Recipes for Solving Constant-Coefficient Linear Nonhomogeneous Ordinary Differential Equations Whose Right Hand Sides are Sums of Products of Polynomials times Exponentials

Let $D = \frac{d}{dx}$. We begin by noting that $(D-r)\left[f(x)e^{sx}\right] = \left[f'(x) + (s-r)f(x)\right]e^{sx}$. That can be rewritten:

(1)
$$(D-r)\left[f(x)e^{sx}\right] = \begin{cases} f'(x)e^{sx} & \text{if} \quad s = r \\ g(x)e^{sx} & \text{if} \quad s \neq r \end{cases}$$

where g is some polynomial of the same degree as f.

This result can be reversed: Given some polynomial g,

(2)
$$\text{If } (D-r)[?] = g(x)e^{sx}, \text{ then } ? = \begin{cases} e^{sx}\int g(x)dx & \text{if} \quad s = r \\ f(x)e^{sx} & \text{if} \quad s \neq r \end{cases}$$

where f is some polynomial of the same degree as g.

Note also that

$$(D-r)^2\left[f(x)e^{rx}\right] = f''(x)e^{rx}$$
$$(D-r)^3\left[f(x)e^{rx}\right] = f'''(x)e^{rx}$$

and in general

$$(D-r)^n\left[f(x)e^{rx}\right] = f^{(n)}(x)e^{rx}.$$

We will look at several cases, with increasing generality.

Case 1: Only one root. We consider equations of the form

$$(D-r)^n y = e^{rx}f(x),$$

where the number r (what I like to call the "root") is the same on both sides of the equation. Here, n is a positive integer, and f is a given polynomial. By applying formula (2) n times, we get an answer of

$$y = e^{rx}\underbrace{\int\int\cdots\int\int}_{n \text{ integrals}} f(x)\,\underbrace{dxdx\cdots dxdx}_{n \text{ differentials}}.$$

For instance,

problem		solution
y	$= (6x+2)e^{5x}$	$y = (6x+2)e^{5x}$
$y' - 5y$	$= (6x+2)e^{5x}$	$y = (3x^2 + 2x + a_1)e^{5x}$
$y'' - 10y' + 25y$	$= (6x+2)e^{5x}$	$y = (x^3 + x^2 + a_1x + a_2)e^{5x}$
$y''' - 15y'' + 75y' - 125y$	$= (6x+2)e^{5x}$	$y = (\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}a_1x^2 + a_2x + a_3)e^{5x}$

The numbers a_1, a_2, a_3 are arbitrary.

Let's take another look at that last example. The problem was

$$\underbrace{(D-5)^3}_{\substack{\text{multiplicity}\\ \text{of root is 3}}} \quad y \quad = \quad \underbrace{(6x+2)}_{\substack{\text{polynomial}\\ \text{with 2}\\ \text{coefficients}}} \quad e^{5x}\,.$$

(I could describe that last polynomial as a polynomial of degree 1, but I find it more helpful to describe it as a polynomial with 2 coefficients, for reasons that will be evident below.) The 5's in $D-5$ and e^{5x} match, which is why we're using the method of Case 1. The answer is

$$y \quad = \quad \Bigg(\underbrace{a_3 + a_2 x + \frac{1}{2}a_1 x^2}_{\substack{\text{3 arbitrary}\\ \text{coefficients}}} + \underbrace{\frac{1}{3}x^3 + \frac{1}{4}x^4}_{\substack{\text{2 particular}\\ \text{coefficients}}} \Bigg) e^{5x}$$

In all problems of this type, we find that

- the arbitrary coefficients are attached to the lower powers of x (such as 1, x, x^2); the particular coefficients are attached to the higher powers of x.
- the multiplicity of the root in the left side of the problem is equal to the number of arbitrary coefficients in the answer.
- the number of coefficients in the polynomial in the right side of the problem is equal to the number of particular coefficients in the answer.

Or, if you prefer, you could say that "one plus the degree of the polynomial in the right side of the problem is equal to the number of particular coefficients in the answer." But I think that's more complicated and harder to remember. — *Caution.* If you're going to follow my system, and count coefficients instead of degrees, don't skip any zero terms. The highest power term doesn't change, but fill in zeros for any missing terms with lower powers of x. For instance, we would count $6x$ as a polynomial with two coefficients, because we would rewrite it as $6x+0$. It's a polynomial of degree 1, in either case.

I'm going to rewrite that last example's answer as

$$y \quad = \quad \Bigg[\quad \underbrace{A(x)}_{\substack{\text{arbitrary}\\ \text{polynomial}\\ \text{of degree 2}}} \quad + \quad \underbrace{\frac{1}{3}x^3 + \frac{1}{4}x^4}_{\substack{\text{2 particular}\\ \text{coefficients}}} \Bigg] e^{5x}.$$

That will make the next discussion easier to understand.

Case 2: Multiple roots on left side. (If you find this too abstract, skip ahead to the "example for case 2.") We consider equations of the form

$$(D-r)^n (D-r_2)^{n_2} \cdots (D-r_k)^{n_k} y = e^{rx} f(x).$$

Here are the ingredients:

- The numbers $r, r_2, \ldots, r_k$ are any constants, but they have to be *different* from one another. (If they are not real numbers, the method I'm about to describe is still correct, but we may want to rewrite our results when we're done.) Actually, you can take $r_1 = r$, if that helps to see the pattern better; but since r_1 will play a special role and will be written far more often than any of the other r_j's, I'm going to define $r_1 = r$ so I don't have to write so many subscripts.
- The number $n = n_1$ is any nonnegative integer, and the numbers $n_2, n_3, \ldots, n_k$ are any positive integers. (Actually, the procedure described below would also work if any of $n_2, n_3, \ldots, n_k$ were zero, but we wouldn't gain anything by doing that — we could simply omit those factors. On the other hand, the case of $n_1 = 0$ is of some importance.)
- The function $f(x)$ is a polynomial of some degree greater than or equal to 0 (where a constant is considered to be a polynomial of degree 0). For our example, we'll assme that $f(x)$ is a polynomial with m coefficients (that is, a polynomial of degree $m - 1$).

Now here's the recipe (don't ask where it came from; we might discuss that later). For the problem we've just described, the answer must be of the form

$$y = \Big[\underbrace{A_1(x)}_{\text{arb.}} + \underbrace{p_1 x^n + p_2 x^{n+1} + \cdots + p_m x^{n+m-1}}_{\substack{m \text{ particular coefficients (same} \\ \text{as number of coefficients in } f)}} \Big] e^{rx}$$

$$+ A_2(x) e^{r_2 x} + A_3(x) e^{r_3 x} + \cdots + A_k(x) e^{r_k x}$$

where

- $A_1(x), A_2(x), \ldots A_k(x)$ are arbitrary polynomials that have $n, n_2, \ldots, n_k$ coefficients respectively (i.e., arbitrary polynomials of degree $n-1$, $n_2 - 1$, …, $n_k - 1$ respectively), and
- $p_1, p_2, \ldots, p_m$ are particular (nonarbitrary) constants that we need to find.

The most obvious way to find the p's is to simply substitute our formula for y into the given problem, and carry out all the differentiations, and see what restrictions that imposes on the p's. (It will not impose any restriction on the "arbitrary" terms — they'll all cancel out.) But that is more work than necessary — sometimes a lot more work than necessary. Here is a more efficient procedure.

First, the A's will not affect our computation of the p's, so temporarily replace all the A's with zeros. Thus, write

$$y = \Big[\underbrace{p_1 x^n + p_2 x^{n+1} + \cdots + p_m x^{n+m-1}}_{\substack{m \text{ particular} \\ \text{coefficients}}} \Big] e^{rx}$$

Be sure you've got the right exponents.

Next, apply $(D-r)^n$ to both sides of the equation. The right side will simplify, by using formula (1) repeatedly (see the first paragraph of this document).

$$\begin{aligned} (D-r)^n y &= (D-r)^n \Big\{ (p_1 x^n + p_2 x^{n+1} + \cdots + p_m x^{n+m-1}) e^{rx} \Big\} \\ &= \Big\{ D^n [p_1 x^n + p_2 x^{n+1} + \cdots + p_m x^{n+m-1}] \Big\} e^{rx} \\ &= \Big\{ \frac{n!}{0!} p_1 + \frac{(n+1)!}{1!} p_2 x + \frac{(n+2)!}{2!} p_3 x^2 + \cdots + \frac{(n+m-1)!}{(m-1)!} p_m x^{m-1} \Big\} e^{rx} \end{aligned}$$

That may look complicated in the abstract case, but in particular examples the quotients of factorials are just constants — in fact, integers.

Finally, what we still need to solve is

$$(D-r_2)^{n_2}\cdots(D-r_k)^{n_k}\left\langle\left\{\frac{n!}{0!}p_1+\frac{(n+1)!}{1!}p_2x+\cdots+\frac{(n+m-1)!}{(m-1)!}p_mx^{m-1}\right\}e^{rx}\right\rangle = e^{rx}f(x).$$

Now, at this point, I recommend that you simply carry out the differentiations, by straightforward and perhaps tedious computations, and see what restrictions that imposes on the p's. (In the remaining steps, there are some shortcuts that still might be used by advanced students, but we're not ready for the explanation of those yet.)

Example for case 2. $y'''' - 3y'' + 2y' = (2x+1)e^x$.

Solution. First rewrite the equation as

$$(D-1)^2D(D+2)y = (2x+1)e^x.$$

That is,

$$(D-1)^2(D-0)^1(D--2)^1y = (2x+1)e^x,$$

which we analyze as follows:

$$\underbrace{\underbrace{(D-1)^2}_{\text{mult.2}}\ \underbrace{(D-0)^1}_{\text{mult.1}}\ \underbrace{(D--2)^1}_{\text{mult.1}}}_{\substack{\text{multiplicities on}\\ \text{left side determine}\\ \text{arbitrary coefficients}\\ \text{in answer}}}\ y \quad = \quad \underbrace{(2x+1)}_{\substack{f \text{ has 2}\\ \text{coefficients}}}\ e^x.$$

Here $r=1$ and $n=2$. The general recipe tells us to expect a solution of the form

$$y \quad = \quad \Big[\quad \underbrace{A_1(x)}_{\substack{\text{2 arb.}\\ \text{coeffs.}}} \ +\ \underbrace{p_1x^2+p_2x^3}_{\substack{\text{2 particular}\\ \text{coefficients}\\ \text{(same as } f)}}\ \Big]\, e^x \ +\ \underbrace{A_2(x)}_{\text{1 arb.}}\, e^{0x} \ +\ \underbrace{A_3(x)}_{\text{1 arb.}}\, e^{-2x}.$$

That is, more simply,

$$y \quad = \quad (a_1+a_2x+p_1x^2+p_2x^3)e^x + a_3 + a_4e^{-2x}$$

where

- a_1, a_2, a_3, a_4 are arbitrary (we don't need to find them), and
- p_1, p_2 are particular (we do need to find them).

Finding the p's — straightforward method. Now, to find the p's, we begin by temporarily replacing the a's with 0's, since the a's will have no effect on our computing the p's. Thus we temporarily simplify to

$$y \quad = \quad (p_1x^2 + p_2x^3)e^x.$$

Now, the problem is $D(D+2)(D-1)^2y = (2x+1)e^x$, which can be rewritten as

$$(D^2 + 2D)\left\langle (D-1)^2\left[(p_1x^2 + p_2x^3)e^x\right]\right\rangle = (2x+1)e^x.$$

I've rearranged the order of the terms, because I want to concentrate on the part inside the $\langle\ \rangle$. That part can be simplified by twice using formula (1) from the first paragraph of this document. The polynomial $p_1x^2 + p_2x^3$ has derivative $2p_1x + 3p_2x^2$, and that in turn has derivative $2p_1 + 6p_2x$. Hence

$$\left\langle (D-1)^2\left[(p_1x^2 + p_2x^3)e^x\right]\right\rangle = \langle [2p_1 + 6p_2x]\, e^x\rangle\,.$$

Substitute this into the problem; thus the problem is to find p_1 and p_2 satisfying

$$(D^2 + 2D)\,\langle [2p_1 + 6p_2x]\, e^x\rangle \overset{?}{=} (2x+1)e^x.$$

That's a bit of an improvement — it's simpler than what we were facing a few sentences ago. (It's because we had $n = 2$ in this problem. In some problems we'll have $n = 0$, and no such simplification will be available.)

Now compute as follows. In the lines below, we must repeatedly use the formula for a derivative of a product of two functions, each time we take the derivative of a mixed term xe^x.

$$\begin{array}{rrclr|l}
\text{Let} & u & = & 6p_2xe^x & +2p_1e^x. & 0 \\
\text{Then} & Du & = & 6p_2xe^x & +(2p_1 + 6p_2)e^x & 2 \\
 & D^2u & = & 6p_2xe^x & +(2p_1 + 12p_2)e^x & 1 \\
\hline
 & \text{multiply and add} & & & & \\
 & (D^2 + 2D)u & = & 18p_2xe^x & +(6p_1 + 24p_2)e^x & \overset{?}{=} 2xe^x + 1e^x
\end{array}$$

Matching corresponding coefficients, that gives us two equations:

$$18p_2 = 2, \qquad 6p_1 + 24p_2 = 1.$$

Those solve to $p_2 = 1/9$ and $p_1 = -5/18$. Putting together all the pieces, our answer is

$$\boxed{y = (a_1 + a_2x - \frac{5}{18}x^2 + \frac{1}{9}x^3)e^x + a_3 + a_4e^{-2x}}\,,$$

where we don't have to find a_1, a_2, a_3, a_4. (Though I should mention that if initial conditions had been specified in the problem, we *would* have to find a_1, a_2, a_3, a_4; that would be our next step now.)

Finding the p's — shortcut method. Following is a method that may save a little bit of computational method in some problems (though in some examples, perhaps including the present one, it really makes very little difference). However, this method is harder to understand. It is recommended only for more advanced students — i.e., only after you have understood the straightforward method. Otherwise, this method is sure to confuse you.

To find the p's, as before, we begin by temporarily replacing the a's with 0's. Thus the solution is (as before) $y = (p_1x^2 + p_2x^3)e^x$, and the problem is

$$D(D+2)(D-1)^2\left[(p_1x^2 + p_2x^3)e^x\right] = (2x+1)e^x.$$

Now, let us temporarily define $E = D - 1$. Then we also have $E + 1 = D$ and $E + 3 = D + 2$. The problem can be rewritten as

$$(E+1)(E+3)E^2\left[(p_1x^2 + p_2x^3)e^x\right] = (2x+1)e^x,$$

or as

$$(E^4 + 4E^3 + 3E^2)\left[(p_1x^2 + p_2x^3)e^x\right] \stackrel{?}{=} (2x+1)e^x,$$

Near the beginning of this document, we mentioned that $(D-r)^n\left[f(x)e^{rx}\right] = f^{(n)}(x)e^{rx}$. This includes as a special case, for example, $(D-1)^n\left[f(x)e^x\right] = f^{(n)}(x)e^x$. That is, $E^n\left[f(x)e^x\right] = f^{(n)}(x)e^x$. We shall use that fact in the following computation. To get each successive line, just differentiate the polynomial from the previous line; we don't need to bother with the derivative of a product of two functions.

$$\begin{array}{rllllll|l}
y & =(& p_2x^3 & +p_1x^2 & & &)e^x & 0 \\
Ey & =(& & 3p_2x^2 & +2p_1x & &)e^x & 0 \\
E^2y & =(& & & 6p_2x & +2p_1 &)e^x & 3 \\
E^3y & =(& & & & 6p_2 &)e^x & 4 \\
E^4y & =0 & & & & & & 1 \\
\hline
\text{multiply and add} & & & & & & & \\
(E^4+4E^3+3E^2)y & =(& & & 18p_2x & +\langle 6p_1 + 24p_2\rangle &)e^x & \stackrel{?}{=}\ 2x+1)e^x
\end{array}$$

This yields the two equations, $18p_2 = 2$ and $6p_1 + 24p_2 = 1$, as in the straightforward method, and we finish the problem as before.

Second example for case 2. $y'''' - 3y'' + 2y' = 7e^{3x}$.

Solution. First rewrite the equation as

$$(D-3)^0(D-1)^2D(D+2)y = 7e^{3x},$$

which we analyze as follows:

$$\underbrace{\underbrace{(D-3)^0}_{\text{mult.0}}\ \underbrace{(D-1)^2}_{\text{mult.2}}\ \underbrace{(D-0)^1}_{\text{mult.1}}\ \underbrace{(D--2)^1}_{\text{mult.1}}}_{\substack{\text{multiplicities on}\\\text{left side determine}\\\text{arbitrary coefficients}\\\text{in answer}}}\ y \quad = \quad \underbrace{(7)}_{\substack{f\text{ has 1}\\\text{coefficient}}}\quad e^{3x}.$$

Here $r = 3$ and $n = 0$. Note that we have introduced a factor of $(D-3)^0$ on the left side, to help see how this equation fits the general pattern described earlier. Multiplying by $(D-3)^0$ has no effect — it's just multiplying by 1. The general recipe tells us to expect a solution of the form

$$y \quad = \quad \Bigg[\ \underbrace{A_1(x)}_{\substack{\text{no arb.}\\\text{coeffs.}}} \ + \ \underbrace{p_1}_{\substack{\text{1 particular}\\\text{coefficient}\\\text{(same as } f\text{)}}}\ \Bigg]\, e^{3x} + \underbrace{A_2(x)}_{\text{2 arb.}}\, e^{1x} \ + \ \underbrace{A_3(x)}_{\text{1 arb.}}\, e^{0x} \ + \ \underbrace{A_4(x)}_{\text{1 arb.}}\, e^{-2x}.$$

The polynomial $A_1(x)$ has *no* coefficients, so it is just zero. So y is, more simply,

$$y \quad = \quad p_1 e^{3x} + (a_1 + a_2 x)e^x + a_3 + a_4 e^{-2x}$$

where

- a_1, a_2, a_3, a_4 are arbitrary (we don't need to find them), and
- p_1 is particular (we do need to find it).

To find the p's, we temporarily replace the a's with 0; thus $y = p_1 e^{3x}$. The computations are fairly simple, because there are no xe^{3x} terms. We compute:

$$\begin{array}{rcr|l} y & = & p_1 e^{3x} & 0 \\ y' & = & 3p_1 e^{3x} & 2 \\ y'' & = & 9p_1 e^{3x} & -3 \\ y''' & = & 27p_1 e^{3x} & 0 \\ y'''' & = & 81p_1 e^{3x} & 1 \\ \hline \text{multiply and add} & & & \\ y'''' - 3y'' + 2y' & = & 60p_1 e^{3x} & \stackrel{?}{=} 7e^{3x} \end{array}$$

which yields $60p_1 = 7$, so $p_1 = 7/60$, and $\boxed{y = \frac{7}{60}e^{3x} + (a_1 + a_2 x)e^x + a_3 + a_4 e^{-2x}}$.

Third example for case 2. $y'''' - 3y'' + 2y' = 6$.

Solution. First rewrite the equation as $D(D-1)^2(D+2)y = 6e^{0x}$, which we analyze as follows:

$$\underbrace{\underbrace{(D-0)^1}_{\text{mult.1}} \underbrace{(D-1)^2}_{\text{mult.2}} \underbrace{(D--2)^1}_{\text{mult.1}}}_{\substack{\text{multiplicities on} \\ \text{left side determine} \\ \text{arbitrary coefficients} \\ \text{in answer}}} y \quad = \quad \underbrace{(6)}_{\substack{f \text{ has 1} \\ \text{coefficient}}} \quad e^{0x}.$$

Here $r = 0$ and $n = 1$. The general recipe tells us to expect a solution of the form

$$y \quad = \quad \Big[\quad \underbrace{A_1(x)}_{\substack{\text{1 arb.} \\ \text{coeffs.}}} \quad + \quad \underbrace{p_1}_{\substack{\text{1 particular} \\ \text{coefficient} \\ \text{(same as } f)}} \quad \Big] e^{0x} + \underbrace{A_2(x)}_{\text{2 arb.}} e^{1x} + \underbrace{A_3(x)}_{\text{1 arb.}} e^{-2x}.$$

That is, more simply, $y = (a_1 + p_1 x)e^{0x} + (a_2 + a_3 x)e^x + a_4 e^{-2x}$. To find the p's, we temporarily replace the a's with 0; thus $y = p_1 x$. The computations are fairly simple, because there are no xe^{3x} terms. We compute:

$$\begin{array}{rcr|r} y & = & p_1 x & 0 \\ y' & = & p_1 & 2 \\ y'' & = & 0 & -3 \\ y''' & = & 0 & 0 \\ y'''' & = & 0 & 1 \\ \hline \text{multiply and add} & & & \\ y'''' - 3y'' + 2y' & = & 2p_1 & \stackrel{?}{=} \ 6 \end{array}$$

so $p_1 = 3$, and $\boxed{y = a_1 + 6x + (a_2 + a_3 x)e^x + a_4 e^{-2x}}$.

Case 3: Right side is a sum of such terms. Here we consider a differential equation of the form

$$Ly \quad = \quad f_1(x)e^{r_1 x} + f_2(x)e^{r_2 x} + \cdots + f_m(x)e^{r_m x},$$

where L is a linear differential operator of the type shown in the previous examples, and each f_j is a polynomial, and the r_j's are different constants.

There is not really much new to explain in this case. The recipe is as follows: Solve each of the separate problems

$$\begin{array}{rcl} Ly_1 & = & f_1(x)e^{r_1 x}, \\ Ly_2 & = & f_2(x)e^{r_2 x}, \\ \vdots & & \vdots \\ Ly_m & = & f_m(x)e^{r_m x}, \end{array}$$

all of which are as in Case 2. Then add up the results;

$$y \quad = \quad y_1 + y_2 + \cdots + y_m$$

is the solution to the problem at hand. There may be some overlap among the arbitrary terms; combine those where possible.

Example for case 3. $y'''' - 3y'' + 2y' = (2x+1)e^x + 7e^{3x} + 6$.

Solution. I already did all the hardest parts of this in my earlier examples; we just have to put the parts together. We have these problems (but note that I'm changing the subscripts on the solutions of the problems):

problem	solution
$y'''' - 3y'' + 2y' = (2x+1)e^x$	$y = (a_1 + a_2 x - \frac{5}{18}x^2 + \frac{1}{9}x^3)e^x + a_3 + a_4 e^{-2x}$,
$y'''' - 3y'' + 2y' = 7e^{3x}$	$y = \frac{7}{60}e^{3x} + (a_5 + a_6 x)e^x + a_7 + a_8 e^{-2x}$,
$y'''' - 3y'' + 2y' = 6$	$y = a_9 + 6x + (a_{10} + a_{11}x)e^x + a_{12}e^{-2x}$.

Adding those results yields

$$y = \Big[\underbrace{(a_1 + a_5 + a_{10})}_{\text{combine these}} + \underbrace{(a_2 + a_6 + a_{11})}_{\text{combine these}}\, x - \frac{5}{18}x^2 + \frac{1}{9}x^3\Big]e^x$$

$$+ \underbrace{(a_3 + a_7 + a_9)}_{\text{combine these}} + \underbrace{(a_4 + a_8 + a_{12})}_{\text{combine these}}\, e^{-2x} + \frac{7}{60}e^{3x} + 6x$$

which simplfies to

$$\boxed{y = \Big[b_1 + b_2 x - \tfrac{5}{18}x^2 + \tfrac{1}{9}x^3\Big]e^x + b_3 + b_4 e^{-2x} + \tfrac{7}{60}e^{3x} + 6x}$$

where b_1, b_2, b_3, b_4 are arbitrary — i.e., we don't have to find them. (Though we would have to find them if some initial values were specified in the problem.)

To be continued soon: About complex roots.