General Recipe for Constant-Coefficient Equations

We want to look at problems like

$$y^{(6)} + 10y^{(5)} + 39y^{(4)} + 76y''' + 78y'' + 36y' = (x+2)e^{-3x} + xe^{-x}\cos x + 2x + 5e^{x}.$$

This example is actually more complicated than any that you're likely to have to do. That's because I chose one example to illustrate several different features. Most of your homework and test problems will be of second order, or at most third order.

This is a *constant-coefficient linear ordinary differential equation.* That refers to the fact that its left side is the sum of constants times derivatives of y, and its right side is just a function of x. It is *nonhomogeneous*; that refers to the fact that the right side is not zero. However, it is a particularly easy type of nonhomogeneous: the right side is a sum of polynomials times exponentials. It might not appear that way, but we can rewrite the right side as

$$(x+2)e^{-3x} + \frac{1}{2}xe^{(-1-i)x} + \frac{1}{2}xe^{(-1+i)x} + 2xe^{0} + 5(x^{0})e^{x}.$$

The first step is to factor the differentiation operators on the left side of the equation. The left side becomes

$$D(D+3)^2(D+1+i)(D+1-i)(D-2)y.$$

(Admittedly, factoring this sixth degree polynomial may be difficult, but you'll be doing easier problems — generally second degree, or third degree with at least one root fairly obvious.)

Next, rewrite the equation to display the "roots," and add dummy terms to both sides as needed to get the roots to match up. The equation can be rewritten:

$$\begin{aligned} & [D-(-3)]^2 \quad [D-(-1-i)]^1 \quad [D-(-1+i)]^1 \quad (D-0)^1\,(D-1)^0\,(D-2)^1\,y \\ = \;& (2+1x)e^{-3x} + \left(0+\tfrac{1}{2}x\right)e^{(-1-i)x} + \left(0+\tfrac{1}{2}x\right)e^{(-1+i)x} + (0+2x)\,e^{0} \quad +5e^{x} \quad +0e^{2x}. \end{aligned}$$

The term $0e^{2x}$ at the end of the equation is a dummy term, introduced to match up with the factor $(D-2)^1$ above it. Similarly, the factor $(D-1)^0$ near the end of the first line of the equation is a dummy factor, introduced to match up with the term $5e^x$ below it. The dummy terms have no effect on the quantifies involved — they just multiply by 1 or add 0 — but they give us matching roots on the left and right sides of the equation, so we can make this table:

root	-3	$-1-i$	$-1+i$	0	1	2
exponent in given left side	2	1	1	1	0	1
number of coefficients in given right side	2	2	2	2	1	0

That may require some explanation; here are a few examples of what the table means:

- The root -3 has exponent 2 in the given left side; this means the exponent is 2 in the expression $[D-(-3)]^2$.

- The root -3 has two coefficients in the given right side; this means that in the expression $(2+1x)e^{-3x}$ the numbers 2 and 1 are coefficients in the polynomial.
- We fill in zero's for missing coefficients in polynomials, and count the number of coefficients up to the highest degree term that is nonzero. For instance, the expression $\frac{1}{2}xe^{(-1-i)x}$ gets rewritten as $(0+\frac{1}{2}x)e^{(-1-i)x}$, which has two coefficients: 0 and $\frac{1}{2}$.
- A constant polynomial other than 0 has one coefficient. Thus, the expression $5e^x$ in the right side of the differential equation tells us that the root 1 has one coefficient in the given right side.
- When the polynomial is the constant 0 (as in $0e^{2X}$, we say it has no coefficients. Thus (for instance) the last entry in the table is 0.

It should also be noted that the number of coefficients is one greater than the degree of the polynomial. For instance, $(2+1x)$ has two coefficients; it is a polynomial of degree 1. We may be tempted to think in terms of degree (because we have a name for it: "degree"), but it turns out that we can work more easily in terms of the number of coefficients.

Each number in the table is found from the given equation, but it turns out that each number in the table also tells us something about the form of the answer we're looking for.

root	-3	$-1-i$	$-1+i$	0	1	2
exponent in given left side = number of arbitrary coefficients in answer	2	1	1	1	0	1
number of coefficients in given right side = number of particular coefficients in answer	2	2	2	2	1	0

Now we can write down a preliminary form of the answer:

$$y = \underbrace{(\underbrace{a_1 + a_2 x}_{\text{2 arb's}} + \underbrace{p_1 x^2 + p_2 x^3}_{\text{2 partic's}})e^{-3x}}_{\text{root}=-3}$$

$$+ \underbrace{(\underbrace{a_3}_{\text{1 arb}} + \underbrace{p_3 x + p_4 x^2}_{\text{2 partic's}})e^{(-1-i)x}}_{\text{root}=-1-i} + \underbrace{(\underbrace{a_4}_{\text{1 arb}} + \underbrace{p_5 x + p_6 x^2}_{\text{2 partic's}})e^{(-1+i)x}}_{\text{root}=-1+i}$$

$$+ \underbrace{(\underbrace{a_5}_{\text{1 arb}} + \underbrace{p_7 x + p_8 x^2}_{\text{2 partic's}})e^{0x}}_{\text{root}=0} + \underbrace{\underbrace{p_9}_{\text{1 partic}}\ e^{1x}}_{\text{root}=1} + \underbrace{\underbrace{a_6}_{\text{1 arb}}\ e^{2x}}_{\text{root}=2}$$

or more briefly

$$
\begin{aligned}
y &= (a_1 + a_2x + p_1x^2 + p_2x^3)e^{-3x} + (a_3 + p_3x + p_4x^2)e^{(-1-i)x} \\
&\qquad + (a_4 + p_5x + p_6x^2)e^{(-1+i)x} + (a_5 + p_7x + p_8x^2) + p_9e^x + a_6e^{2x},
\end{aligned}
$$

where the a's are arbitrary and the p's are particular (non-arbitrary). The abstract theory tells us that the answer is of this form. It is a sum of polynomials times exponentials. Each polynomial consists of several arbitrary terms, which are (in order of increasing exponent on x) first the arbitrary terms, then the particular terms.

Now, in all the problems we're going to consider, all the given data involves only real numbers. Any polynomial with real coefficients has its complex roots occurring in complex conjugate pairs — that is, if $r + is$ is a root, then $r - is$ is also a root. Consequently, any term $e^{rx}e^{isx}$ is accompanied by an $e^{rx}e^{-isx}$ term. Those two terms (preceded by arbitrary coefficients and some power of x) can be transformed into the two terms $e^{rx}\cos(sx)$ and $e^{rx}\sin(sx)$ (preceded by different arbitrary coefficients and the same power of x). For instance, our lengthy example given above can now be rewritten as

$$
y = (a_1 + a_2x + p_1x^2 + p_2x^3)e^{-3x} + (b_3 + q_3x + q_4x^2)e^{-x}\cos x
$$

$$
+(b_4 + q_5x + q_6x^2)e^{-x}\sin x + (a_5 + p_7x + p_8x^2) + p_9e^x + a_6e^{2x}
$$

which makes no explicit mention of non-real numbers. Here the a's and b's are arbitrary and the p's and q's are particular (nonarbitrary).

To finish the problem, we still must take these steps:

- Plug this formula for y back into the original equation, and see what it tells us about the a's, b's, p's, and q's. If we've done it correctly, all the a's and b's will cancel out, and we'll find out nothing about them; they remain arbitrary. But the p's and q's will not cancel out; we'll obtain several equations just involving p's and q's. Solve those equations to find *particular numbers* for the p's and q's. The result is an equation of the form $y =$ (some function), where the function may involve x, a's, b's, and numbers.

- (If the problem had initial conditions, then we plug those into our results, and they now yield information about the a's and b's. Solve those equations, to find the values of the a's and b's. In the result, the answer is $y =$ (some function of just x); it does not involve any other unknowns. But we'll ignore this case for the moment.)

When we're looking for the p's and q's, we can ignore the a's and b's. Thus, as our problem presently stands, we are to find p's and q's to satisfy

$$
\begin{aligned}
&D(D+3)^2(D^2+2D+2)(D-2)\Big[(p_1x^2 + p_2x^3)e^{-3x} \\
&\qquad + (q_3x + q_4x^2)e^{-x}\cos x + (q_5x + q_6x^2)e^{-x}\sin x + (p_7x + p_8x^2) \\
&\qquad\qquad + p_9e^x\Big] \quad = \quad (x+2)e^{-3x} + xe^{-x}\cos x + 2x + 5e^x.
\end{aligned}
$$

But the operator $D(D+3)^2(D^2+2D+2)(D-2)$ is linear. Consequently, it suffices to solve these four slightly simpler problems: Find p's and q's so that

$$\begin{aligned}
D(D+3)^2(D^2+2D+2)(D-2)[\,(p_1x^2+p_2x^3)e^{-3x}\,] &= (x+2)e^{-3x} \\
D(D+3)^2(D^2+2D+2)(D-2)[\quad (p_7x+p_8x^2) \quad] &= 2x \\
D(D+3)^2(D^2+2D+2)(D-2)[\qquad p_9e^x \qquad] &= 5e^x
\end{aligned}$$

and

$$\begin{aligned}
D(D+3)^2(D^2+2D+2)(D-2)[(q_3x+q_4x^2)e^{-x}\cos x& \\
+(q_5x+q_6x^2)e^{-x}\sin x] &= xe^{-x}\cos x.
\end{aligned}$$

However, **that's as far as I'm going to go** with the first example that I've been developing for the last couple of pages. It's a good example for illustrating how to do the problem up to this point, but a poor example for illustrating how to finish the last few steps of the problem, because it is far too messy. Instead I'll show how to finish the problem on a few easier examples. We'll see that the last few steps use techniques that vary from one problem to another.

Example 2. $y''+4y'+4y=7e^{-2x}+5e^{3x}$.

Solution. This equation gets rewritten as

$$(D-(-2))^2(D-3)^0y=7e^{-2x}+5e^{3x},$$

so it has this chart:

root	-2	3
exponent in given left side = number of arbitrary coefficients in answer	2	0
number of coefficients in given right side = number of particular coefficients in answer	1	1

and this preliminary form of solution:

$$y=(a_0+a_1x+p_0x^2)e^{-2x}+p_1e^{3x}.$$

The problem can be broken into these two simpler problems:

$$\begin{aligned}
(D+2)^2\left[p_0x^2e^{-2x}\right] &= 7e^{-2x}, \\
(D+2)^2\left[p_1e^{3x}\right] &= 5e^{3x}.
\end{aligned}$$

For the first of those two problems, we can use a shortcut mentioned in class: We can make use of the fact that

$$(D-k)[f(x)e^{kx}] = f'(x)e^{kx}$$

for any function f and any constant k. That is, applying $D-k$ to a function times the corresponding exponential e^{kx} has the same effect as replacing the function $f(x)$ with its derivative. This procedure can be repeated as many times as we like — we have $(D-k)^2[f(x)e^{kx}] = f''(x)e^{kx}$ and $(D-k)^3[f(x)e^{kx}] = f'''(x)e^{kx}$, etc. In particular, taking $k=-2$, we look for the second derivative of p_0x^2. That's easy to compute; it is $2p_0$. Hence the equation $(D+2)^2[p_0x^2e^{-2x}] = 7e^{-2x}$ simplifies to $2p_0e^{-2x} = 7e^{-2x}$. Thus $p_0 = 7/2$.

The equation $(D+2)^2[p_1e^{3x}] = 5e^{3x}$ will take a little more work. Here is a tedious but fairly straightforward method:

$$\begin{array}{rl|l} \text{if} \quad u = & p_1e^{3x} & 4 \\ \text{then} \quad u' = & 3p_1e^{3x} & 4 \\ u'' = & 9p_1e^{3x} & 1 \\ \hline u''+4u'+4u = & 25p_1e^{3x} & \\ \stackrel{?}{=} & 5e^{3x} & \end{array}$$

and so p_1 must be $1/5$. Thus we arrive at the answer

$$\boxed{y \quad = \quad \left(a_0 + a_1x + \frac{7}{2}x^2\right)e^{-2x} + \frac{1}{5}e^{3x}}.$$

Example 3. Same as the preceding example, but also with initial conditions $y(0)=2$ and $y'(0)=0$.

Solution. We begin as in the preceding example. Then continue as follows:

$$\begin{array}{rcllll} & y & = & a_0e^{-2x} & +a_1xe^{-2x} & +\frac{7}{2}x^2e^{-2x} & +\frac{1}{5}e^{3x} \\ & y' & = & (-2a_0+a_1)e^{-2x} & +(-2a_1+7)xe^{-2x} & -7x^2e^{-2x} & +\frac{3}{5}e^{3x} \\ \\ 2 \stackrel{?}{=} & y(0) & = & a_0 & & & +\frac{1}{5} \\ 0 \stackrel{?}{=} & y'(0) & = & -2a_0+a_1 & & & +\frac{3}{5} \end{array}$$

Solve the third equation for a_0; thus $a_0 = 2 - \frac{1}{5} = 9/5$. Then solve the fourth equation for a_1; thus $a_1 = 2a_0 - \frac{3}{5} = \frac{18-3}{5} = \frac{15}{5} = 3$. Thus we arrive at the answer

$$\boxed{y \quad = \quad \left(\frac{9}{5} + 3x + \frac{7}{2}x^2\right)e^{-2x} + \frac{1}{5}e^{3x}}.$$

Example 4. $y'' - 4y' + 4y = (1+x)e^{2x}$.

Solution. This equation gets rewritten as

$$(D-2)^2 y = (1+x)e^{2x},$$

so it has this chart:

root	2
exponent in given left side = number of arbitrary coefficients in answer	2
number of coefficients in given right side = number of particular coefficients in answer	2

and this preliminary form of solution:

$$y = (a_0 + a_1 x + p_0 x^2 + p_1 x^3)e^{2x}.$$

We are looking for p_0 and p_1 to satisfy

$$(D-2)^2[(p_0 x^2 + p_1 x^3)e^{2x}] \quad \stackrel{?}{=} \quad (1+x)e^{2x}.$$

Again we make use of the formula $(D-k)^n[f(x)e^{kx}] = f^{(n)}(x)e^{kx}$. With $k=2$ and $n=2$ and $f(x) = p_0 x^2 + p_1 x^3$, that tells us $(D-2)^2[(p_0 x^2 + p_1 x^3)e^{2x}] = (2p_0 + 6p_1 x)e^{2x}$. Hence the problem we're trying to solve reduces to

$$(2p_0 + 6p_1 x)e^{2x} \quad \stackrel{?}{=} \quad (1+x)e^{2x}$$

which yields $2p_0 = 1$ and $6p_1 = 1$. Thus $p_0 = 1/2$ and $p_1 = 1/6$, so the answer is

$$\boxed{y \quad = \quad \left(a_0 + a_1 x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^{2x}}.$$

Example 5. $y'' - 4y' + 4y = 12e^{2x} \sin 3x$.

Solution. This equation gets rewritten as

$$(D-2)^2 y = -6ie^{(2+3i)x} + 6ie^{(2-3i)x},$$

but actually that's more than we need to know. With less work, we can determine that the equation is of the form

$$(D-2)^2 y = (\text{some constant})e^{(2+3i)x} + (\text{some constant})e^{(2-3i)x}$$

(where the two constants are not necessarily the same). That's enough to tell us that the problem has this chart:

root	2	$2+3i$	$2-3i$
exponent in given left side = number of arbitrary coefficients in answer	2	0	0
number of coefficients in given right side = number of particular coefficients in answer	0	1	1

and this preliminary form of solution:

$$y = (a_0 + a_1 x)e^{2x} + p_1 e^{(2+3i)x} + p_2 e^{(2-3i)x}$$

which can be rewritten as

$$y = (a_0 + a_1 x)e^{2x} + q e^{2x} \cos 3x + r e^{2x} \sin 3x.$$

We must find the constants q and r. Unfortunately, there are no shortcuts here; the method for finding q and r is long and tedious.

$$\begin{array}{rcll|l}
\text{Let} \quad u & = & qe^{2x}\cos 3x & +re^{2x}\sin 3x & 4 \\
\text{Then} \quad u' & = & qe^{2x}(2\cos 3x - 3\sin 3x) & +re^{2x}(2\sin 3x + 3\cos 3x) & -4 \\
u'' & = & qe^{2x}(-5\cos 3x - 12\sin 3x) & +re^{2x}(-5\sin 3x + 12\cos 3x) & 1 \\
\hline
u'' - 4u' + 4u & = & qe^{2x}(-9\cos 3x) & +re^{2x}(-9\sin 3x) & \\
 & \stackrel{?}{=} & & 12e^{2x}\sin 3x &
\end{array}$$

Note that each differentiation uses the rule for the product of the derivative of two functions. For the term in the u' row and the q column is $\frac{d}{dx}\left[e^{2x}\cos 3x\right] = (e^{2x})'(\cos 3x) + (e^{2x})(\cos 3x)' = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$. The last line of the table yields $q = 0$ and $r = -12/9 = -4/3$, so

$$\boxed{y = (a_0 + a_1 x)e^{2x} - \frac{4}{3}e^{2x}\sin 3x}.$$

Here is a minor variant: Instead of sorting q and r into separate columns, we can sort cos and sin into separate columns. It's the same computation, with the same result, but it looks a little different:

$$\begin{array}{rcrr|l}
\text{Start with} \quad u & = & qe^{2x}\cos 3x & +re^{2x}\sin 3x & 4 \\
\text{Then} \quad u' & = & (2q+3r)e^{2x}\cos 3x & +(2r-3q)e^{2x}\sin 3x & -4 \\
u'' & = & (-5q+12r)e^{2x}\cos 3x & +(-5r-12q)e^{2x}\sin 3x & 1 \\
\hline
u'' - 4u' + 4u & = & (-9q)e^{2x}\cos 3x & +(-9r)e^{2x}\sin 3x & \\
 & \stackrel{?}{=} & & 12e^{2x}\sin 3x, &
\end{array}$$

with the same result as the preceding method. In some problems this variant may be slightly easier.

Here is a more substantial variant: What if we retained the complex numbers until the end of the problem? The solution of

$$(D-2)^2 y = -6ie^{(2+3i)x} + 6ie^{(2-3i)x}$$

is of the form

$$y = (a_0 + a_1 x)e^{2x} + se^{(2+3i)x} + te^{(2-3i)x}$$

where we must find the constants s and t. Compute as follows:

$$\begin{array}{rcrr|l}
u & = & se^{(2+3i)x} & +te^{(2-3i)x} & 4 \\
u' & = & (2+3i)se^{(2+3i)x} & +(2-3i)te^{(2-3i)x} & -4 \\
u'' & = & (2+3i)^2 se^{(2+3i)x} & +(2-3i)^2 te^{(2-3i)x} & \\
 & = & (-5+12i)se^{(2+3i)x} & +(-5-12i)te^{(2-3i)x} & 1 \\
\hline
u'' - 4u' + 4u & = & -9se^{(2+3i)x} & -9te^{(2-3i)x} & \\
 & \overset{?}{=} & -6ie^{(2+3i)x} & +6ie^{(2-3i)x} &
\end{array}$$

which yields $s = 6i/9 = 2i/3$ and $t = -6i/9 = -2i/3$. Then

$$\begin{array}{rcll}
y & = & (a_0 + a_1 x)e^{2x} & +se^{(2+3i)x} + te^{(2-3i)x} \\
 & = & (a_0 + a_1 x)e^{2x} & +\frac{2i}{3}e^{(2+3i)x} - \frac{2i}{3}e^{(2-3i)x} \\
 & = & (a_0 + a_1 x)e^{2x} & +\frac{2i}{3}e^{2x}\left(e^{3ix} - e^{-3ix}\right) \\
 & = & (a_0 + a_1 x)e^{2x} & +\frac{2i}{3}e^{2x}\left(2i \sin 3x\right) \\
 & = & (a_0 + a_1 x)e^{2x} & -\frac{4}{3}e^{2x}\sin 3x,
\end{array}$$

the same answer obtained by the previous methods.