A SURVEY OF LOCAL EXISTENCE THEORIES FOR ABSTRACT NONLINEAR INITIAL VALUE PROBLEMS

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Abstract. This paper surveys the abstract theories concerning local-in-time existence of solutions to differential inclusions, $u'(t) \in F(t, u(t))$, in a Banach space. Three main approaches assume generalized compactness, isotonicity in an ordered Banach space, or dissipativeness. We consider different notions of "solution," and also the importance of assuming or not assuming that $F(t,x)$ is continuous in x. Other topics include Carathéodory conditions, uniqueness, semigroups, semicontinuity, subtangential conditions, limit solutions, continuous dependence of u on F, and bijections between u and F.

1. Introduction. In this paper we consider differential inclusions of the form

$$u'(t) \in F(t, u(t)) \qquad (0 \le t \le T). \tag{1.1}$$

Here $u(t)$ takes values in a Banach space $(X, \| \quad \|)$, and F is a mapping from some subset of $[0,T] \times X$ into the set of all subsets of X. We ask what hypotheses on X, F and a given initial value $u(0)$ guarantee that a solution of (1.1) exists for some $T > 0$. We permit T to be small, and to depend on $u(0)$ — we are concerned primarily with *local* existence, as explained further in §2. Intuitively, t represents time, and $u(t)$ represents the state of some system which is evolving as time passes, according to a rule or environment described by F. Thus u is sometimes called an *evolution*, and (1.1) an *evolution equation*. The operator F is said to be the *generator* of the evolution. An important special case to which we shall devote some additional attention is the *autonomous* problem

$$u'(t) \in G(u(t)) \qquad (0 \le t \le T) \tag{1.2}$$

(i.e., where $F(t,x) = G(x)$ is independent of t).

Note that F may be point-valued, or interval-valued; thus (1.1) includes differential equations and differential inequalities as special cases. On the other hand, F may be set-valued; thus implicit differential equations of the form $H(t, u(t), u'(t)) = 0$ are also included as special cases, even if $H(t, x, \cdot)$ is not invertible for some t and x. Also F may be discontinuous and X may be infinite-dimensional; thus (1.1) may represent a partial differential equation. Problems of the form (1.1) also include problems from control theory, integral equations, functional differential equations, population models, and numerous other applications; but we are concerned here with the abstract theory, not the applications.

We are far from a complete understanding of local existence of solutions to (1.1), or even (1.2). Some of the known sufficient conditions for existence have partial converses, but we are still quite far from knowing full necessary and sufficient conditions. A handful of examples are known in which (1.1) has no solution, but most of these examples are variants of a single example of Dieudonné (discussed in §2). They are not diverse enough to adequately motivate the known sufficient conditions.

We are even far from a *unified* understanding of (1.1) or (1.2). The literature includes several essentially separate schools of thought, which use definitions and hypotheses so different in nature that it is difficult to make comparisons between them. Three main approaches assume F satisfies a condition of generalized compactness, isotonicity, or dissipativeness (introduced in §7-10); the dissipativeness school of thought also has several separate subschools. Various papers also differ substantially in their definitions of "solution" (see §3, 7, 13, and 14), as well as in their other assumptions about F — e.g., whether F is assumed continuous, semicontinuous, or not continuous at all (see especially §5 and 12).

Wide gaps exist between the theories. However, this survey will show certain similarities and analogies between the several theories. These suggest that deeper connections may be uncovered in the future; that is a goal of this author's future research. This paper is intended as preparation for that research.

This survey contains no new results; it is merely a review of the literature. However, it is our hope that this survey may convey to some readers a broad perspective including some new insights. This survey is ordered pedagogically — i.e., simpler topics first — with the intention that it can be read by newcomers to the subject. Mathematicians who are already familiar with one or more of the theories of existence may wish to skip ahead to §11 and 12, which unite many of the simpler ideas of §5-10 into a single framework.

The literature concerning (1.1) is enormous, and so this survey is necessarily incomplete. We omit much historical background, and concentrate on the most recent results. We make no attempt to deal with applications. Aside from an occasional indication of some of the simpler ideas involved, we omit all proofs. For brevity, we give only the simplest and/or most general versions of certain ideas or theorems, omitting many other versions. We sometimes mention technicalities (boundedness, measurability, etc.) without giving their details. We omit some important and widely-studied topics — e.g., additional existence results which use linear operators or Hilbert spaces or the dual space X^* — because these topics are too specialized for the purposes of this survey. Decisions of what to include and what to omit were guided by the goal of a broad perspective, mentioned above, but were inevitably biased by the present author's own interests and ignorances. If we have omitted the reader's favorite result, or some cherished technical details from that result, we apologize. The present author would be grateful for communications about such omissions and about more recent results; some of these might be reported in an addendum to this survey a year or two hence.

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2. Boundedness, local existence, and nonexistence. We begin by recalling two classical existence theorems, with G single-valued and continuous:

2.1. THEOREM. *Given any* $u(0) \in X$, *the initial value problem* (1.2) *has a unique solution for all* $t \geq 0$, *if* G *satisfies the Lipschitz condition*

$$\|G(x) - G(y)\| \leq \omega \|x - y\| \qquad (x, y \in X) \tag{2.2}$$

for some constant ω.

Such a function G is said to be *Lipschitzian*, and ω is called the *Lipschitz constant* of G.

2.3. THEOREM (Peano, 1885). *If* X *is a finite-dimensional Banach space and* $G : X \to X$ *is continuous, then for each* $u(0) \in X$ *the initial value problem* (1.2) *has at least one solution for some* $T > 0$.

Both of these results have many different proofs which can be found in many books; see for instance [32]. We sketch one proof: If F is single-valued, continuous, and defined everywhere on $[0, T] \times X$, then (1.1) is equivalent to the integral equation

$$u(t) = u(0) + \int_0^t F(s, u(s))\, ds \qquad (0 \leq t \leq T). \tag{2.4}$$

A solution of (2.4) is the same as a fixed point of the operator Φ defined by

$$(\Phi u)(t) = u(0) + \int_0^t F(s, u(s))\, ds \qquad (0 \leq t \leq T). \tag{2.5}$$

Under the hypotheses of Theorems 2.1 or 2.3, for sufficiently small T, Φ has a fixed point, by Banach's fixed point theorem for contraction mappings or by Schauder's fixed point theorem for compact mappings, respectively. (See [116] for an introduction to fixed point theory.) Under the hypotheses of Theorem 2.1, or under those of 2.3 with G bounded, we can repeat this argument on the intervals $[0, T]$, $[T, 2T]$, $[2T, 3T]$, etc., and thus obtain a solution for all positive t.

We shall use (2.4) and (2.5) again occasionally in this paper, but we shall not survey the theories of integral equations or fixed points. Those theories give an elegant approach to (1.1), but it is not the only approach, and it is generally less successful than other approaches which focus more directly on initial value problems. When F is permitted to be discontinuous and set-valued, and when the domain of $F(t, \cdot)$ is permitted to vary with t, then the conditions corresponding to (2.4) and (2.5) become very complicated.

The hypotheses of Peano's Theorem (2.3) do not guarantee global existence. For instance, the equation $u'(t) = u(t)^2$ with initial value $u(0) = 1$ has unique solution $u(t) = 1/(1-t)$, which blows up as t increases to 1. This behavior is not pathological, but actually typical of nonlinear differential equations, and so we concern ourselves with *local* existence of solutions. Global continuability versus finite-time blowup is a separate question which has received much attention in the literature but will not be studied in depth here. See [6] and references cited therein, for an introduction to this subject. We remark that for continuability of solutions, the crucial question is not whether $\|u(t)\|$ blows up as $t \uparrow T$, but whether $\{u(t) : 0 \le t < T\}$ is a relatively compact set. Dieudonné [33] gives a simple example of noncontinuability without blowup.

Most abstract existence theorems can be stated in either a local or global form. For instance, the Lipschitz hypothesis (2.2) could be weakened to a *local* Lipschitz condition: we could replace the constant ω with a continuous function of x and y; then the conclusion would be just local existence. On the other hand, if we add to Peano's theorem the assumption that G is uniformly bounded, then we gain the conclusion that existence is guaranteed for all positive t. (Locally this is no real change in the theorem, since any continuous G must be bounded at least on some neighborhood of $u(0)$.) Most abstract existence results in the literature can be modified similarly. In effect, most global existence results can be decomposed into a local existence result plus a global continuability result, the latter being obtained from a global estimate.

The verification of such global estimates in specific problems is a basic part of applied mathematics. The methods needed for such verification vary greatly from one differential equation to another; they do not seem to fit into just a few abstract theories. In fact, the main question addressed in this paper — what conditions on F are sufficient for local existence — may seem alien or even trivial to the applied mathematician, who generally begins with an observed physical phenomenon for which existence is already known.

On the other hand, local existence does fit into just a few abstract theories. A pure mathematician pursuing abstract existence theory may choose to simply assume as a hypothesis that F is uniformly bounded, and focus his or her attention on other difficulties not substantially affected by that assumption. Of course, some researchers choose to investigate the subtleties of weaker, more complicated, and more appropriate boundedness hypotheses on F. For instance, Himmelberg and Van Vleck [57] say that F is "locally weakly integrably bounded" if for each $\rho > 0$, there exists a function $m_\rho \in L^1[0, T]$ such that $\inf\{\|y\| : y \in F(t, x)\} \le m_\rho(t)$ for all x, t with $\|x\| \le \rho$. For another example, Dollard and Friedman [34] work with (1.1) where $F(t, \cdot)$ is continuous and linear for each t; their boundedness assumption is that the upper integral

$$\overline{\int_0^T} \|F(s, \cdot)\|\, ds \equiv \inf\{\textstyle\int_0^T \psi(s)\, ds \,:\, \psi \in L^1[0, T]; \quad \|F(s, \cdot)\| \le \psi(s) \text{ for all } s\}$$

be finite. The upper integral coincides with the ordinary (Lebesgue) integral if $\|F(s, \cdot)\|$ is a measurable function of s. A similar estimate is applied to nonlinear problems in [108].

With the abstract viewpoint indicated above, we see local and global existence theorems as superficially different presentations of the same basic ideas. The global formulation is usually simpler in notation than the local formulation, and so it is usually the version used in the literature. In this respect 2.1 is typical, while 2.3 is not. This paper is concerned with existence results (and closely related results, such as uniqueness and continuous dependence) that are local *in essence*, even if they are sometimes formulated in global terms for simplicity of notation; we shall not pursue the verification of the global estimates. For our purposes, nonexistence of a solution to (1.1) means that no solution exists *no matter how small we choose T.*

Our questions of local existence can be studied in a very abstract setting, and so they may be simple in appearance, but they are by no means trivial. For instance, Peano's existence result 2.3 *fails* in infinite-dimensional Banach spaces. Dieudonné [33] gave a simple counterexample in the space c_0 of sequences converging to 0: Let $G(\{x_n\}) = \{(1/n) + \sqrt{|x_n|}\}$ and $u(0) = 0$. Then G is continuous from c_0 into c_0, but (1.2) has *no* solution in c_0, no matter how small we choose T. Several other authors subsequently gave examples in other spaces, by modifying Dieudonné's example, and finally Godunov [45] showed that Peano's conclusion fails in *every* infinite-dimensional Banach space.

Numerous different notions of "solution" can be found in the literature; these will be discussed in §3, 7, 13, and 14. But the particular choice of the definition of "solution" does not matter in the examples of Dieudonné *et al.* A "solution" $u(t)$ for Dieudonné's initial value problem can be explicitly constructed componentwise in the space ℓ^∞ of bounded sequences. This function $u(t)$ is uniquely determined, and is the only possible candidate for a "solution" in any reasonable sense. But this function $u(t)$ takes its values outside of the chosen Banach space c_0, and so is disqualified as a solution of (1.2). Similar remarks apply to the examples of Godunov *et al.*, which are variants of Dieudonné's example.

3. Carathéodory conditions, Carathéodory solutions, and other differentiable solutions. Numerous different notions of "solution" are used in the literature. In this section we discuss those solutions $u(t)$ which are differentiable, i.e. which actually satisfy the differential inclusion (1.1) or (1.2) in some fairly direct sense. Other notions of "solution," more general and useful but less directly appealing to our intuition, will be discussed in §7, 13, and 14.

If F or G is continuous, then a "solution" of (1.1) or (1.2) usually means a continuously differentiable function $u(t)$ which satisfies the differential equation. Such solutions are sometimes called *Newton solutions*, or "strong" or "classical" solutions; however, those last two terms also have other meanings in the literature. If F or G is not continuous, then a more general "solution" may be needed. For motivation we first consider some classical notions of Carathéodory:

We say that a single-valued function $F(t,x)$ satisfies *Carathéodory conditions* if it is continuous in x and if $\sup\{\|F(t,x)\| : x \in S\}$ is majorized by some locally integrable function $m_S(t)$ for each bounded set S or each relatively compact set S. Both definitions — with S bounded or with S relatively compact — are used in the literature, and of course they coincide when X is finite-dimensional. For an arbitrary Banach space, the two definitions still agree locally, i.e. on a sufficiently small neighborhood of $u(0)$; see Theorem 4.7 in [108]. The condition involving compact sets generalizes joint continuity of F.

3.1. THEOREM (Carathéodory, 1927). *Assume $F : [0,\infty) \times X \to X$ satisfies Carathéodory conditions, as defined above. Let $u(0) \in X$. If X is finite-dimensional, then* (2.4) *has a solution u on $[0,T]$ for some $T > 0$.*

This theorem can be found in [24], for instance. Of course, the conclusion fails when X is infinite-dimensional, but variants of this theorem are valid in arbitrary Banach spaces, as we shall see later.

The "solution" u whose existence is asserted above need not be continuously differentiable, since F may be discontinuous. In an arbitrary Banach space, for single-valued F, we say u is a *Carathéodory solution* of the initial value problem (1.1) if u is the solution of the corresponding integral equation (2.4). The integral is understood in the sense of Bochner integrals, i.e., Banach-space-valued Lebesgue integrals. (For an introduction to such integrals, see [35], [74], [129].) Of course, if F is jointly continuous, then a Carathéodory solution is the same thing as a Newton solution.

More generally, whether F is single-valued or not, we say that a *Carathéodory solution* of the differential inclusion (1.1) is a function $u(t)$ which is absolutely continuous on $[0,T]$, which is differentiable almost everywhere on $[0,T]$, which satisfies the differential equation almost everywhere on $[0,T]$, and which satisfies the initial condition if one is given. (For single-valued F, this is equivalent to (2.4). In finite-dimensional spaces, absolute continuity of u implies existence of $u'(t)$ for almost all t, but that implication is not valid in an arbitrary Banach space.) This notion of "solution" is used widely, and in §4-9 of this paper a "solution" will mean a Carathéodory solution unless specified otherwise. Carathéodory solutions are also sometimes referred to in the literature as "strong" or "classical" solutions, but those terms also sometimes refer to Newton solutions, or have still other meanings in the literature.

Because integrals occur naturally in (2.4), Carathéodory solutions are in some sense more natural than Newton solutions, and Carathéodory solutions have been studied extensively in the literature. An interesting result is that of Binding [12], who considers the autonomous differential equation $u'(t) = G(u(t))$ for a single-valued function $G : \mathbf{R} \to \mathbf{R}$ in one dimension. Among other results, Binding gives *necessary and sufficient* conditions on G for existence of solutions, given an initial value. These conditions are as follows. If it is not constant, the solution for such a differential equation cannot cross itself, hence it must be monotone; hence G

must be nonnegative (respectively, nonpositive) almost everywhere on an interval to the right (respectively, left) of the initial value. Also, the set where G is infinite or undefined must have measure zero. Finally, if G is not zero almost everywhere, then $1/G$ must be integrable on that interval on one side of the initial value.

Though the notion of Carathéodory solutions is fairly simple and intuitively appealing, it is not adequate for discontinuous G, or for $F(t,x)$ discontinuous in x. For example, the differential equation

$$u'(t) = G(u(t)) = \begin{cases} -1 & \text{if } u(t) \geq 0, \\ 1 & \text{if } u(t) < 0, \end{cases}$$

with initial condition $u(0) = 0$, has no Carathéodory solutions. This problem exhibits what Binding [12] calls *jamming.* Intuitively, we might feel that $u(t) \equiv 0$ *should* be a solution, and indeed it is in the sense of Krasovskij or Filippov:

A *Krasovskij solution*, respectively a *Filippov solution* of (1.1) is a Carathéodory solution of $u'(t) \in \mathbf{K}F(t,u(t))$, respectively $u'(t) \in \mathbf{F}F(t,u(t))$, where

$$\mathbf{K}F(t,x) = \bigcap_{\varepsilon>0} \overline{\mathrm{conv}} F(t, x+\varepsilon B),$$

$$\mathbf{F}F(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\text{null } Z} \overline{\mathrm{conv}} F(t, (x+\varepsilon B)\backslash Z).$$

Here B is the open unit ball, and $\overline{\mathrm{conv}}$ means convex closure. In both of these definitions, "bad" points which are in some sense isolated and atypical of the behavior of F or u are discarded. In the definition of $\mathbf{F}F$, sets $Z \subset X$ having Lebesgue measure zero are discarded. This definition is only meaningful when the Banach space X is finite-dimensional, since Lebesgue measure has no natural analogue on infinite-dimensional spaces. In finite dimensions, $\mathbf{F}F(t,x) \subseteq \mathbf{K}F(t,x)$, so any Filippov solution is also a Krasovskij solution. Krasovskij and Filippov solutions, as well as Carathéodory solutions and Hermes solutions (introduced in §11 of this paper) are surveyed by Hájek [48], at least for the finite-dimensional case.

Still other "solutions" weaken the notion of derivative. The *contingent derivative* of a function $u(t)$ at a point t_0 is the set of all limits (or, in some papers, all weak limits) of sequences of the form $\{[u(t_0 + h_n) - u(t_0)]/h_n\}_{n=1}^{\infty}$, where $h_n \to 0$. That set is denoted by $Du(t_0)$. A solution of the *contingent differential equation* $Du(t) \subset F(t,u(t))$ is a function u which satisfies that relation for almost all t (or, in some papers, for all but at most denumerably many t). Some results on contingent differential equations in Banach spaces are given by Chow and Schuur [23].

If $F(t,\cdot)$ is linear for each t, and $F(t,x)$ is written $F(t)x$, then a *weak solution*, or $*$*-solution*, of (1.1) is a function u satisfying

$$< u(t), y > \;=\; < u(0), y > + \int_0^t < u(s), F(s)^* y > ds \qquad (0 \leq t \leq T),$$

for every y in the dual space X^* or in some dense subset of X^*. (The term "weak solution" also has other meanings.) For some recent results concerning $*$-solutions, see Dawson and Gorostiza [31].

A $*$-solution need not be differentiable in the topology of the norm of X. In §11 we shall consider some "solutions" u which need not be differentiable in any sense at all. Thus, the solution $u(t)$ of (1.1) need not actually satisfy (1.1) in any direct sense. Equation (1.1) is only used as an abbreviation for a much longer and more complicated definition of "solution" which does involve u and F. Although we shall not discuss such solutions in any detail until §11, they should be kept in mind in the discussion of evolution operators at the end of §4.

4. Uniqueness, Kamke functions, and semigroups. The hypotheses of Peano's Theorem (2.3) do not guarantee uniqueness. For instance, the equation $u'(t) = 2\sqrt{|u(t)|}$ with initial value $u(0) = 0$ has solution $u(t) = (\max\{0, t-b\})^2$ for any number $b \geq 0$. Among the three major hypotheses of generalized compactness, isotonicity, or dissipativeness, introduced in §7-10 below, only dissipativeness guarantees uniqueness of solutions — and even that uniqueness is lost when we consider some generalizations in §15. However, even without uniqueness, the theory associated with (1.1) is rich and interesting; see for instance inequality (7.5) and the remarks about continuous and semicontinuous dependence in §7 and 13. Thus, uniqueness is not essential to the theory of existence of solutions. Still, some of the concepts of uniqueness theory will be useful in our study of existence, and so we briefly introduce them here. For a more detailed introduction to uniqueness, see [51].

A function $\omega : [0,T] \times [0,+\infty) \to [0,+\infty)$ is a *Kamke function* (or *uniqueness function*) if ω satisfies Carathéodory conditions, $\omega(t,0) = 0$ for all t, and ω has the property that the only Carathéodory solution of $p'(t) \le \omega(t,p(t))$ on $[0,T]$ with $p(0) = 0$ is the trivial solution $p \equiv 0$. Examples of Kamke functions are $\omega(t,r) = kr$ or $\omega(t,r) = kr\ln(1+r)/\sqrt{t}$ (k = constant) or $\omega(t,r) = r/t$, but *not* $\omega(t,r) = 2r/t$. (Some papers on uniqueness use slightly different definitions; [9] gives a comparison of some of the different classes of Kamke functions.) One variant (given by [24]) of Kamke's classical uniqueness result is as follows: if ω is a Kamke function, F satisfies Carathéodory conditions, and

$$\|F(t,x) - F(t,y)\| \le \omega(t, \|x-y\|) \tag{4.1}$$

for all t, x, y, then (1.1) has at most one Carathéodory solution for each initial value $u(0)$. (The main idea of the proof is that if u_1 and u_2 are solutions of (1.1), then we may apply the definition of the Kamke function with $p(t) = u_1(t) - u_2(t)$.)

Hypothesis (4.1) can be generalized substantially. For instance, let ω be a Kamke function, but instead of (4.1) assume that F satisfies

$$\liminf_{h\downarrow 0} \frac{V(t,x,y) - V(t-h, x-hF(t,x), y-hF(t,y))}{h} \le \omega(t, V(t,x,y)) \tag{4.2}$$

where V is locally Lipschitzian in x and y, V is nonnegative, and $V(t,x,y) = 0$ if and only if $x = y$. Then (1.1) has at most one solution u for each initial value $u(0)$. Roughly, the idea of the proof is that $D_t V(t, u_1(t), u_2(t)) \le \omega(t, V(t, u_1(t), u_2(t)))$ for any solutions u_1, u_2. The relation between (4.1) and (4.2) will be discussed further at the end of §9. Many other uniqueness results, more general and more complicated, can be found in the literature. One particularly general and recent result is [104]; see its bibliography for earlier results.

In initial value problems where uniqueness is known, we can use the notation of semigroups and evolution operators. An *evolution* (or *evolution operator*) on a set Ω is a two-parameter family of self-mappings of Ω:

$$U(t,s) : \Omega \to \Omega \qquad (-\infty < s \le t \le +\infty), \tag{4.3}$$

such that

$$U(t,t) = \text{identity} \qquad \text{and} \qquad U(t,s) \circ U(s,r) = U(t,r) \qquad (r \le s \le t). \tag{4.4}$$

The evolution is *linear* if Ω is a linear subspace of X and $U(t,s)$ is linear for each t and s.

An important special case is that in which $U(t,s)$ depends on t,s only through the value $t-s$. Then we can write $U(t,s) = S(t-s)$, where S is a *semigroup* on Ω, i.e., a one-parameter family of mappings

$$S(t) : \Omega \to \Omega \qquad (t \ge 0) \tag{4.5}$$

with the properties

$$S(0) = \text{identity}, \qquad S(t+s) = S(t) \circ S(s). \tag{4.6}$$

If Ω is a subset of a Banach space, then the semigroup S on Ω is *of type* ω if $(t,x) \mapsto S(t)x$ is a jointly continuous function on $[0,+\infty) \times \Omega$ and

$$\|S(t)x - S(t)y\| \le \|x-y\| \exp(t\omega) \tag{4.7}$$

for all t, x, y. If this is true for $\omega = 0$, we say S is *nonexpansive* (or, in some papers, *contractive*). If G satisfies (2.2), then the semigroup generated by G (in the sense of (4.8), below) is of type ω.

Joint continuity is more natural in semigroups $S(t)$ than in temporally inhomogeneous evolutions $U(t,s)$. Indeed, Ball [5] has shown that if S is a semigroup on a metric space Ω, and $S(t)x$ is measurable in t and continuous in x, then $(t,x) \mapsto S(t)x$ is jointly continuous on $(0,+\infty) \times \Omega$. (Joint continuity at $t=0$ does not necessarily follow, even if S is separately continuous at $t=0$; see [22].) Measurability does not imply continuity for temporally inhomogeneous evolutions, even bounded linear ones. For instance, take $U(t,s)x = \exp[ig(t) - ig(s)]x$, where $g : \mathbf{R} \to \mathbf{R}$ is measurable but not continuous. Thus, temporally homogeneous and inhomogeneous evolutions differ substantially: a temporally inhomogeneous evolution may have jumps, but a semigroup (if measurable) may not.

Evolution operators arise in the study of (1.1) as follows: Let F be some operator — possibly discontinuous and set-valued — in a Banach space X. Assume some notion of "solution" has been specified for the differential inclusion $u'(t) \in F(t,u(t))$ — e.g., one of the notions discussed in any of §3, 11, 13, 14. Assume that the notion of "solution" is such that if u is a solution on an interval I_1, and I_2 is an interval contained in I_1, then the restriction of u to I_2 is a solution on I_2. For simplicity, assume global existence on some set $\Omega \subseteq X$ — i.e., assume that for each initial time $a \in \mathbf{R}$ and each initial value $x \in \Omega$, there exists a solution $u(t) = u(t;a,x)$ for the initial value problem

$$\begin{cases} u'(t) \in F(t,u(t)) & (a \le t < +\infty), \\ u(a) = x, & \end{cases}$$

with $u(t)$ taking values in Ω. Also assume forward uniqueness — i.e., assume that whenever $u_1(t)$ and $u_2(t)$ are solutions of $u'(t) \in F(t,u(t))$ on some interval $[a,b]$, and $u_1(a) = u_2(a)$, then $u_1(t) = u_2(t)$ for all $t \in [a,b]$. Under these conditions, it follows that $U(t,s)x \equiv u(t;s,x)$ defines an evolution operator on Ω. We say that $U(t,s)$ (or $u(t)$) is the evolution *generated* by F, and that F is the *generator* of U.

An important special case is that in which $F(t,x) \equiv G(x)$ does not depend on t. For that case, it follows that $U(t,s) = S(t-s)$ defines a semigroup S, called the semigroup *generated* by G, and G is called the *generator* of that semigroup. That semigroup is often denoted $S(t) = \exp(tG) = e^{tG}$, because for many G's we have the *exponential formula*

$$\exp(tG)x = \lim_{n\to\infty} \left(I - \frac{t}{n} G \right)^{-n} x, \tag{4.8}$$

generalizing a familiar formula from undergraduate calculus. When G is continuous and linear, then (4.8) and the other familiar formulas $\exp(tG) = \lim_{n\to\infty}(I + (t/n)G)^n$ and $\exp(tG) = \sum_{n=0}^{\infty} t^n G^n / n!$ are valid; but (4.8) is also valid for many G's which are discontinuous and/or nonlinear — see Theorems 10.1 and 14.1.

For the benefit of newcomers who are unfamiliar with semigroup notation, here are two very elementary examples for study: If $X = \mathbf{R}$ and $G(x) = -(x+5)^3$, then $\exp(tG)x = [2t + (x+5)^{-2}]^{-1/2} - 5$. Continuity of G is not required: If X is a space of functions $x(\theta)$ from $\mathbf{R}$ into $\mathbf{R}$, and $G(x) = 2x + 6 + 5dx/d\theta$ (with domain equal to some suitable subspace of X), then $[\exp(tG)x](\theta) = \exp(2t)x(\theta + 5t) + 3\exp(2t) - 3$. These examples are atypical in that we are able to give explicit formulas for $\exp(tG)$. In many applications, the most we can give is an approximating formula such as (4.8), and perhaps also some other properties of the limit.

Notations like those above are also sometimes used when existence of solutions to (1.1) or (1.2) is only known locally, not globally; but then the notation must be modified slightly and it becomes somewhat more complicated. Even for locally defined solutions, however, the exponential notation remains the simplest way to express certain ideas, e.g. the Trotter-Lie-Kato product formula (13.3); see for instance [111]. The exponential notation is used less often in the study of temporally inhomogeneous evolutions (i.e., (4.3)-(4.4)), but it can be extended to that context too; see for instance [34].

5. Generalizations of continuity. The literature concerning (1.1) varies considerably in its assumptions about continuity of F. Some papers using dissipativeness conditions (discussed in §10 below) make no assumption of continuity at all; these results have direct applications to partial differential equations. Most other existence results for (1.1) assume $F(t,x)$ is continuous in x, or satisfies some condition generalizing continuity, as discussed below. Still, the continuous theory might well be of interest even to researchers concerned primarily with discontinuous problems, for at least a couple of reasons. First, many of the difficulties in the discontinuous theory are still present, albeit in simpler form, when we assume continuity. Second, it may be possible to weaken or remove the continuity hypotheses from some of the continuous theory, particularly by methods discussed in this section or in §12.

Numerous papers deal with (1.1) or (1.2) using assumptions of semicontinuity. A set-valued function G is *upper semicontinuous* (respectively, *lower semicontinuous*) at a point x_0 if for each open set V which contains (respectively, meets) $G(x_0)$, the set $\{x : V \text{ meets } G(x)\}$ (respectively, $\{x : V \text{ contains } G(x)\}$) is a neighborhood of x_0.

Some consequences: If the values of G are compact sets, then G is both upper and lower semicontinuous if and only if G is continuous with respect to the Hausdorff metric on closed bounded sets. If the values of G are closed subsets of a single compact set, then upper semicontinuity is equivalent to closed graph. If the values of G are points, then either upper or lower semicontinuity in the sense above is equivalent to continuity in the usual sense of functions.

Most of the results for semicontinuous set-valued functions assume the values of G are closed nonempty subsets of $\mathbf{R}^n$. An introduction to this theory can be found in the book [4]. The main existence results are as follows: Let F be a mapping from $[0,T] \times \mathbf{R}^n$ into the set of all nonempty closed subsets of $\mathbf{R}^n$. Also assume F satisfies some conditions of measurability and boundedness (which vary from one paper to another; see §2). Then (1.1) has a Carathéodory solution if either

(5.1) for each t, $F(t,\cdot)$ is lower semicontinuous, or

(5.2) for each t, $F(t,\cdot)$ is upper semicontinuous and has convex sets for its values.

Condition (5.1) is used in [15], [77]; condition (5.2) is surveyed in [4]. Convexity cannot be omitted from hypothesis (5.2), as the following simple example from [87] shows: With $X = \mathbf{R}$, let

$$G(x) = \begin{cases} \{1\} & \text{when } x < 0, \\ \{1,-1\} & \text{when } x = 0, \text{ and} \\ \{-1\} & \text{when } x > 0. \end{cases}$$

Then G is upper semicontinuous and has closed values, but (1.2) has no Carathéodory solution for $u(0) = 0$. (However, (1.2) has a Krasovskij solution; see §3.)

The resemblance between upper and lower semicontinuity is only superficial. Hypotheses (5.1) and (5.2) lead to two separate theories, whose solutions have fundamentally different properties. For instance, under assumption (5.2), the solution set is closed and connected, and depends on the initial value in an upper semicontinuous fashion; see [4]. Analogous conclusions are not valid under assumption (5.1).

Some attempts have been made to unify these two approaches, but so far the "unifications," though more general, have also been more complicated. Lojasiewicz [78] assumed, in addition to measurability and boundedness conditions, that

> for almost every t, for each x, either (a) $F(t,x)$ is convex and $F(t,\cdot)$ has closed graph at x, or (b) $F(t,\cdot)$ is lower semicontinuous on some neighborhood of x.

Similarly, Himmelberg and Van Vleck [57] assumed, in addition to measurability and boundedness conditions, that

> for each t, $F(t,\cdot)$ has closed graph and, at each x, either (a) $F(t,x)$ is convex, or (b) $F(t,\cdot)$ is lower semicontinuous at x.

It is not yet known whether a single sufficient condition can be given which is weaker than both (5.1) and (5.2) and which is also *simpler*. A recent step in that direction was taken in [17]:

5.3. THEOREM (Bressan, 1987). *Let $F : [0,T] \times \mathbf{R}^n \to \{$compact nonempty subsets of $\mathbf{R}^n\}$ be bounded and lower semicontinuous. Then there exists an upper semicontinuous function H with compact convex* *nonempty values such that every Carathéodory solution of $x'(t) \in H(t, x(t))$ is also a Carathéodory solution of $x'(t) \in F(t, x(t))$.*

Bressan's proof involves "directional continuity," which is also of interest for its own sake. Directional continuity is a weakened version of continuity, for single-valued functions. It was first used by [19]. One simple application of it is as follows:

5.4. THEOREM (Pucci, 1971). *Let δ be a positive constant. Let $G : \mathbf{R}^n \to \mathbf{R}^n$ be a bounded mapping which is continuous at each x where $G(x) = 0$. At each x where $G(x) \neq 0$, assume the following "directional continuity" condition:*

$$x_k \to x, \quad x_k \neq x, \quad \left\| \frac{x_k - x}{\|x_k - x\|} - \frac{G(x)}{\|G(x)\|} \right\| < \delta \qquad \Rightarrow \qquad G(x_k) \to G(x).$$

Let $u(0) \in \mathbf{R}^n$ be given. Then (1.2) has a Carathéodory solution for some $T > 0$.

Finally, we mention one other way of generalizing continuity. Discontinuous operators become continuous if we restrict them to smaller domains; in some cases this restriction does not entirely destroy their usefulness. For instance, the differential operator d/dx is uniformly continuous from the metric of $L^2(\mathbf{R})$ to the metric of $L^2(\mathbf{R})$, when restricted to a bounded subset of the Sobolev space $H^2(\mathbf{R})$; see [109]. Consequently, some techniques of ordinary differential equations can be applied to some partial differential equations; see [109], [110], [111].

6. Subtangential conditions. In some of the literature concerning (1.2), it is assumed that G is defined on all of X, or at least on some neighborhood of the given initial value $u(0)$. That assumption does simplify some of the resulting local existence theory, but it precludes many important applications, especially to partial differential equations. If the domain of G does not include a neighborhood of $u(0)$, then generally we must assume some condition at the boundary of the domain of G. Such conditions — known variously as *subtangential conditions*, *inwardness conditions*, *range conditions*, and *Nagumo conditions* — are satisfied trivially if G is defined everywhere, and so such conditions are not even mentioned in most papers concerned with everywhere-defined functions.

We now introduce the main subtangential conditions, although some of them will not be used until §10 and 12. In the conditions below, "Ran" stands for range, "Dom" for domain, "cl" for closure, and "dist" for distance. Some subtangential conditions, in order of increasing generality, are:

(6.1) $\qquad \mathrm{Ran}(I - hG) = X$ for all $h > 0$ sufficiently small;

(6.2) $\qquad \mathrm{Ran}(I - hG) \supseteq \mathrm{cl}(\mathrm{Dom}(G))$ for all $h > 0$ sufficiently small; and

(6.3) $\qquad \liminf_{h\downarrow 0} \frac{1}{h}\mathrm{dist}(x, \mathrm{Ran}(I - hG)) = 0$ for all $x \in \mathrm{cl}(\mathrm{Dom}(G))$.

That last condition is the case $N = 1$ of the following condition:

(6.4) $\qquad$ for each $\varepsilon > 0$ and each $x_0 \in \mathrm{cl}(\mathrm{Dom}(G))$, there exist a $\delta \in (0, \varepsilon]$, an integer N, and numbers $h_i > 0$ and points $y_i \in G(x_i)$ such that $\sum_{i=1}^N h_i = \delta$ and $\sum_{i=1}^N \|x_i - x_{i-1} - h_i y_i\| < \varepsilon\delta$.

If G is single-valued, continuous, and bounded, then (6.3) is equivalent to this classical subtangential condition:

$$\liminf_{h\downarrow 0} \frac{\mathrm{dist}(x + hG(x), D)}{h} = 0, \tag{6.5}$$

where D is the domain of G. This condition is appealing because of its obvious geometrical interpretation: it says, roughly, that the vector starting at x and pointing in the direction of $G(x)$ points "into" — or at least not out of — the domain. Condition (6.5), or one equivalent to it, is used in many different papers which assume G is single-valued and continuous. In some of these papers, D is a closed set. A few papers assume, more generally, that D is a *locally closed set*, i.e., the intersection of an open set and a closed set — equivalently, that each point $x \in D$ has a neighborhood N such that $D \bigcap N$ is closed.

It is easy to see that if (1.2) has a continuously differentiable solution u which begins at $u(0) = x$ and which remains in D, then (6.5) must be satisfied at x. Indeed, we have then

$$\frac{\operatorname{dist}(x + hG(x), D)}{h} \leq \frac{\|[x + hG(x)] - u(h)\|}{h} = \frac{\|[u(0) + hu'(0)] - u(h)\|}{h}$$

which vanishes as $h \downarrow 0$. That is, the tangent to the trajectory points into the domain. Thus, conditions like (6.5) are *necessary* for the existence of solutions to (1.2). In the presence of certain other hypotheses, such as dissipativeness or compactness (discussed in §7, 9), a condition like (6.5) is both necessary and sufficient for the existence of solutions to (1.1). In finite dimensions, the earliest such result apparently was due to Nagumo (1942); see [63] for further discussion and references.

Variants of (6.5) also apply to set-valued operators. For instance:

6.6. THEOREM (Bressan, 1983). *Let D be a compact subset of a Banach space X, and let B be a bounded subset of X. Let $G : D \to \{$nonempty closed subsets of $B\}$ be lower semicontinuous. Also suppose that*

(6.7) $$\text{for all } x \in D \text{ and all } y \in G(x), \qquad \lim_{h \downarrow 0} \frac{\operatorname{dist}(x + hy, D)}{h} = 0.$$

Let $u(0) \in D$ be given. Then $u'(t) \in G(u(t))$ has a solution for all $t \geq 0$.

Martin [82] shows the importance of (6.5) without any assumptions of compactness or dissipativeness. Let D be a locally closed subset of a Banach space X, and let $G : D \to X$ be single-valued, bounded and uniformly continuous. Then (6.5) is necessary and sufficient for the existence of a class of *approximate solutions* for (1.2). Approximate solutions will be discussed further in §11.

Temporally inhomogeneous conditions corresponding to (6.5) and (6.3) are, respectively,

(6.8) $$\liminf_{h \downarrow 0} \frac{\operatorname{dist}(x + hF(t, x), D_{t+h})}{h} = 0 \qquad \text{for all } (t, x) \in \operatorname{Dom}(F(t, \cdot));$$

(6.9) $$\liminf_{h \downarrow 0} \frac{\operatorname{dist}(x, \operatorname{Ran}(I - hF(t + h, \cdot)))}{h} = 0 \qquad \text{for all } x \in \operatorname{cl}(\operatorname{Dom}(F(t, \cdot))).$$

These will be used in §9 and 10, respectively.

Closely related to subtangential conditions is the notion of invariant sets. Invariance has at least two slightly different meanings. Let S be a subset of X, not necessarily the domain of G. Generally, we say S is *invariant* for (1.2) if either

(6.10) every solution of (1.2) which is in S at some time t must remain in S at all later times, or

(6.11) for each initial value $u(0) \in S$ there exists at least one solution u of (1.2) remaining in S for some $T > 0$.

Clearly, if uniqueness of solutions is known, then (6.11) $\Rightarrow$ (6.10); if existence of solutions is known, then (6.10) $\Rightarrow$ (6.11). In most of the literature on invariance, both uniqueness and existence are known, so the two notions of "invariance" coincide. See [50] for a theorem concerning (6.11) without uniqueness. See [99] for results concerning (6.10) with a discussion of different kinds of uniqueness functions.

For solutions which remain in S, the behavior of G outside S is irrelevant, so G might as well be undefined outside S. Thus, for most definitions of "solution," (6.11) is equivalent to the existence of solutions of (1.2) if we replace G with its restriction to the set $\operatorname{Dom}(G) \bigcap S$. Invariance generally is guaranteed by conditions similar to the subtangential conditions described earlier in this section. An important problem is to identify invariant sets by some of their other properties. For further discussion of invariant sets see [21] [50] [63] [95] [99] [100] [120] and other papers cited therein.

7. Generalized compactness. As Dieudonné's example (§2) showed, Peano's local existence theorem 2.3 fails in infinite-dimensional Banach spaces. However, that theorem can be extended to arbitrary Banach spaces in a natural way, if we restate the the finite-dimensional results a bit differently. Let us add to Peano's theorem the hypothesis that G is a *compact mapping*; i.e., G maps bounded sets to relatively compact sets. (More generally, we could assume G maps some neighborhood of each point to a relatively compact set.) Then Peano's theorem remains unchanged in finite-dimensional spaces, since bounded sets in such spaces are relatively compact; but Peano's conclusion becomes valid in infinite-dimensional spaces as well. Peano's theorem can be generalized further:

For bounded sets S in a Banach space X, we define *Kuratowski's measure of noncompactness*,

$$\alpha(S) = \inf\{r \ : \ S \text{ can be covered by finitely many sets with diameter} \le r\}.$$

Slightly less often used is the *Hausdorff* (or *ball*) *measure of noncompactness*,

$$\beta(S) = \inf\{r \ : \ S \text{ can be covered by finitely many sets with radius} \le r\}.$$

These functions measure how far S is from being compact; they vanish precisely when S is relatively compact. In fact, $\beta(S)$ is the distance from $\mathrm{cl}(S)$ to the nearest compact set, in the Hausdorff metric (a metric on the space of closed bounded sets). These two measures are equivalent in the sense that $\beta(S) \le \alpha(S) \le 2\beta(S)$. Other measures of noncompactness, and more general notions of such measures, can be found in the book [8].

Now let ω be a Kamke function (see §4). Assume that $F : [0,T] \times X \to X$ is bounded and satisfies Carathéodory conditions, and that

$$\alpha(F(t,S)) \le \omega(t, \alpha(S)) \tag{7.1}$$

for all bounded sets S. Under various mild additional assumptions (discussed below), it follows that (1.1) has a Carathéodory solution. The proofs make use of the Kamke function ω in different ways. A typical argument (from [82]) is roughly as follows: A sequence of approximate solutions $\{u_n\}$ is carefully constructed, with $u_n(0) \to u(0)$. Then (7.1) is used to show that the function $p(t) = \alpha(\{u_1(t), u_2(t), u_3(t), \ldots\})$ satisfies $p'(t) \le \omega(t, p(t))$. Since $p(0) = 0$, it follows that $p(t) = 0$ for all t. Thus the sequence $\{u_n\}$ is relatively compact, and some subsequence converges to a limit, which is then shown to satisfy (1.1).

Some of the earlier results in this direction were by Ambrosetti, Goebel and Rzymowski; we omit the details. Among more recent and more general theorems, some of the most interesting results are as follows: Szufla [119] showed that a solution exists if $F(t,x)$ is uniformly continuous as a function of (t,x) and (7.1) holds. Pianigiani [91] showed that a solution exists if F satisfies Carathéodory conditions and (7.1) is replaced by this slightly stronger assumption:

$$\lim_{\delta\downarrow 0} \alpha(F([t-\delta, t+\delta] \times S)) \le \omega(t, \alpha(S)).$$

Li [76] observed that (7.1) can be weakened slightly, to

$$\frac{\alpha(S) - \alpha([I - hF(t,\cdot)](S))}{h} \le \omega(t, \alpha(S)) \qquad (h > 0), \tag{7.2}$$

if F is uniformly continuous and bounded. Martin [82] used a similar condition for the autonomous problem (1.2): Assume $D = \mathrm{Dom}(G)$ is closed and bounded, and G is bounded and uniformly continuous and satisfies

$$\frac{\alpha(S) - \alpha([I - hG](S))}{h} \le \omega\alpha(S) \qquad (h > 0) \tag{7.3}$$

with ω constant; then (1.2) has a solution. Martin also obtained some other interesting consequences of (7.3): Define

$$W(t)z = \{u(t) : u \text{ satisfies (1.2) with initial value } z\}. \tag{7.4}$$

Then W is a semigroup (in the sense of (4.5), (4.6)) on $\Omega = \{\text{subsets of } D\}$, and

$$\alpha\left(\bigcup_{z\in S} W(t)z\right) \leq \alpha(S)\exp(t\omega) \qquad (t \geq 0), \tag{7.5}$$

generalizing (4.7). Also Martin showed that $W(t)z$ depends on z in an upper semicontinuous fashion.

Mönch and von Harten [83] showed that (7.1) suffices for existence if $F(t,x)$ is jointly continuous as a function of (t,x) and ω is 1/2 times a Kamke function. In general, it is not known whether the factor of 1/2 can be omitted. This depends on certain questions of measurability which are pursued further by Heinz [52] but are still not fully resolved.

As we noted in §4, the literature varies slightly on the precise definition of a "Kamke function." For uniformly continuous F's, Banaś [7] discusses the requirements on the Kamke function ω . He observes that

$$\omega(t,r) = \sup\{\alpha(F(t,S)) : \alpha(S) = r\}$$

is the smallest function which satisfies (7.1). He shows that this function satisfies certain regularity conditions; hence, assuming fewer regularity hypotheses about ω does not permit wider choices of F. See [7] for the technical details.

We have attempted to indicate the main ideas in this theory, but the literature abounds with generalizations, mostly more complicated. We give two indications of the literature's diversity: Tolstogonov [121] uses measure of noncompactness with set-valued functions F, using some of the concepts discussed in §5. Rzepecki [103] introduces Banach-space-valued measures of noncompactness, and then uses them to solve some systems of ordinary differential equations in Banach spaces.

8. Isotonicity. Let K be a cone (closed, convex, and invariant under multiplication by positive scalars) in a Banach space X. Then K is the *positive cone* for a partial ordering on X defined by: $x \leq y$ if and only if $y - x \in K$. We say K is *regular* (in the sense of Kransnosel'skiĭ) if every monotone, order-bounded sequence is convergent in norm to an element of X. A mapping $G : X \to X$ is *isotone* if $x \leq y \Rightarrow G(x) \leq G(y)$. (Such mappings are also called *order-preserving*, or *monotone*; but the latter term also has another meaning indicated in §9 below.)

Let Y be an ordered Banach space. (Below, we shall take Y to be a space of functions from $[0,T]$ into X.) It is easy to show that if $\Phi : Y \to Y$ is continuous (or, more generally, continuous from above or from below) and isotone, and Φ leaves invariant some order-interval $J = \{y : a \leq y \leq b\}$, then Φ has at least one fixed point in that order-interval. Indeed, we have $\Phi(a), \Phi(b) \in J$, and $\Phi(a) \leq \Phi(b)$; hence $a \leq \Phi(a) \leq \Phi(b) \leq b$. Continuing in this fashion, we obtain sequences

$$a \leq \Phi(a) \leq \Phi^2(a) \leq \ldots \leq \Phi^n(a) \leq \Phi^n(b) \leq \ldots . \leq \Phi^2(b) \leq \Phi(b) \leq b.$$

Since the cone is regular, both the sequences $\{\Phi^n(a)\}$ and $\{\Phi^n(b)\}$ must converge to limits. Continuity (or one-sided continuity) of Φ implies that both of those limits (or one of those limits) are fixed points of Φ.

This argument and some variants can be found in [69]. Also, some variants of this argument have been used in solving a number of partial differential equations, especially elliptic ones, using the maximal principle; see for instance [1] or [106].

In the paragraphs below, however, we shall only consider the abstract initial value problem (1.1). A solution of (1.1) is the same thing as a fixed point for Φ, if Φ is the integral operator given by (2.5). For Φ to be isotone, it suffices that $F(t,\cdot)$ be isotone for each t. The remaining problem is to guarantee that Φ leaves invariant some order-interval.

If F is bounded, then Φ maps into a bounded set. If the positive cone K has nonempty interior, then every bounded set is contained in an order-interval. Thus we have sketched a proof of:

8.1. THEOREM (Stecenko, 1961). *Let X be a Banach space partially ordered by a regular cone with nonempty interior. Suppose $F : [0,T] \times X \to X$ is single-valued, continuous, and bounded, and*

$$F(t,x) \leq F(t,y) \qquad \textit{whenever} \qquad x \leq y. \tag{8.2}$$

Then (1.1) *has at least one solution.*

Stecenko's hypotheses on K are really very strong. They are not satisfied by the usual positive cones in any of the most familiar Banach spaces of real-valued functions. Indeed, in $L^\infty(a,b)$ and in $C[a,b]$, the usual positive cone does have nonempty interior, but it is not regular. The same is true for $B(S)$, the space of bounded functions on a set S, if that set S is infinite. In c_0 and in $L^p(a,b)$ $(1 \le p < \infty)$, the usual positive cone is regular but has empty interior.

Moreover, it is not possible to include those Banach spaces by proving a better theorem. Volkmann [123], [124] gives examples showing that Stecenko's conclusion fails when K is the usual positive cone in c_0 or in $C[-1,1]$ — i.e., Volkmann gives examples of bounded, continuous, isotone F for which (1.1) has no solution. Volkmann's examples are variants of Dieudonné's example, mentioned in §2. Like Dieudonné, Volkmann determines solutions componentwise and then shows that they do not lie in an appropriate space; hence the particular choice of a notion of "solution" (discussed in §3, 11, 13, 14) is not at issue here.

What about using some cone K other than the usual cone of nonnegative-valued functions? Trivial, degenerate choices will not satisfy Stecenko's hypotheses: a cone which is too small has no interior, and a cone which is too big is not regular. But Volkmann [124] notes that Stecenko's conditions are satisfied in *any* Banach space X, using a cone K constructed as follows: Let q be any nonzero element of X, and let ρ be any constant in $(0, \|q\|)$. Let S be the closed ball centered at q with radius ρ, and let $K = \bigcup_{\lambda \ge 0} \lambda S$. Then it can be shown that Stecenko's hypotheses are satisfied.

Volkmann [123] also notes that Stecenko's regularity hypothesis can be omitted at least in one important case: Let S be an arbitrary set, and let X be the space $B(S)$ of bounded, real-valued functions from S, equipped with the supremum norm. Let K be the usual positive cone, i.e., those functions which are nonnegative everywhere on S. Then K is regular precisely when S is finite; but (1.1) has a solution even if S is infinite.

Volkmann [123] also observes that condition (8.2) can be weakened to the assumption that

$$F(t,y) - F(t,x) \ge -\omega(y-x) \qquad \text{whenever} \qquad x \le y \tag{8.3}$$

for some real constant ω, since then we can apply the preceding theorem to the function

$$H(t,x) = e^{\omega t} F(t, e^{-\omega t}x) + \omega x$$

and obtain existence of the function $v(t) = e^{\omega t}u(t)$.

Because their hypotheses are so strong, Stecenko's theorem and related results have few known applications. We have included these results simply because isotonicity is so very different from generalized compactness or dissipativeness (§7 and 9). Two other related results, in finite dimensions, are also simple enough to deserve mention:

Biles [11] has shown that if $F : [0,T] \times \mathbf{R} \to \mathbf{R}$ is bounded and measurable, and $F(t,x)$ is both right-continuous and upper semicontinuous in x for each fixed t, then (1.1) has a solution. Biles' two assumptions of one-sided continuity may be restated as follows:

$$\limsup_{y \to x-} F(t,y) \le F(t,x) = \lim_{y \to x+} F(t,y).$$

When $X = \mathbf{R}$, Biles' result extends Stecenko's. We wonder if Biles' result or one like it might be extended to spaces of higher dimension.

Wend [127] considers $\mathbf{R}^n$ with its usual positive cone. He shows that if $F : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ takes values in the positive cone, and $F(t,x)$ is a nondecreasing function of both t and x, then (1.1) has a Carathéodory solution for some $T > 0$. No continuity or semicontinuity of F is assumed explicitly, but Wend makes use of the fact that any isotone function from $\mathbf{R}$ into $\mathbf{R}^n$ is continuous almost everywhere.

9. Dissipativeness. Some introductions to dissipativeness are [26] and [73]. For completeness we give a brief introduction here, with different emphases than those two sources.

Let G be any map from the Banach space X into the set of all subsets (not necessarily nonempty) of X. We say G is *single-valued*, or *point-valued*, if $G(x)$ contains at most one point for each x. The *effective domain*, the *range*, and the *resolvent* of G are defined by

$$\mathrm{Dom}(G) = \{x \in X : G(x) \text{ is nonempty}\},$$

$$\mathrm{Ran}(G) = \bigcup_{x \in X} G(x) = \bigcup_{x \in \mathrm{Dom}(G)} G(x),$$

$$J_\lambda(x) = (I - \lambda G)^{-1} x = \{y : x \in y - \lambda G(y)\} \quad (\lambda > 0).$$

The operator G is *dissipative* if

> for each $\lambda > 0$, the resolvent J_λ is single-valued on $\mathrm{Dom}(J_\lambda) = \mathrm{Ran}(I - \lambda G)$ and is Lipschitzian with Lipschitz constant ≤ 1.

Equivalently, we say $-G$ is *accretive* — or, in Hilbert spaces, $-G$ is *monotone*, but that word also has another meaning (see §8). Much of the related literature is written in terms of accretive operators, rather than dissipative operators, because the resulting notation is more consistent with the traditional notation of hyperbolic partial differential equations — one of the most important applications of accretive/dissipative operators. But we shall use the dissipative notation, since it is more consistent with the other theories surveyed in this paper.

Locally Lipschitz mappings satisfy a local dissipativeness condition; hence so do continuously differentiable mappings. Thus, in most studies of ordinary differential equations, both compactness and dissipativeness are available for existence proofs. Many nonlinear partial differential operators are dissipative, or satisfy a similar condition, in suitably normed Banach spaces; see [25], [37], [38], [109]. For motivation, consider that differential operators tend to be discontinuous under most natural topologies, but in many cases their inverses — or their resolvents, which are inverses of perturbations of those differential operators — are integral operators, which tend to be continuous under most natural topologies. For this reason, many of the inequalities and other conditions in the dissipative theory are formulated in terms of the resolvent J_λ, rather than directly in terms of the generator G.

Dissipativeness can be restated in terms of the directional derivative of the norm:

$$[p, q]_- \equiv \sup_{h<0} \frac{\|p + hq\| - \|p\|}{h} = \lim_{h \uparrow 0} \frac{\|p + hq\| - \|p\|}{h}. \tag{9.1}$$

The last equality follows from the fact that $\|p+hq\|$ is a convex function of h, and hence $(\|p+hq\|-\|p\|)/h$ is a nondecreasing function of h. Slight variants of (9.1) are used in various papers; here we follow the notation of [73]. A consequence of (9.1) is

$$|\,[p, q]_-\,| \leq \|q\|. \tag{9.2}$$

For motivation note that if X is a Hilbert space with inner product $(\ ,\ )$, then $[p, q]_- = \mathrm{Re}(p, q)/\|p\|$.

For any $\omega \in \mathbf{R}$, we say G is *ω-dissipative* if $G - \omega I$ is dissipative. That condition holds if and only if

$$[x_1 - x_2, y_1 - y_2]_- \leq \omega \|x_1 - x_2\| \tag{9.3}$$

for all $x_1, x_2 \in \mathrm{Dom}(G)$ with $x_1 \neq x_2$, and all $y_1 \in G(x_1)$, $y_2 \in G(x_2)$. A few papers call this "quasi-dissipative," but that term usually has another meaning: We say G is *ω-quasi-dissipative* if

$$[x_1 - x_2, y_1]_- + [x_2 - x_1, y_2]_- \leq \omega \|x_1 - x_2\| \tag{9.4}$$

for all $x_1, x_2 \in \mathrm{Dom}(G)$ with $x_1 \neq x_2$, and all $y_1 \in G(x_1)$, $y_2 \in G(x_2)$. This is generally weaker than ω-dissipativeness (but not if X is a Hilbert space). A stronger condition than ω-dissipativeness, used in a few papers, is *strict ω-dissipativeness*; this condition is obtained by using $t \downarrow 0$ instead of $t \uparrow 0$ in (9.1).

Dissipativeness is sometimes referred to as a "one-sided Lipschitz condition," because if (2.2) holds then (by (9.2)) both G and $-G$ are ω-dissipative. However, the converse is not valid. For instance, if A is a linear, self-adjoint operator in a complex Hilbert space, then both iA and $-iA$ are dissipative, but A need not be Lipschitz or even continuous. Dissipativeness does generalize another property of Lipschitz operators mentioned in §4: If G is ω-dissipative for some constant ω, then each solution of (1.2) is uniquely determined by its initial value, and the semigroup so determined is of type ω — i.e., satisfies (4.7). More generally, if $F(t,\cdot)$ is $\omega(t)$-dissipative for each t, and u and v are solutions of (1.1), then

$$\|u(t)-v(t)\| \le \|u(0)-v(0)\| \exp[\textstyle\int_0^t \omega(s)\,ds]. \tag{9.5}$$

For continuous F or G, these inequalities actually characterize dissipativeness.

The existence theory takes its simplest form when F is continuous:

9.6. THEOREM. *Let D be a locally closed subset of a Banach space X. Let $F:\mathbf{R}\times D\to X$ be single-valued and jointly continuous. For each t, suppose $F(t,\cdot)$ satisfies* (6.5) *and is $\omega(t)$-dissipative, where $\omega(\cdot)$ is a continuous function. Then for each $u(0)\in D$,* (1.1) *has a solution at least for some $T>0$. If D is closed, then* (1.1) *has a solution $u(t)$ for all $t\ge 0$.*

The local existence result is essentially due to Martin [81], although he assumed strict ω-dissipativeness. That additional assumption was dropped in later papers which extended Martin's result in other ways as well. Most notably: [108] weakens the assumption of joint continuity to an assumption that F satisfies Carathéodory conditions. In that paper, the domain of F is still "cylindrical," i.e., of the form $\mathbf{R}\times D$. Several other papers assume joint continuity, but permit F to have a noncylindrical domain. It is assumed, roughly, that the set $D_t=\mathrm{Dom}(F(t,\cdot))$ is upper semicontinuous from the left, as a function of t. Also, the subtangential condition (6.5) is replaced with (6.8). Kenmochi and Takahashi [67] observe that, in the presence of the other hypotheses, this subtangential condition is not only sufficient but also necessary for the existence of solutions. Iwamiya [59] weakens the dissipativeness hypothesis, assuming instead only that

$$[x-y, F(t,x)-F(t,y)]_- \le \omega(t, \|x-y\|) \tag{9.7}$$

for some Kamke function ω.

Note that (9.7) is a special case of (4.2), with $V(t,x,y)=\|x-y\|$. Also (9.7) $\Rightarrow$ (4.1), in view of (9.2).

10. Dissipativeness without continuity. We now turn to the existence results in which $F(t,\cdot)$ is dissipative but not necessarily single-valued or continuous. In most of these results, the solution obtained is not a Carathéodory solution (defined in §3), but a still weaker "limit solution." It may be rather ill-behaved, but that is not necessarily a disadvantage to this theory, for it means that the theory can even be applied to differential equations whose solutions are known to be ill-behaved — for instance, hyperbolic partial differential equations with shocks. Limit solutions will be discussed in §11.

10.1. THEOREM (Crandall and Liggett, 1970). *Suppose G is an ω-dissipative operator in a Banach space X, satisfying* (6.2). *Then G generates a strongly continuous semigroup $S(t)=\exp(tG)$, defined as in* (4.8), *on* $\mathrm{cl}(\mathrm{Dom}(G))$*; and that semigroup is of type ω (i.e., satisfies* (4.7)*).*

The Crandall-Liggett theorem has been extended in a few ways:

Kobayashi [68] replaces hypothesis (6.2) with the weaker subtangential condition (6.3). This requires that we replace the simple product of resolvents in (4.8) with a more complicated product of approximate resolvents, such as that in (11.4) below. More recently, Kobayashi (unpublished result; mentioned in [26]) has shown that (6.3) can be weakened further to (6.4). In fact, for G dissipative, (6.4) is *necessary and sufficient* for existence of a solution to (1.2).

The operator G need not be ω-dissipative; it suffices for G to be ω-quasi-dissipative, as in (9.4). This weaker hypothesis was used by Kobayashi [68]. The extension from dissipative to quasi-dissipative is primarily of theoretical, not applied, interest. Kobayashi gives an example of an ω-quasi-dissipative operator which is not ω-dissipative, but the example is quite artificial. Apparently no examples are yet known which have real application.

A different direction of generalization is taken by Picard [92]. Let ϕ be a convex mapping from a topological vector space X into $\mathbf{R}$. The vector space X is *ϕ-complete* if $\phi(x_m - x_n) \to 0$ implies that $\{x_n\}$ converges to a limit in X. An operator U in X is *ϕ-nonexpansive* if $\phi(Ux - Uy) \leq \phi(x - y)$ for all $x, y \in \mathrm{Dom}(U)$. An operator G is *ϕ-dissipative* if the resolvent $J_\lambda = (I - \lambda G)^{-1}$ is ϕ-nonexpansive for all $\lambda > 0$. (Actually, Picard calls such U "ϕ-contractive," and says that $-G$ is "ϕ-accretive.") Picard finds that if X is a normed vector space, $\phi : X \to \mathbf{R}$ is convex and Lipschitzian, X is ϕ-complete, G is ϕ-dissipative, and G satisfies (6.2), then the limit $S(t)$ defined in (4.8) exists on $\mathrm{cl}(\mathrm{Dom}(G))$, and $S(t)$ is ϕ-nonexpansive for each t.

The Crandall-Liggett Theorem 10.1 and most of its extensions assert global existence, i.e., existence for all $t \geq 0$. This follows from the fact that, for simplicity, we have taken ω to be constant, and also from the global nature of our "range condition" (6.2) or (6.3). Weakening these conditions to local ones yields local existence results. However, for simplicity of notation, most of the abstract theory of dissipative operators has been developed with global estimates. In fact, much of the literature only considers the case of $\omega = 0$. These restrictions are without substantial loss of generality, since most of the statements and proofs can be extended to more general choices of ω without great difficulty. Of course, for some applications, ω must be nonzero or even nonconstant; see for instance [109]. In most of the remainder of this survey, we shall only discuss dissipativeness for the case of $\omega = 0$.

We shall have more to say about the autonomous problem (1.2) in later sections, but now let us turn to the nonautonomous problem (1.1). The theory for this problem, with $F(t, x)$ dissipative in x, is not unified; there are at least three substantially separate approaches to it. One approach, mentioned earlier, is descended from Martin's result, Theorem 9.6. A second approach is discussed below; a third approach will be discussed in §14.

The Crandall-Liggett Theorem 10.1 has been generalized to temporally inhomogeneous problems (1.1) — i.e., with $F(t, \cdot)$ dissipative and discontinuous — in a number of different ways, none of them wholly satisfactory. All such generalizations involve regularity assumptions about the dependence of $F(t, x)$ on t which, though fairly weak, seem somewhat unnatural and hard to motivate. Research in this area continues. A good introduction to this area is given by the survey paper [86] and the book [89].

The earliest result in this direction is that of Crandall and Pazy [28]. They make regularity assumptions of the form

$$\|J_\lambda(t)x - J_\lambda(s)x\| \leq \lambda |g(t) - g(s)| B(s, x), \tag{10.2}$$

where $J_\lambda(t) = (I - \lambda F(t, \cdot))^{-1}$. They give two versions. In one version (condition (C.1) in [28]), they assume g is continuous and $B(s, x) = L(\|x\|)$ for some increasing function L independent of s. In the other version (condition (C.2)), they assume g is continuous and of bounded variation, and $B(s, x)$ depends on $\|x\|$ and on the behavior of $J_\lambda(s)$ near x as $\lambda \downarrow 0$; the details are complicated and are omitted here. Evans [36] keeps the form of (10.2) but replaces continuity of g with integrability. The hypotheses of Crandall and Pazy and of Evans imply that $\mathrm{cl}(\mathrm{Dom}(F(t, \cdot)))$ is independent of t, but that restriction is weakened in later papers mentioned below.

Some subsequent papers have departed in form from (10.2). Hypotheses used are of roughly the following form:

$$\|J_\lambda(t_1)x_1 - J_\lambda(t_2)x_2\| \leq \|x_1 - x_2\| + f(t_1, t_2) L(\|x_1\|) \tag{10.3}$$

or, alternatively,

$$[x_1 - x_2, y_1 - y_2]_- \leq f(t_1, t_2) L(\|x_1\|) \qquad \text{whenever} \quad y_i \in F(t_i, x_i) \quad (i = 1, 2), \tag{10.4}$$

in both cases for $t_1 \leq t_2$. Here f is assumed integrable on $[0, T] \times [0, T]$, and f is also assumed to satisfy some other conditions. Condition (10.2) is included by taking $f(t, s) = g(t) - g(s)$. Iwamiya, Oharu, and Takahashi [62] have recently proven existence assuming f is integrable on $[0, T] \times [0, T]$, $f(t, t) = 0$, and f is continuous on the diagonal. Pavel [88] uses a quasi-dissipativeness condition, replacing the left side of (10.4) with $[x_1 - x_2, y_1]_- + [x_2 - x_1, y_2]_-$. The papers [61], [93], [94] assume regularity conditions similar to (10.3), but applied to the "tangential" semigroups $\exp(hF(t,\cdot))$ rather than to the resolvents $J_\lambda(t) = (I - \lambda F(t,\cdot))^{-1}$.

In some papers, the factor $L(\|x_1\|)$ is omitted. That omission might seem to be a severe restriction, in that it requires the regularity of $F(t,x)$ in t to be uniform for all x in $\mathrm{Dom}(F(t,\cdot))$. But the factor $L(\|x_1\|)$ is unnecessary, as explained in [60]: If Crandall and Pazy's condition (10.2) is satisfied, then — once one has selected the initial value $u(0)$ — one can restrict F to a smaller set, still large enough to contain the solution and all needed approximate solutions, but small enough so that — with a suitable choice of f — the quantity $[x_1 - x_2, y_1 - y_2]_-/f(t_1,t_2)$ (or other relevant quantity) remains bounded.

Even with Crandall and Pazy's condition (10.2), the regularity in t (as $t \to s$, with s fixed) is uniform for x in sets where $B(s,x)$ is bounded, and those may be rather large sets. In contrast, if F is jointly continuous (or more generally, satisfies Carathéodory conditions), then F behaves regularly in t uniformly on compact sets of x — but compact sets are not "large," in infinite-dimensional spaces. It appears that Theorem 9.6 is not a special case of the theories discussed in the last few paragraphs. Indeed, a simple example is given in the introduction of [108], in which $F : \mathbf{R} \times X \to X$ satisfies the hypotheses of Theorem 9.6 (and moreover $F(t,\cdot)$ is a bounded linear operator for each t), but F does not satisfy $\|F(t,x) - F(s,x)\| \le |g(t) - g(s)| B(s,x)$ for any choice of B and integrable g. An easy modification of the computations in [108] shows that this same F also does not satisfy (10.2). One wonders whether this F can be made to fit (10.2) by a suitable restriction of domain.

Conditions (10.2)-(10.4) are sufficient to make various existence proofs work, but motivation for those hypotheses is not entirely clear. In light of the example just noted, those hypotheses seem somewhat unnatural. It is not yet clear what hypotheses would be more natural. A first guess would be to impose Carathéodory conditions on the resolvents $J_\lambda(t) = [I - \lambda F(t,\cdot)]^{-1}$, rather than on F itself. However, an example given by Freedman [41] shows that that does not work. In Freedman's example, $F(t,\cdot)$ is a bounded linear operator for each t, and the resolvent $J_\lambda(t)x$ is a jointly continuous function of λ, t, and x. Yet (10.2) is not satisfied and (1.1) has no solution. Freedman's example is also noteworthy in that he proves nonexistence using a technique substantially different from that of Dieudoné [33] (sketched in §2). Most of the examples of nonexistence to be found in the literature are slight variants of Dieudonné's example.

11. Limit solutions. There are a number of initial value problems, arising in control theory or in fluid mechanics, for which a "solution" is known to exist by physical considerations, but for which that "solution" is not differentiable [25], [37], [38]. An abstract theory to support these applications is desirable, even if it requires somewhat complicated notions of "solution." As we remarked at the end of §3, such a solution need not satisfy (1.1) in any direct sense; the relation (1.1) is kept merely as an abbreviation for the more complicated notion of "solution" to be used.

We say $u(t)$ is a *limit solution* of (1.1) if there exist a sequence of numbers $\varepsilon_n \downarrow 0$ and a sequence of functions $v_n(t) \to u(t)$, such that v_n is an ε_n-approximate solution of (1.1). A function $v(t)$ is an *ε-approximate solution* of (1.1) if v comes within ε of satisfying (1.1) in some suitable sense. The precise definition of "ε-approximate solution" varies from one paper to another; some typical versions are indicated below. We shall not attempt to give all the variants in technical detail. (In fact, many of the papers surveyed do not use the terminology given here, but their results can easily be reformulated in this terminology.) Usually the definitions require that $u(t)$ be continuous and that the convergence $v_n(t) \to u(t)$ be uniform over all $t \in [0,T]$, but these requirements are weakened in some papers discussed below. The existence theorems usually do not assert that every sequence of ε-approximate solutions converges to u as $\varepsilon \downarrow 0$, but only that some such convergent sequence exists. In some results with compactness, the limit solution u is not unique, and different approximating sequences $\{v_n\}$ may converge to different limit solutions u.

Even when a Carathéodory solution exists, one common method of constructing that solution is by taking the limit of a sequence of approximate solutions. Generally, if F is continuous in its second argument, then every limit solution is a Carathéodory solution. If F is not continuous in its second argument, then a limit solution $u(t)$ need not be a Carathéodory solution, and in fact $u(t)$ need not be differentiable for any t. A simple example of a nowhere differentiable solution is given in [27]. Moreover, $u(t)$ need not take values in $\mathrm{Dom}(F(t,\cdot))$ — but under most definitions of "limit solution," $u(t)$ does take values in $\mathrm{cl}(\mathrm{Dom}(F(t,\cdot))$.

The simplest type of approximate solution is an "outer perturbation": v is an ε-approximate solution of (1.1) in this sense if v is absolutely continuous, differentiable almost everywhere, and satisfies

$$\|v'(t) - F(t, v(t))\| \le \varepsilon \qquad \text{for} \qquad 0 \le t \le T.$$

This condition generalizes to multivalued F as well: use the distance from $v'(t)$ to $F(t, v(t))$ instead of the norm of the difference. Equivalently, v is an ε-approximate solution of (1.1) if v is a Carathéodory solution of

$$v'(t) \in F(t, v(t)) + \varepsilon B \tag{11.1}$$

where B is the closed unit ball.

A more general approach involves "inner perturbations;" we replace (11.1) with

$$v'(t) \in F(t + [-\varepsilon, \varepsilon], v(t) + \varepsilon B) \qquad (0 \le t \le T). \tag{11.2}$$

It is easy to see that any limit of outer perturbations is a limit of inner perturbations. The converse holds if F is single-valued and continuous, but not in general. In the terminology of Hájek [48] (modified slightly), u is a *Hermes-$*$ solution* of (1.1) if u is a uniform limit of ε-approximate solutions in the sense of (11.2).

In many of the constructions found in the literature, the interval $[0, T]$ is (for fixed ε) partitioned into subintervals:

$$0 = t_0 < t_1 < t_2 < \ldots < t_m = T,$$

where $h_j = t_j - t_{j-1} < \varepsilon$ for all j. Then the approximate solution $v(t_j)$ is defined at each partition point t_j, typically by some recursion on j, using some subtangential condition such as (6.8) or (6.9). The approximate solution $v(t)$ is then defined either as a step-function which is constant on each open subinterval (t_{j-1}, t_j), or as a piecewise-linear function which is affine on each closed subinterval $[t_{j-1}, t_j]$. These two methods for defining $v(t)$ differ only superficially, since both methods must yield the same uniform limit $u(t)$ if that limiting function is continuous. The method of step-functions — in particular, the backward difference method, discussed further below — has been especially popular among papers with discontinuous $F(t, \cdot)$ (§10), perhaps because this brings the definition of "solution" closer to the method of proof. On the other hand, the piecewise-linear approach has an advantage in that it makes the definition of "approximate solution" simpler and more intuitive: Each approximate solution is itself the exact (Carathéodory) solution of an approximating differential equation, such as (11.2).

Generally, $v(t_j)$ is chosen recursively from $v(t_{j-1})$ so that $[v(t_j) - v(t_{j-1}]/h_j$ is exactly or approximately equal to an element of one of the following sets:

> $F(t_{j-1}, v(t_{j-1}))$, used in *forward difference schemes*, also known as *Euler polygonal approximations*;
>
> $F(t_j, v(t_j))$, used in *backward difference schemes*;
>
> $(t_j - t_j - 1)^{-1} \int_{t_{j-1}}^{t_j} F(s, v(t_{j-1}))\, ds$, used in *Euler-Lebesgue approximation*;

or some other function involving $F(\sigma, v(\tau))$ for one or more values of σ and τ in the interval $[t_{j-1}, t_j]$.

The backward difference method uses a product of resolvents, as in the Crandall-Liggett Theorem 10.1, or an approximate product of resolvents, as in the example below. In the papers discussed in §10, the definition of "solution" is very complicated, and must be bewildering to newcomers to this subject. A typical definition is as follows: v is an ε-approximate solution to (1.1) (with initial value $u(0)$ given) if there exists a partition

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T \tag{11.3}$$

and points $x_k \in \mathrm{Dom}(F(t_k, \cdot))$ and $p_k \in X$, such that

$$\begin{cases} \|x_0 - u(0)\| < \varepsilon; \\ t_k - t_{k-1} < \varepsilon \qquad (k = 1, 2, \ldots, N); \\ v(0) = x_0, \text{ and } v(t) = x_k \text{ for } t \in (t_{k-1}, t_k] \quad (k = 1, 2, \ldots, N); \\ (x_k - x_{k-1})/(t_k - t_{k-1}) - p_k \in F(t_k, x_k) \qquad (k = 1, 2, \ldots, N); \text{ and} \\ \sum_{k=1}^{N} (t_k - t_{k-1}) \|p_k\| < \varepsilon. \end{cases} \tag{11.4}$$

It may be helpful to observe that

$$x_k \in \{I - (t_k - t_{k-1})[p_k + F(t_k, \cdot)]\}^{-1}(x_{k-1}) \qquad (k = 1, 2, \ldots, N),$$

and thus in a sense $x_k \approx J_\lambda(t_k)x_{k-1}$ with $\lambda = t_k - t_{k-1}$.

A "solution" in these papers is a continuous function $u(t)$ which is a uniform limit of such approximations $v(t)$, as $\varepsilon \downarrow 0$. The existence of such approximate solutions follows easily from a subtangential condition such as (6.9). The hard part of this theory is proving their convergence to a limit (which proof we shall *not* attempt to sketch here).

Any such limit solution is also a Hermes-$*$ solution, as we shall now show. For each ε-approximate solution v and error terms p_k as in (11.4), let w be the piecewise-linear function which is affine on each interval $[t_{k-1}, t_k]$ and agrees with v at the endpoints of that interval; then $w'(t) \in p_k + F(t_k, x_k)$ on that interval. Let $p(t)$ be the step-function which satisfies $p(t) = p_k$ on the interval (t_{k-1}, t_k), and let $z(t) = w(t) - \int_0^t p(s)\, ds$. Then $\int_0^t \|p(s)\|\, ds \leq \varepsilon$, hence $\|z(t) - w(t)\| \leq \varepsilon$ for all t, and $z'(t) \in F(t_k, x_k) \subseteq F([t - \varepsilon, t + \varepsilon], x_k)$. As $\varepsilon \downarrow 0$, the v's converge uniformly to u; hence so do the w's; hence so do the z's; hence $\|v - z\|_{\sup} \downarrow 0$. Since u is continuous, $\sup\{\|z(t) - x_k\| : t \in [x_{k-1}, x_k]\} \downarrow 0$ also. Thus $z'(t) \in F([t - \delta, t + \delta], z(t) + \delta B)$ for some δ which goes to 0 when $\varepsilon \downarrow 0$.

Thus, any backward difference scheme limit solution is also a Hermes-$*$ solution; perhaps this observation will be helpful to beginners trying to come to grips with (11.4). It is not known (at least, to this author) whether the converse holds — i.e., whether every Hermes-$*$ solution is also a backward difference scheme limit solution. Such a converse would follow if dissipativeness of $F(t, \cdot)$ is enough to guarantee uniqueness of Hermes-$*$ solutions of (1.1).

With most definitions of "limit solution" — including all of those discussed above — the limit $u(t)$ is required to be a continuous function of t. In §13 and 14 we shall discuss some other approaches to (1.1) which permit discontinuities in u for various theoretical reasons. Moreau [84] permits discontinuities for more practical reasons: For some differential equations with applications to elastoplastic mechanical systems, the physically "natural" solution $u(t)$ may have jump discontinuities. Moreau [85] observes that if $u(t)$ is discontinuous, then it is not natural to require approximate solutions $v_n(t)$ to converge uniformly in t. Indeed, if $u(t)$ has a jump at t_0 of size r or larger, and if $\|u - v_n\|_{\sup} < r$, then v_n must also have a jump (possibly of a different size) at the exact same location t_0. It is more natural to require v_n to have a jump *near* t_0. This effect is achieved by requiring the sequence $\{v_n\}$ to converge to u *in graph*, rather than uniformly.

All the notions of "solution" discussed so far depend only on the topological vector space structure of the Banach space X. If we replace the norm on X with an equivalent norm, then a solution of (1.1) remains a solution. Intuitively, we expect the same to be true for our methods of proving existence of solutions. But our methods of proof are very norm-dependent. To apply existence theorems such as those discussed in §7-10, we must choose precisely the right norm on X; an equivalent norm will not necessarily work. For instance, if an operator G is dissipative with respect to one norm, the same operator G is not necessarily dissipative (or ω-dissipative, or even ω-quasi-dissipative) with respect to any other, equivalent norm.

Some interesting approaches have been used to get around this difficulty. For instance, in studying nonautonomous problems (1.1), [64] and [126] use a whole family of norms $\|\quad\|_t$, parametrized by the time variable t; each instantaneous operator $F(t, \cdot)$ is shown to satisfy dissipativeness and other conditions with respect to $\|\quad\|_t$. Ideally, we would prefer to find some topological vector space condition, not dependent on particular norms, which generalizes dissipativeness and which is satisfied by $F(t, \cdot)$ for all t. The notion of "dissipative," though very useful for applications, is not completely satisfactory from a purely theoretical point of view.

In the *linear* theory of dissipative operators, this is not a problem. The norm on the Banach space X can always be replaced by an equivalent norm, so that a semigroup S satisfying $\|S(t)\| \leq Me^{\omega t}$ for some constant M becomes a semigroup of type ω. Thus, M gets replaced by 1 (see [46]). An analogous trick does not work for nonlinear semigroups. The best we can do is replace the norm with an equivalent *metric*, making the Banach space X into a Banach manifold. That approach has been investigated by Marsden [80], but it requires more smoothness than is customary in the theory of nonlinear semigroups. The smoothness requirements have recently been weakened somewhat by Pimbley [97].

12. Identification of the limit. Usually, a proof of existence of a solution to (1.1) can be divided into three main steps — approximability, convergence, and identification — discussed in greater detail below. This outline is indicated in [73], for instance. (A variant of this outline is needed for proofs which use fixed point theorems; see the discussion after (13.4).) Of course, even in papers where these three steps are followed, they may not be mentioned explicitly; most papers in existence theory combine the steps. But we may understand these three steps more clearly if we study them separately; they involve different techniques and different hypotheses. The first two steps can be described briefly, since they involve ideas which we have already discussed at some length:

(1) Existence of approximate solutions. We must show that ε-approximate solutions v exist, for arbitrarily small values of ε. Generally the approximate solutions must be constructed with some care, so that the next two steps will be possible. Existence of approximate solutions follows from a subtangential condition such as those discussed in §6.

(2) Convergence to a limit. We must show that some sequence of approximate solutions v_n (with $\varepsilon_n \downarrow 0$) converges to some limit u. This usually follows from a hypothesis of generalized compactness, isotonicity, or dissipativeness — discussed in §7-10. Thus, those three substantially different hypotheses play analogous roles. Hence there is some hope of unifying their three separate theories, as discussed in §15.

The third step will require a lengthier discussion:

(3) Identification of the limit. Roughly, the goal of this step is to show that the limit $u(t)$ obtained in Step 2 is really a "solution" in some sense, i.e., to establish some connection between $u(t)$ and the differential inclusion (1.1). Step 3 is optional: Some connection is implicit in the definition of ε-approximate solution used in Step 1, and so a further connection is not absolutely necessary. However, that implicit connection is rather indirect, and not fully satisfactory. If one's notion of "solution" is too general, then *everything* becomes a solution, and existence theory becomes meaningless. It is desirable to show that the solution $u(t)$ obtained in Step 2 is uniquely determined by (1.1), or at least that the set of solutions is narrowly restricted by some properties connecting it more closely to the differential equation (1.1).

The simplest case is that in which $F(t,x)$ is single-valued, bounded, and continuous in its second argument. Then, generally, the limit solution is also a Carathéodory solution. A typical proof runs thus: We observe that the approximate solutions satisfy an approximate version of the integral equation (2.4). Applying Lebesgue's Dominated Convergence Theorem, we find that the limit $u(t)$ satisfies (2.4). Variants of this argument sometimes work when F is set-valued and semicontinuous, as discussed in §5.

Thus, one of the chief uses of a hypothesis of continuity of F (in conjunction with hypotheses of compactness or isotonicity) is for identification of the limit. If we are satisfied with the identification property already implicit in our definition of ε-approximate solutions, then continuity of F becomes less important. For instance, Hájek [48] observes that if $F : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$ is bounded, then there exists a convergent sequence of Euler-polygonal approximations for (1.1). We need not assume F is continuous — or even measurable! These results extend easily to multivalued F in infinite dimensional spaces, if we assume $\mathrm{Ran}(F)$ is compact. We speculate that, analogously, the continuity hypotheses in some other known existence theorems might be weakened or removed if we weaken the method of identification of the limit (as suggested in the paragraphs below). The present author hopes to research this idea further in the near future.

Even if the solution $u(t)$ is not differentiable, other methods of identifying the limit may be available. Such methods have been developed especially for $F(t,\cdot)$ dissipative and discontinuous.

The concept of "envelope solutions" was introduced by Pierre [93], [94] and developed further by Iwamiya, Oharu, and Takahashi [61]. A function u is an *envelope solution* of (1.1) if

$$\lim_{h\downarrow 0} \frac{1}{h} \int_0^{T-h} \|u(t+h) - \exp(hF(t,\cdot))u(t)\| \, dt = 0.$$

Roughly, the idea is that at each instant t the trajectory $u(t)$ is "tangent" to the semigroup $\exp(hF(t,\cdot))$, and thus behaves like that semigroup, at least momentarily. The solution $u(t)$ need not be differentiable, but the tangential semigroups need not be differentiable either.

Another approach is that of integral inequalities. An evolution operator $U(t,s)$ is the *integral solution* of (1.1) if it satisfies

$$\|U(t,r)x_1 - x_2\| \le \|x_1 - x_2\| - \int_r^t \left\{[U(s,r)x_1 - x_2, -w]_- + f(s,r)\right\} ds \tag{12.1}$$

for all $[r,t] \subseteq [0,T]$, all $x_1 \in \mathrm{Dom}(F(r,\cdot))$, all $q \in [0,T]$, and all $w \in F(q,z)$. Here f is the function which appears in (10.3) or (10.4). Under appropriate hypotheses on F, (12.1) determines $U(t,r)$ uniquely. This inequality is given in [86], for instance. It generalizes an inequality which was developed by Bénilan for the quasiautonomous problem, i.e., the problem in which F can be written in the form $F(t,x) = G(x) + g(t)$.

Another approach to identification involves continuous dependence (discussed further in the next section). For simplicity, suppose that for each $(u(0),F)$ in some problem space $\mathcal{P}$, the construction described above as Steps 1 and 2 yields a unique solution u which is a point in some solution space $\mathcal{S}$. Define a mapping $\Gamma : \mathcal{P} \to \mathcal{S}$ by $u = \Gamma(u(0),F)$. Suppose that some topologies on $\mathcal{P}$ and $\mathcal{S}$ can be described simply and naturally, and that they make the mapping Γ continuous. Moreover, suppose that u is a Carathéodory solution (or other easily-motivated solution) of (1.1), for all $(u(0),F)$ in some dense set $\mathcal{P}_0 \subset \mathcal{P}$. Then it is reasonable to call $u = \Gamma(u(0),F)$ a "solution" of (1.1) in a generalized sense, even for $(u(0),F)$ outside that dense set. If Γ is *uniformly* continuous, then we can replace $\mathcal{P}$ with its completion, thus motivating solutions for a possibly larger class of initial value problems. Even if the solution u is not uniquely determined by $u(0)$ and F, the preceding argument is applicable if we can show that the *set of solutions* depends continuously on the data, as indicated in the next section.

13. Continuous dependence. Assume (for the moment) that for each F in some suitable class of operators, and each $u(0)$ in some suitable subset of X, the initial value problem (1.1) has a solution u. Does u depend continuously on the initial value $u(0)$ and the generator F? This question is important for many reasons: (i) As we noted at the end of the previous section, continuous dependence results can be viewed as a method of identification of the limit. (ii) Continuity of a map such as $(u(0),F) \mapsto u$ is one of the principal ingredients in applications of the Schauder Fixed Point Theorem — see for instance the discussion after (13.4), below. (iii) The data $u(0)$ and F may be based on a physical experiment, and so the specification of $u(0)$ and F may involve inexact measurements; we would like to know that small errors in the data cause only small errors in u. (iv) If F is discontinuous or otherwise badly behaved — or if $u(0)$, which itself may lie in some function space, is badly behaved — we may wish to replace F or $u(0)$ with a "nearby" choice which is better behaved, to facilitate computations or proofs; we would like to know that this will not change u greatly either. (v) The convergence of approximate solutions to a limit can be viewed as a special case of continuous dependence results, since — as we noted in (11.2) — approximate solutions to an initial value problem are themselves excact solutions to approximating problems. (vi) Continuous dependence results can be applied in existence proofs in other ways, too. For instance, in [108], a continuous dependence result is used to show that a certain "nice" class of initial value problems has solutions taking values in a separable subset of X. That implies that a certain "bad" set of values of t has Lebesgue measure 0. That fact, in turn, is used to prove convergence of some of those "nice" solutions, and hence to prove existence of limit solutions to some not-so-nice initial value problems.

Let us survey first the continuous dependence of u on $u(0)$. We have already seen that if $F(t,\cdot)$ satisfies a dissipativeness condition for each t, then $u(t)$ depends on $u(0)$ in a Lipschitz fashion (9.5). Without dissipativeness, other hypotheses may be needed to guarantee uniqueness, as noted in §4. In finite dimensions, for F bounded but possibly discontinuous, it is known that uniqueness of solutions implies continuous dependence on initial values; see [49]. In infinite dimensions, however, this result fails, even for continuous F; see [42].

Continuous dependence is meaningful even without uniqueness of solutions. If F is a bounded, upper semicontinuous map from $[0,T] \times \mathbf{R}^n$ into the nonempty compact convex subsets of $\mathbf{R}^n$, then the set of solutions of (1.1) is a compact set which depends in an upper semicontinuous fashion on the initial data; see [4]. Similar results apply to at least some initial value problems in infinite-dimensional spaces; for instance, see the remark after (7.5).

Next we shall survey the dependence of u on F. We shall not give complete details here, but shall try to indicate some of the most interesting ideas. The theory of dependence on F is unavoidably more complicated than that of the dependence on $u(0)$, for $u(0)$ is just a point in X, while F is a function of two arguments, possibly discontinuous and set-valued. Moreover, as we vary the choice of F, even the set $\mathrm{Dom}(F) \subseteq [0,T] \times X$ may vary.

For simplicity of notation, let $F = F(t,x,\theta)$, where θ takes values in some parameter space. For each fixed θ, assume $F(\cdot,\cdot,\theta)$ is a generator, i.e., its initial value problem is "solvable" in some sense. For motivation, at first we shall assume that $F(\cdot,\cdot,\theta)$ is single-valued and defined everywhere on $[0,T]\times X$, and that the solution to the initial value problem is unique. For part of the discussion below, we shall need to vary the initial time, so let us rewrite the initial value problem (1.1) as

$$\begin{cases} u'(t) \in F(t,u(t),\theta) & (a \le t \le b) \\ u(a) = z. \end{cases}$$

Let the unique solution be denoted $u(t) = u(t;a,z,\theta)$ to display its dependence on the various arguments. The continuous dependence problem now is to show that if F depends continuously on θ in some appropriate sense, then so does u.

It is not hard to choose a topology for u. In most of the literature, u is a continuous function of t, and so we consider $u(\cdot;a,z,\theta)$ as an element of the Banach space $C([a,b];X)$ with the usual supremum norm. We want $\theta \mapsto u(\cdot;a,z,\theta)$ to be continuous from the parameter space into $C([a,b];X)$ for each choice of a and z.

The choice of a topology for F is not so simple. Numerous different topologies have been used in the literature (and the choice will become more complicated when we permit F to be set-valued). The simplest choice is *pointwise continuity* — i.e., $F(t,x,\theta)$ is a continuous function of θ, separately for each t. This assumption can be made separately for each x, or uniformly on certain sets of x's — e.g., for bounded x, or for x lying in a compact set, etc. This kind of hypothesis is found in some textbooks [51]. It is sufficient to guarantee continuous dependence of u on θ, in the presence of various additional technical assumptions about $F(\cdot,\cdot,\theta)$. In fact, variants of this pointwise continuity assumption are sufficient even if F is discontinuous in t and x, set-valued and not everywhere defined. For instance, Crandall and Pazy [28] assume that $F(t,x,\theta)$ is dissipative in x for each t and θ, and that the resolvent $J_\lambda(t,\theta)x = (I-\lambda F(t,\cdot,\theta))^{-1}x$ depends continuously on θ for each fixed λ, t, and x. Under these assumptions, plus some mild technical hypotheses, it follows that u depends continuously on θ.

For another topology on F, consider the mapping $t \mapsto F(t,x,\theta)$ as an element in $L^1([a,b];X)$ for each fixed x and θ. We shall say F depends *L^1-continuously* on θ if the mapping $\theta \mapsto F(\cdot,x,\theta)$ is continuous from the parameter space $\{\theta\}$ into $L^1([a,b];X)$, for each fixed x (or uniformly on certain sets of x's). This hypothesis does not differ greatly from pointwise continuity: any L^1-convergent sequence of functions on $[a,b]$ has a pointwise-convergent subsequence, while any bounded, pointwise convergent sequence of functions on $[a,b]$ converges also in L^1.

One example of L^1-continuity has been studied extensively: An operator G is *m-dissipative* if it is dissipative and satisfies (6.1). Let G be an m-dissipative operator in a Banach space X. Then for each $z \in \mathrm{cl}(\mathrm{Dom}(G))$ and each $g \in L^1([0,T];X)$, the *quasiautonomous problem*

$$\text{(13.1)} \qquad \begin{cases} u'(t) \in G(u(t)) + g(t) & (0 \le t \le T), \\ u(0) = z \end{cases}$$

has a unique solution $u(t) = u(t;z,g)$. Using (12.1), it can be shown that

$$\|u(t;z_1,g_1) - u(t;z_2,g_2)\| \le \|z_1 - z_2\| + \int_0^t \|g_1(s) - g_2(s)\|\,ds,$$

and hence $u(t;z,\cdot)$ is continuous on $L^1([0,T];X)$. (We remark that this quasiautonomous problem in X can be reduced to an autonomous problem in $X \times L^1([0,+\infty);X)$; see [29].)

For a weaker hypothesis than pointwise- or L^1-continuity, we shall say F depends *integral-continuously* on θ if

$$\text{(13.2)} \qquad \Phi(t,x,\theta) = \int_0^t F(s,x,\theta)\,ds$$

is a continuous function of θ. For many classes of F's (discussed below), it can be shown that u depends continuously on θ for all choices of t, a, and z *if and only if* F depends integral-continuously on θ for all choices of t and x. (It is necessary to permit the initial time a to vary, since the behavior of $F(s,x,\theta)$

for $s < a$ is unrelated to the behavior of $u(t; a, z, \theta)$.) Of course, the "if" part of this statement is the practical part for most applications, since we generally can verify conditions on F more directly and more easily than conditions on u. The "only if" part is of interest because it tells us our theory is headed in the right direction: in some sense, the practical part can't be improved. Integral-continuity yields a topology on the F's weak enough so that many sets are compact; this compactness has some applications mentioned below. The fact that the topology can't be improved — i.e., weakened further — follows from the theorem in point-set topology that no Hausdorff topology on a set can be strictly weaker than a compact Hausdorff topology.

Obviously, L^1-continuity implies integral-continuity. Integral-continuity is in fact a weaker hypothesis, especially in cases where F oscillates rapidly as t varies. For instance, suppose $H(s,x)$ is periodic in s with period 1 (e.g., let $H(s,x) = \sin(s/2\pi)$). Let

$$F(t,x,\theta) = \begin{cases} H(t/\theta, x) & \text{when } \theta \neq 0, \\ G(x) \equiv \int_0^1 H(s,x)\,ds & \text{when } \theta = 0. \end{cases}$$

Then $F(t,x,\theta)$ generally is discontinuous at $\theta = 0$ in both the pointwise- and L^1-senses, but $\Phi(t,x,\theta)$ is continuous there. In cases where integral-continuity is sufficient for continuous dependence of u, this tells us that the solution of $u'(t) \in H(t/\theta, u(t))$, with $u(0)$ given, converges as $\theta \to 0$ to the solution of $u'(t) \in G(u(t))$.

An interesting special case is that in which $H(s,x) = B(x)$ for $0 < s < 1/2$ and $H(s,x) = A(x)$ for $1/2 < s < 1$, where A and B are two m-dissipative operators. Then the convergence result just described reduces to the *Trotter-Lie-Kato product formula*:

$$e^{t(A+B)}x = \lim_{n\to\infty} \left[\exp\left(\frac{t}{n}A\right)\exp\left(\frac{t}{n}B\right)\right]^n x. \tag{13.3}$$

This formula is valid for many, but not all, choices of A and B. See [65] [66] [70] [75] [96] for some recent discussions of this formula. A particularly simple example in which the formula fails is given by [112]: Let X be the complex Banach space of bounded sequences of complex numbers, with the supremum norm; let $A(\{x_k\}) = \{ikx_k + 1\}$ and $B(\{x_k\}) = \{ikx_k - 1\}$. Then (13.3) can be shown to fail at $x = 0$.

The assumption of integral-continuity apparently was first used by Gihman [44], to prove continuous dependence for a very simple class of ordinary differential equations in finite dimensions. The converse ("only if") part apparently was first noted by Artstein [3]. Gihman's continuous dependence principle has been extended to many other classes of initial value problems; the rather strong assumptions about F suggested above can be weakened considerably in numerous different directions. We note a few of those directions below, but we omit the many technical details, which vary from one paper to another. Although the several existence theories for (1.1) are far from unified, each has a version of Gihman's principle which is true for at least some F's. These different versions are proved separately by different methods, but their similarity suggests that there may be a single theory underlying them all. The present author hopes to research this idea further in the near future.

The paper [108] proves a version of Gihman's continuous dependence result in infinite dimensions, assuming $F(t,x,\theta)$ continuous and dissipative in x. The paper [110] extends Gihman's result to a case where $F(t,\cdot,\theta)$ is actually discontinuous, but is uniformly continuous and dissipative when restricted to suitable subsets of the Banach space X. These hypotheses are weak enough to apply to some partial differential equations with smooth coefficients.

A variant of Gihman's principle applies to the quasiautonomous problem (13.1), at least for some choices of G. Following the terminology of [112], we say an m-dissipative operator G has *Gihman's property* if the solution $u(t; z, g)$ of (13.1) depends continuously on the indefinite integral of g, as g varies over a weakly compact set of integrable functions. Many, but not all, m-dissipative operators have this property; see [112]. This continuous dependence property has consequences for existence theory, as follows: Let G be an m-dissipative operator (not necessarily satisfying any compactness condition), and let $H : X \to X$ be a continuous mapping with relatively compact range (not necessarily satisfying any dissipativeness condition). Let $z \in \mathrm{cl}(\mathrm{Dom}(G))$. Does the "dissipative plus compact" problem

$$\text{(13.4)} \qquad \begin{cases} v'(t) \in (G+H)(v(t)) & (0 \le t \le T), \\ v(0) = z \end{cases}$$

necessarily have any solutions? It does if we impose some additional mild hypotheses on G or H or X, but in general the answer is not yet known. The best results in this direction use the following observation: A solution of (13.4) is the same thing as a fixed point of the map $v \mapsto u(\cdot; z, H \circ v)$, where $u(t; z, g)$ is the solution of (13.1). Such a fixed point exists by Schauder's Fixed Point Theorem (or a variant thereof) if G has Gihman's property (or some variant thereof). Here we make use of the many compact sets made available by the topology of integral-convergence, mentioned earlier. The "dissipative plus compact" problem will be discussed further in §15.

For applications of Gihman's principle, the solution u of (1.1) need not be uniquely determined by F. Nonuniqueness can be dealt with in at least a couple of ways. In [30] [117], the functions F and Φ are set-valued. Continuity is defined for Φ and for the set of solutions $\{u\}$ by using the Hausdorff metric to measure the distance between two sets. Again, a variant of Gihman's integral convergence is shown sufficient for continuous dependence.

Another approach to nonuniqueness is taken by [11] [107]. In those papers, the Banach space X considered is the real line. The generator F considered — discontinuous in both papers — does not uniquely determine the solution u, but among the solutions is a *maximal* (i.e., largest) solution. One-sided continuous dependence results (i.e., involving lim sup's and inequalities, instead of limits and equations), analogous to Gihman's result, are proved for the maximal solution.

Variants of Gihman's hypothesis also suffices for continuous dependence for some stochastic differential equations; see [43] [122].

Continuous dependence results such as those described above can be taken as a basis for extending the notion of "differential equation." The following discussion is based on Kurzweil [71]. (We now drop the θ from our notation.) If we substitute (13.2), then the integral equation (2.4) can be restated as

$$\text{(13.5)} \qquad u(t) - u(0) = \int_0^t d\,\Phi(s, u(s)) \equiv \lim \sum_{k=1}^{N} \big[\Phi(t_k, u(\tau_k)) - \Phi(t_{k-1}, u(\tau_k))\big]$$

where $\tau_k \in (t_{k-1}, t_k)$ and the limit is taken over partitions (11.3) partially ordered by refinement. Many of the main equations, inequalities, and theorems about existence and continuous dependence — including a version of Gihman's principle — can then be restated in terms of Φ and Kurzweil's integral (13.5), without ever mentioning the original function F, and much of the theory then takes a simpler form. Now define u to be a *solution of the generalized differential equation*

$$u'(t) = D_t \Phi(t, u(t))$$

if u satisfies (13.5). Then the theory of existence and continuous dependence of such solutions can be developed entirely in terms of Φ. We have mentioned F only for motivation; we do not need to assume that Φ arises as in (13.2). Some assumptions must be made about Φ, of course, but the natural hypotheses on Φ are in many cases simpler and weaker than those implicit in (13.2). In particular, neither $\Phi(t, x)$ nor $u(t)$ need be a continuous function of t. Schwabik [114] applies these ideas to differential equations with impulses.

Kurzweil's integral is a variant of the Stieltjes integral. Schwabik [115] compares Kurzweil's integral with those of Perron and Young. For equations using the usual Stieltjes integral, Binding [13] [14] develops a theory including some results on existence (generalizing Caratheódory's Theorem 3.1), uniqueness, continuous dependence, and other results. (However, Binding does not give an analogue of Gihman's convergence principle.)

Finally, we note that integral-continuity plays an important role in the study of nonautonomous differential equations as dynamical systems (invariance principles, stability, etc.): Let Ω be a subset of a Banach space X, and let $\mathcal{F}$ be some family of mappings from $\mathbf{R} \times \Omega$ into X. Assume that each $F \in \mathcal{F}$ generates an evolution on Ω, in the sense of (4.3)-(4.4); denote that evolution by $U(t, s; F)$. Also assume that $F_h \in \mathcal{F}$ whenever $h > 0$ and $F \in \mathcal{F}$, where $F_h(t, x) \equiv F(t+h, x)$. Then it follows that $U(t, s; F_h) = U(t+h, s+h; F)$. Hence

$$\mathcal{S}(h)\begin{pmatrix} x \\ F \end{pmatrix} = \begin{pmatrix} U(h,0;F)x \\ F_h \end{pmatrix}$$

defines a semigroup (in the sense of (4.5)-(4.6)) on $\Omega \times \mathcal{F}$. For some choices of $\mathcal{F}$, the topology given by integral-convergence is compact. Now stability results, invariance results, etc. can be applied to the semigroup $\mathcal{S}$; they yield corresponding results for the evolution U. The semigroup $\mathcal{S}$ is called the *skew product semidynamical system.* It was first studied by Miller and Sell, and later by Wakeman, Artstein, and others; an introduction to this subject is given by [105].

14. Bijections. In the continuous dependences $\Gamma : F \mapsto u$ described in the preceding two sections, we know the domain $\mathcal{P}$ (i.e., the set of F's) and the codomain $\mathcal{S}$ (i.e., a space *containing* the set of u's), but generally we do not have a characterisation of the *range* $\Gamma(\mathcal{P})$. Under stronger h ypotheses than those of the previous section, we are sometimes able to give bijections $F \leftrightarrow u$, where both domain anad range are known. All such results are motivated by the following classical result:

14.1. THEOREM (Hille and Yosida, 1948). *Formula* (4.8) *gives a one-to-one correspondence between linear, strongly continuous semigroups* $S(t) = \exp(tG)$ *of type* ω *(i.e., satisfying* (4.7)*) on a Banach space* X*, and linear, densely defined operators* G *on* X *such that* $G - \omega I$ *is m-dissipative. Moreover,* G *can be recovered from* S *by the formula*

$$G(x) = \lim_{t \downarrow 0} \frac{S(t)x - x}{t}$$

with $\mathrm{Dom}(G)$ *consisting of those* x *for which the limit exists.*

The theory of linear semigroups is extensive; see [46] for a thorough introduction. Part of the Hille-Yosida Theorem extends to nonlinear operators in an arbitrary Banach space, in Theorem 10.1. However, the bijection between $S(t)$ and G does not extend to the setting of that theorem. Crandall and Liggett [27] give an example with $X = \mathbf{R}^2$, normed by $||(x_1, x_2)|| = \max\{|x_1|, |x_2|\}$, in which many different generators G yield the same semigroup S.

All the results stated in 14.1 do extend to nonlinear semigroups in a sufficiently nice Banach space — e.g., a Hilbert space, or more generally a uniformly smooth Banach space, or still more generally a reflexive Banach space with uniformly Gateaux differentiable norm. The nonlinear semigroup is defined on a closed convex nonexpansive retract C of the Banach space, and the limit defined in (4.8) exists for all x in a dense subset of C; see Reich [101] [102] for further details.

We turn now to the temporally inhomogenous theory. A number of bijections have been established between classes of evolution operators U and their generators F. In most of these results, $F(t, \cdot)$ and $U(t, s)$ take their values in the space $B(X)$ of bounded linear operators on the Banach space X. However, $U(t, s)x$ need not be a continuous function of t. In the temporally inhomogeneous case, discontinuity in t is quite natural, as we noted in §4 and 13.

Many of the bijection results use the following notation, or a close variant: Let W be a mapping, not necessarily multiplicative, from $\{(t, s) : -\infty < s \le t < \infty\}$ into $B(X)$. We define the "sum integral" and "product integral," respectively, as

$$\sum_a^b W = \lim\big[W(t_n, t_{n-1}) + \ldots + W(t_2, t_1) + W(t_1, t_0)\big]$$

$$\prod_a^b W = \lim\big[W(t_n, t_{n-1}) \circ \cdots \circ W(t_2, t_1) \circ W(t_1, t_0)\big]$$

when these limits exist; the limits are taken over of partitions of $[a, b]$:

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b, \qquad (14.2)$$

with those partitions partially ordered by refinement. (In some papers, the symbol $\int$ is used in place of Σ.) Note that if $V(t, s) = \sum_s^t W$ exists for all $[s, t] \subseteq [a, b]$, then it is additive — i.e., $V(t, s) + V(s, r) = V(t, r)$.

Similarly, if $U(t,s) = \prod_s^t W$ exists for all $[s,t] \subseteq [a,b]$, then it is multiplicative (i.e., an evolution, in the sense of (4.3), (4.4)). We say W has *bounded variation* on an interval $[a,b]$ if

$$\sup\{\|W(t_n,t_{n-1})\| + \ldots + \|W(t_2,t_1)\| + \|W(t_1,t_0)\|\} < \infty,$$

where the supremum is taken over all refinements (14.2) of some partition of $[a,b]$. (Clearly, if W is an additive function given by $W(t,s) = w(t) - w(s)$, then W has bounded variation in the sense above if and only if w has bounded variation in the usual sense of real analysis.) Helton [53] has proven that if W has bounded variation on $[a,b]$, then $\sum_a^b W$ exists if and only if $\prod_s^t (I+W)$ exists for all $[s,t] \subseteq [a,b]$.

To relate these results to differential equations, take (13.5) as a starting point, but with $\Phi(s,\cdot)$ linear. Then we write $\Phi(s,u(s)) = \Phi(s)u(s)$, and the condition we wish to satisfy is the Stieltjes integral equation

$$u(t) - u(r) = \int_r^t d\Phi \cdot u \tag{14.3}$$

for all r, t in the time-interval being considered. The solution is given by $u(t) = U(t,r)u(r)$, where $U(t,r)$ is a multiplicative function (i.e., an evolution). Thus (14.3) can be restated in terms of U:

$$U(t,r) = I + \int_r^t d\Phi \cdot U(\cdot, r).$$

Define an additive function $V(t,r) = \Phi(t) - \Phi(r)$; then the function Φ or V can be retrieved from the evolution U by

$$V(t,r) = \Phi(t) - \Phi(r) = \int_r^t U(s,b)\, dU(b,s).$$

Here s is the variable of integration, and b is an arbitrary constant; the value of the Stieltjes integral can be shown independent of the choice of b. This formula is equivalent to

$$V(t,r) = \sum_r^t [U - I]. \tag{14.4}$$

Conversely, we can obtain U from V by the formula

$$U(t,r) = \prod_r^t [I + V]. \tag{14.5}$$

In fact, formulas (14.4) and (14.5) give a bijection between additive V's with bounded variation and multiplicative U's with $U - I$ having bounded variation. For an introduction to this subject and proof of this bijection see MacNerney [79].

A number of subsequent papers have extended MacNerney's bijection; we mention a few of the simplest and most interesting extensions. Operators are continuous and linear except where otherwise noted.

Herod [54] permitted the operators to be nonlinear. However, the other conditions involved in Herod's bijection are complicated.

Freedman [40] added an assumption of continuity, but weakened MacNerney's hypothesis of bounded variation to a hypothesis of bounded p-variation. The *p-variation* of a function W over an interval $[a,b]$ (in the sense of Wiener [128]) is the supremum of the quantities

$$\left[\|W(t_n,t_{n-1})\|^p + \ldots + \|W(t_2,t_1)\|^p + \|W(t_1,t_0)\|^p\right]^{1/p}$$

over all partitions (14.2) of the interval. For further references concerning the p-variation, see [40].

Herod and McKelvey [55] permitted their operators $V(t,s)x$ to be discontinuous in x, in a somewhat weak sense. They did this by using a scale of Banach spaces $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_N$. Each generator $V(b,a)$ is assumed to be continuous from X_p to X_{p+1}, but possibly discontinuous from X_p to X_p. Their somewhat complicated bijection has the virtue of including the classical Hille-Yosida Theorem 14.1 as a special case. Freedman [39] extended this result further, so as to include evolutions $U(t,s)x$ which may be discontinuous in t.

The results of Hinton [58] take us a bit further from the simple initial value problem (1.1), but weaken MacNerney's hypotheses slightly and also strengthen the symmetry: the limit procedures in (14.4) and (14.5) are special cases of a single mapping. Let $\mathcal{F}$ be the class of all functions $F : [a,b] \times [a,b] \to B(X)$ with these properties: (i) $F(t,t) = I$ for all t; (ii) for each fixed r, $F(\cdot,r)$ has left- and right-hand limits at every point of (a,b), and one-sided limits at a and at b; and (iii) for each fixed t, $F(t,\cdot)$ has bounded variation. (Note that F is not assumed multiplicative or additive.) Hinton shows that for each $F \in \mathcal{F}$ there is a unique $M \in \mathcal{F}$ satisfying the Stieltjes-Volterra integral equation

$$M(t,r) = I + (L)\int_r^t dF(t,s) \cdot M(s,r)$$

for all $[r,t] \subseteq [a,b]$. (Here $(L)\int$ denotes a left Cauchy integral, with s being the variable of integration; see [58].) Thus $M = \mathcal{C}(F)$ defines a mapping from $\mathcal{F}$ into $\mathcal{F}$. Hinton shows that this mapping $\mathcal{C}$ is in fact a bijection from $\mathcal{F}$ onto $\mathcal{F}$, and moreover $\mathcal{C}\mathcal{C}(F) = F$ for each $F \in \mathcal{F}$. When U and V are an evolution and generator satisfying the hypotheses of MacNerney [79] and related by (14.4) and (14.5), then Hinton shows that $U, V \in \mathcal{F}$ and that

$$U = \mathcal{C}(I - V) \qquad \text{and} \qquad V = I - \mathcal{C}(U).$$

15. Unification. The ultimate problem in existence theory is to find necessary and sufficient conditions on $u(0)$ and F for the existence of a solution to (1.1). This problem is too hard for the near future, but it does suggest various subproblems which are worth pursuing. Foremost of these is the unification of the known sufficient conditions for existence of solutions.

As we have indicated in §12, the conditions of generalized compactness, isotonicity, and dissipativeness play analogous roles; each is a hypothesis used in proving convergence of approximate solutions. These three conditions are quite different, and have led to three largely separate theories. Still, those three theories have some analogous structures, as we have tried to indicate by the use of the same letter ω in inequalities (7.1), (8.3), (9.7). Does a single, weaker condition underlie the three hypotheses for convergence — or at least two of them?

Little has been done to find hypotheses weaker than that of isotonicity (§8). One approach worth noting is that of Calvert [18], who defines K-dissipativeness, analogous to dissipativeness, in terms of a cone K. Existence of solutions is proved under the assumption that the cone is normal, i.e., that the ordering satisfies $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$. However, Picard [92] shows that if G is K-dissipative and K is normal, then $\| \;\; \|$ can be replaced by an equivalent norm which makes G dissipative. Of less interest for the goals of this survey, but still worth mentioning, are some results with *stronger* hypotheses, involving both isotonicity and dissipativeness [2] [10], or involving both isotonicity and compactness [72].

More has been done in unifying compactness and dissipativeness. Martin [82] and Li [76] observed that (7.3) is satisfied if G is ω-dissipative *or* if G satisfies a generalized compactness condition of type (7.1). A drawback to the approach of Martin and Li is that they require their operators to be uniformly continuous, thus excluding the many applications of the dissipative theory to partial differential equations. It would be interesting to weaken or remove that uniform continuity assumption.

The sum of two uniformly continuous operators satisfying (7.3) is another such operator (add the ω's — see [82]), and so the existence results with hypothesis (7.3) include, as a special case, some results on the existence of solutions to differential inclusions of the form (13.4), where G is m-dissipative and H satisfies some sort of generalized compactness condition such as (7.1). However, stronger results for (13.4) have been proved by other methods not involving (7.3).

Without further assumptions, does (13.4) necessarily have a solution? In general, the answer is not yet known. Various partial answers can be divided into two main classes:

(a) Assume G is continuous, and defined on all of X. Without further assumptions, it is not known whether (13.4) necessarily has a solution. But (13.4) is known to have a solution if G or H is uniformly continuous, or if $G + H$ is uniformly continuous, or if the Banach space X is uniformly smooth [113]. See the survey [124] for further references.

(b) Assume H has relatively compact range. Again, existence is known if any one of several additional assumptions holds, but not otherwise. For details see the surveys [47] [112], and other papers cited therein; for more recent results not mentioned in those surveys see also [56] [125]. A chief method here is that of fixed point theory, as discussed after (13.4).

Even if we can unify the three known hypotheses for convergence, we are still a long way from a full understanding of the autonomous problem (1.2). To see this, note that any nonautonomous problem (1.1) in a Banach space X can be transformed to an autonomous problem $v'(t) \in G(v(t))$ in $\mathbf{R} \times X$, via the transformation

$$v(t) = \begin{pmatrix} t \\ u(t) \end{pmatrix}, \qquad G\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ F(t,x) \end{pmatrix}. \tag{15.1}$$

However, when we apply this transformation to existence theorems for (1.1), it yields existence theorems for (1.2) which are not at all understood except via this transformation. For instance, some of the theorems discussed in §9-10 assume, roughly, that $F(t,x)$ is dissipative in x, and integrable in t in some sense. For $t_1 \neq t_2$, the functions $F(t_1,\cdot)$ and $F(t_2,\cdot)$ may be almost entirely unrelated. Hence, although the operator G given by (15.1) must satisfy some conditions sufficient for existence of solutions, those conditions have very little resemblance to dissipativeness, compactness, or isotonicity.

16. References. For brevity, we omit some older papers which, though important, are cited by more recent papers listed here.

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