. Combining this with (11.10), it follows that the determinant A is positive, and we have shown that H_k has property C-3 in $C_b(T)$. $\square$

Example 8.6 shows that the analog of Theorem 11.1 does not hold for the properties W-1 or W-1′. The proof of Theorem 11.1 is modeled after the proof of the classical Jones-Karlovitz Theorem 1.4 which deals with the interval $[a, b]$ (see e.g. [11, p. 83]). This result is of a slightly different nature than the others in this paper in the sense that the approximating subspaces H_k here are not in $C_0(T)$, but instead lie in the larger space $C_b(T)$. It is natural to ask whether H_k can be constructed to lie in $C_0(T)$. While this may be possible, it cannot be done using the method of convolution with the Gauss kernel. Indeed, if we define $h_{k,i}$ by convolution with the kernel K_k in (11.2), we get functions that lie in $C_b(T)$ but not in $C_0(T)$. To see this, take for example $T = (0, 1]$ and let $g_1(t) = t$. Then $G := \text{span}\{g_1\}$ has property W-3 in $C_0(T)$. Now $\widetilde{T} = [0, 1]$, and (no matter how $\widetilde{g}_i$ is defined on $\mathbb{R} \setminus \widetilde{T}$ as long as it remains bounded), the function

$$h_{k,1}(t) := \int_{-\infty}^{\infty} K_k(t-s)\widetilde{g}_1(s)\, ds, \qquad t \in T,$$

does *not* tend to 0 as $t \to 0+$, which implies that $h_{k,1} \notin C_0(T)$.

Using Theorem 11.1, it is easy to prove the following generalization of the Jones-Karlovitz theorem.

Corollary 11.2. *Suppose T is a compact subset of $\mathbb{R}$ and that G is a finite-dimensional subspace of $C(T)$. Then G is weak Chebyshev if and only if G is approximable by subspaces having property C-3.*

Proof: If G is weak Chebyshev, then Theorem 11.1 imples that G is approximable by C-3 subspaces in $C_b(T) = C(T)$. Conversely, if G is approximable by subspaces having property C-3, then (see Figure 2) G is approximable by subspaces having property W-2. But limits of W-2 subspaces are also W-2 subspaces by Theorem 9.1. Hence G has property W-2. $\square$

Since a finite closed interval $[a, b]$ is compact, Corollary 11.2 implies the classical Jones-Karlovitz Theorem 1.4. In [6], Jones and Karlovitz also showed that in $C[a, b]$, properties W-2′, W-3, W-4, and A-2 are all equivalent. In this connection we have the following three results concerning the approximability of spaces having the properties A-1, A-2, or A-3. Recall that in this section we are working under the assumption that T is a locally compact subset of $\mathbb{R}$.

Proposition 11.3. *There exists a subspace G in $C_0(T)$ having property A-1 that is not approximable by C-3 subspaces.*

Proof: This follows from Example 8.6, since A-1 and W-1 are equivalent from Figure 2. $\square$

Theorem 11.4. *Let G be a finite-dimensional subspace of $C_0(T)$.*

1) *If G is approximable by C-3 subspaces, then G has property A-2.*
2) *If T is compact, then G has property A-2 if and only if G is approximable by C-3 subspaces.*

Proof: If G is approximable by C-3 subspaces, then, by Figure 2, G is approximable by W-2 subspaces. Since G is a limit of W-2 subspaces, G must also be a W-2 subspace by Theorem 9.1. But W-2 is equivalent to A-2 by Figure 2. Thus G must be an A-2 subspace. If T is compact, the result follows from Corollary 11.2. $\square$

Theorem 11.5. *Let G be a finite-dimensional subspace of $C_0(T)$.*

1) *If G has property C-1 and is approximable by C-3 subspaces, then G has property A-3.*
2) *If T is compact, then G has property A-3 if and only if G has property C-1 and is approximable by C-3 subspaces.*

Proof: If G is C-1 and is approximable by C-3 subspaces, then G is approximable by W-2 subspaces by Figure 2. By Theorem 9.1, G is W-2. By Figure 2, G is A-3. If T is compact and G is A-3, then by Figure 2, G is C-1 and W-2. By Corollary 11.2, G is approximable by C-3 subspaces. $\square$

It would be interesting to know whether the "only if" part of statement 2 in each of the preceding two theorems is valid without the compactness of T.

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