ANSWERS

Math 250 Test 3, Wednesday 12 November 2008, 3 pages, 50 points, 50 minutes

Our first two problems will be concerned with these three formulas:

Not-elimination	19.2.a	$\vdash \overline{\overline{A}} \to A.$
Third contrapositive law	19.2.b	$\vdash (\overline{A} \to B) \to (\overline{B} \to A).$
Fourth contrapositive law	19.2.c	$\vdash (\overline{A} \to \overline{B}) \to (B \to A).$

Adding any one of those formulas, as an additional axiom, to basic logic makes the other two provable. A proof of 19.2.a $\Rightarrow$ 19.2.b is given in the textbook, so we'll skip that; we'll show the other two proofs.

(12 points) Show that adding 19.2.b as an axiom makes 19.2.c provable, by filling in the details in the following proof:

(1)	$B \to \overline{\overline{B}}$	not-introduction (14.9)
(2)	$(\overline{\overline{B}} \to A) \to (B \to A)$	(1), suffixing-detached (13.5.a or 13.4.δ)
(3)	$(\overline{A} \to \overline{B}) \to (\overline{\overline{B}} \to A)$	3rd contrapositive (19.2.b)
(4)	$(\overline{A} \to \overline{B}) \to (B \to A)$	(3), (2), transitivity (13.5.b)

(9 points) Show that 19.2.c as an axiom makes 19.2.a provable. Give a full 3-column proof of that. *Hint*: Substitute $B = \overline{\overline{A}}$; then what?

Solution.	(1)	$(\overline{A} \to \overline{\overline{\overline{A}}}) \to (\overline{\overline{A}} \to A)$	19.2.c
	(2)	$\overline{A} \to \overline{\overline{\overline{A}}}$	not-introduction (14.9)
	(3)	$\overline{\overline{A}} \to A$	(2), (1), detachment (13.2.a)

or	(1)	$\overline{A} \to \overline{\overline{\overline{A}}}$	not-introduction (14.9)
	(2)	$\overline{\overline{A}} \to A$	(1), 4th contrapos. det. (19.2.c.δ)

Our remaining problems involve the following three formulas:

Coassertion	20.3.a	$\vdash [(A \to B) \to B] \to A$
Meredith's permutation	20.3.b	$\vdash [(P \to Q) \to R] \to [(R \to Q) \to P]$
Suffix cancellation	20.3.c	$\vdash [(Y \to Z) \to (X \to Z)] \to (X \to Y)$

Adding any one of those formulas, as an additional axiom, to basic logic makes the other two provable. A proof of 20.3.a $\Rightarrow$ 20.3.b is given in the textbook, so we'll skip that; we'll show the other proofs.

(11 points) Show that assuming suffix cancellation (20.3.c) makes coassertion (20.3.a) provable, by writing a 3-column proof. *Hints*: Substitute $Y = A$, $Z = B$, $X = (A \to B) \to B$ into suffix cancellation. The right half of the resulting formula is coassertion. The left half is a theorem of basic logic, proved (how?) using identity and permutation. My proof of this one only took four steps — one of them was an application of 20.3.c, and the other three steps were from chapter 13 — but the steps are not obvious, and one of the formulas in the proof is fairly long.

Solution. Here is the four-step proof that I had in mind:

(1)	$[(A \to B) \to (\{(A \to B) \to B)\} \to B)]$ $\to (\{(A \to B) \to B)\} \to A)$	suffix cancellation (20.3.c)
(2)	$\{(A \to B) \to B)\} \to [(A \to B) \to B]$	identity (13.2.b)
(3)	$(A \to B) \to (\{(A \to B) \to B)\} \to B)$	(2), permut-det (13.2.c.δ)
(4)	$\{(A \to B) \to B)\} \to A$	(3), (1), detachment (13.2.a)

But I subsequently found a couple of simplifications. The formula in step (3) is actually a specialization of assertion, so that one justification replaces both steps (2) and (3) above. And if we replace 20.3.c with its detachmental corollary, we save another step. Thus we obtain this much shorter proof:

(1)	$(A \to B) \to (\{(A \to B) \to B)\} \to B)$	assertion (13.8.a)
(2)	$\{(A \to B) \to B)\} \to A$	(1), suff.cancel.det. (20.3.c.δ)

(9 points) Show that assuming coassertion (20.3.a) makes suffix cancellation (20.3.c) provable, by writing a 3-column proof. Hints: Apply $\to$-prefixing to coassertion (in what form?) to prove $\{X \to [(Y \to Z) \to Z]\} \to (X \to Y)$; then what?

(1)	$[(Y \to Z) \to Z] \to Y$	coassertion (20.3.a)
(2)	$\{X \to [(Y \to Z) \to Z]\} \to (X \to Y)$	(1), $\to$-prefix.detach. (13.2.d.δ)
(3)	$\{(Y \to Z) \to (X \to Z)\} \to$ $\{X \to [(Y \to Z) \to Z]\}$	permutation (13.2.c)
(4)	$[(Y \to Z) \to (X \to Z)] \to (X \to Y)$	(3), (2), transitivity (13.5.b)

(9 points) Show that assuming Meredith's permutation (20.3.b) makes coassertion (20.3.a) provable, by writing a 3-column proof. Hints: Substitute $P = A$, $Q = B$, $R = A \to B$.

(1)	$[(A \to B) \to (A \to B)] \to$ $[((A \to B) \to B) \to A]$	Meredith's permutation (20.3.b)
(2)	$(A \to B) \to (A \to B)$	identity (13.2.b)
(3)	$((A \to B) \to B) \to A$	(2), (1), detachment (13.2.a)

or, more briefly,

(1)	$(A \to B) \to (A \to B)$	identity (13.2.b)
(2)	$((A \to B) \to B) \to A$	(1), Mer.perm.det. (20.3.b.δ)