ANSWERS

Math 250 Final Exam, Thurs 11 Dec 2008, 4 pages, 50 points, 120 minutes
Open book, open notes

(11 points) On page 447, one proof that we didn't do in class or for homework was that for (h): $(Q \to S) \to \{(R \to S) \to [(Q \vee R) \to S]\}$. You should do that proof now. That is, using the same style as the other proofs on that page, prove

$$\left[\!\!\left[\ (Q \to S) \to \{(R \to S) \to [(Q \vee R) \to S]\}\ \right]\!\!\right] = 1,$$

where $[\![\ \ ]\!]$ is the Meyer valuation.

Solution:

(a)	$[\![\ (Q \to S) \to \{(R \to S) \to [(Q \vee R) \to S]\}\ ]\!] = 0$	assumption
(b)	$\mathcal{H} \vdash (Q \to S) \to \{(R \to S) \to [(Q \vee R) \to S]\}$	axiom
(1)	$[\![Q \to S]\!] = 1$	(a), (b), 27.5
(2)	$[\![\ (R \to S) \to [(Q \vee R) \to S]\ ]\!] = 0$	(a), (b), 27.5
(3)	$\mathcal{H} \vdash (R \to S) \to [(Q \vee R) \to S]$	(a), (b), 27.5
(4)	$[\![R \to S]\!] = 1$	(2), (3), 27.5
(5)	$[\![\ (Q \vee R) \to S\ ]\!] = 0$	(2), (3), 27.5
(6)	$\mathcal{H} \vdash (Q \vee R) \to S$	(2), (3), 27.5
(7)	$[\![Q \vee R]\!] = 1$	(5), (6), 27.5
(8)	$[\![S]\!] = 0$	(5), (6), 27.5
(9)	$[\![Q]\!] = 0$	(1), (8), 27.4
(10)	$[\![R]\!] = 0$	(4), (8), 27.4
(11)	$\max\{[\![Q]\!], [\![R]\!]\} = 1$	(7), 27.2
(12)	$\max\{[\![Q]\!], [\![R]\!]\} = 0$	(9), (10)
(13)	contradiction	(11), (12)

Steps (a) and (b) are optional. Variants are possible, particularly in the order of the steps and in the combining of the last few steps. Except for a few students who were clueless, and a few students adding some justifiable but superfluous steps (no penalty for those) most students had nearly identical solutions in steps (1)–(8). Getting that far was worth 8 points. After step (8), the remaining steps must include references, not necessarily in this order, to 27.4, 27.2, and the steps that are in my proof labeled (1), (4), (7), and (8).

(3 points) Show that $\mathcal{C} = \Big\{\varnothing, \{2\}, \{2,5\}, \{5,7\}, \Omega\Big\}$ is not a topology on the set $\Omega = \{2,5,7\}$.

Answer: $\boxed{\{2,5\} \cap \{5,7\} = \{5\} \notin \mathcal{C}.}$ That's the only correct answer, though I gave full credit if you expressed it less concisely. I was not able to make any sense out of the few answers that were unrelated to this one, and so I gave them no credit.

For the remaining problems on this page, use $\Sigma = \Big\{\varnothing, \{2\}, \{5\}, \{2,5\}, \{5,7\}, \Omega\Big\}$, which *is* a topology on the set $\Omega = \{2,5,7\}$. (You don't have to prove that Σ is a topology; you can take my word for it.)

(3 points) List all the subsets of Ω that are not open. For each of those sets, give its interior.

S	$\{7\}$	$\{2,7\}$
$\text{int}(S)$	$\varnothing$	$\{2\}$

(4 points) Using the topological interpretation, give the negation of each semantic value.

$[\![A]\!]$	$\varnothing$	$\{2\}$	$\{5\}$	$\{2,5\}$	$\{5,7\}$	Ω
$[\![\neg A]\!]$	Ω	$\{5,7\}$	$\{2\}$	$\varnothing$	$\{2\}$	$\varnothing$

Common errors: Some students gave a "negation" for all 8 subsets of Ω, not just for the six that are semantic values. Some students conflused "negation" with "complement." And some students wrote $\{\varnothing\}$ when they should have written $\varnothing$; those are *not* the same.

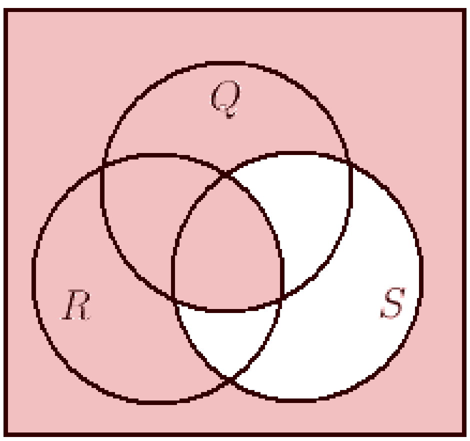
(4 points) Find a suitable value of $[\![A]\!] = \boxed{}$, and the resulting value of $[\![A \vee \neg A]\!] = \boxed{}$, to show that $A \vee \neg A$ is not tautological in this topological interpretation.

Solution. Since $\vee$ is given by $\cup$, we're looking for a column in the previous question's answer where the union of $[\![A]\!]$ and $[\![\neg A]\!]$ is not Ω. There are two such columns, so there are two correct answers to this problem:

$$\boxed{[\![A]\!] = \{5\},\ \ [\![A \vee \neg A]\!] = \{2, 5\}} \quad \text{or} \quad \boxed{[\![A]\!] = [\![A \vee \neg A]\!] = \{2, 5\}}$$

Most students got this one right. In addition to the right answer, I also gave full credit for this one if a student gave a value of $[\![A]\!]$ such that its union with the incorrect value that the student gave for $[\![\neg A]\!]$ was equal to something other than Ω, and was given as the value for $[\![A \vee \neg A]\!]$.

(3 points) Draw a Venn diagram of $(Q \cap R) \cup \complement S$.



A common error was to omit the surrounding box, and to omit shading the part of that box lying outside the three circles. That cost 2 points.

(4 points) The expression below is not a formula, because it doesn't have enough parentheses. Add parentheses in two different ways, to get two different correct formulas. Then draw the tree diagram for each of those formulas.

$$\pi_1 \quad \vee \quad \pi_2 \quad \to \quad \pi_2 \qquad\qquad\qquad \pi_1 \quad \vee \quad \pi_2 \quad \to \quad \pi_2$$

Solution:

$$\begin{array}{ccccc} (\pi_1 & \vee & \pi_2) & \longrightarrow & \pi_2 \\ & & | & & | \\ (\pi_1 & \vee & \pi_2) & & \pi_2 \\ | & & | & & | \\ \pi_1 & & \pi_2 & & \pi_2 \end{array} \qquad\qquad \begin{array}{ccccc} \pi_1 & \vee & (\pi_2 & \longrightarrow & \pi_2) \\ | & & | & & \\ \pi_1 & & (\pi_2 & \longrightarrow & \pi_2) \\ | & & | & & | \\ \pi_1 & & \pi_2 & & \pi_2 \end{array}$$

(4 points) (This is half of 8.30.a(iii).) Find suitable values of

$$[\![A]\!] = \boxed{}, \qquad [\![B]\!] = \boxed{}, \qquad [\![\, A \to (A \wedge (B \vee \neg B)) \,]\!] = \boxed{}$$

to show that $A \to (A \wedge (B \vee \neg B))$ is not tautological in the comparative interpretation.

Solution: In the comparative interpretation, we have $[\![\neg B]\!] = -[\![B]\!]$, and $\vee$ and $\wedge$ are given by max and min, respectively. Thus $[\![B \vee \neg B]\!] = \max\{[\![B]\!], -[\![B]\!]\} = \big|[\![B]\!]\big|$, and therefore

$$[\![A \to (A \wedge (B \vee \neg B))]\!] = \min\left\{[\![A]\!], \big|[\![B]\!]\big|\right\} - [\![A]\!].$$

So we are looking for values of $[\![A]\!]$ and $[\![B]\!]$ to make

$$\min\left\{[\![A]\!], \big|[\![B]\!]\big|\right\} - [\![A]\!] < 0$$

$$\min\left\{0, \big|[\![B]\!]\big| - [\![A]\!]\right\} < 0$$

$$\big|[\![B]\!]\big| - [\![A]\!] < 0$$

$$\big|[\![B]\!]\big| < [\![A]\!]$$

and so the answer has $\boxed{\text{any integers } [\![A]\!] \text{ and } [\![B]\!] \text{ such that } \big|[\![B]\!]\big| < [\![A]\!]}$. And then we have $\boxed{[\![A \to (A \wedge (B \vee \neg B))]\!] = \big|[\![B]\!]\big| - [\![A]\!]}$.

(7 points) Using just 13.1-13.18 and 14.1-14.8.a, give a 3-column proof of

(14.8.b) $$\{A \vee B,\ A \to X,\ B \to Y\} \vdash\ X \vee Y$$

Solution: Here is the proof that I had in mind:

(1)	$A \vee B$	hypoth.
(2)	$A \to X$	hypoth.
(3)	$B \to Y$	hypoth.
(4)	$X \to (X \vee Y)$	left $\vee$-intro (14.1.d)
(5)	$Y \to (X \vee Y)$	right $\vee$-intro (14.1.e)
(6)	$A \to (X \vee Y)$	(2), (4), trans (13.5.b)
(7)	$B \to (X \vee Y)$	(3), (5), trans (13.5.b)
(8)	$(A \vee B) \to (X \vee Y)$	(6), (7), pf by cases (14.4.b)
(9)	$X \vee Y$	(1), (8), detachment (13.2.a)

Many students replicated that proof, or that proof with minor variants. A few students found a shorter proof:

(1)	$A \vee B$	hypoth.
(2)	$A \to X$	hypoth.
(3)	$B \to Y$	hypoth.
(4)	$(A \vee B) \to (X \vee B)$	(2), wk $\vee$-suff (14.7.d)
(5)	$(X \vee B) \to (X \vee Y)$	(3), wk $\vee$-pref (14.7.b)
(6)	$(A \vee B) \to (X \vee Y)$	(4), (5), trans (13.5.b)
(7)	$X \vee Y$	(1), (6), detachment (13.2.a)

(However, several students attempted to give a proof along these lines, and fouled it up.)

(7 points) Using just 13.1-13.18 and 14.1-14.8.g(ii), give a 3-column proof of

(14.8.g(iii)) $$\vdash\ [(A \vee B) \to (A \wedge B)] \to (A \to B)$$

Solution: Many variants of this are possible.

(1)	$A \to (A \vee B)$	left $\vee$-intro (14.1.d)
(2)	$(A \wedge B) \to B$	right $\wedge$-elim (14.1.c)
(3)	$[A \to (A \wedge B)] \to (A \to B)$	(2), prefixing detached (13.2.δ)
(4)	$[(A \vee B) \to (A \wedge B)]$ $\to [A \to (A \wedge B)]$	(1), suffixing detached (13.5.a or 13.4.δ)
(5)	$[(A \vee B) \to (A \wedge B)] \to (A \to B)$	(4), (3), trans (13.5.b)