

Name (please print):

Math 198 Test [3], 2 Apr 2010, 2+ pages, 25 points, 50 minutes.

(This is actually Test 3, but it said “Test 2”; that was a typographical error.)

(7 points) $x^2y'' - 4xy' - 14y = 0$

Solution. This is a Cauchy-Euler equation. Look for solutions of the form $y = x^k$; then $0 = k(k - 1) - 4k - 14 = k^2 - 5k - 14$. That can be solved by the quadratic formula

$$k = \frac{5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot (-14)}}{2} = \frac{5 \pm \sqrt{25 + 56}}{2} = \frac{5 \pm \sqrt{81}}{2} = \frac{5 \pm 9}{2}$$

or by factoring: $k^2 - 5k - 14 = (k - 7)(k + 2)$. Either way, k is 7 or -2 , so the solution is $y = c_1x^7 + c_2x^{-2}$.

Partial credit:

- 1 point for recognizing that this is a problem in which the solutions should be of the form x^k .
 - 2 points for correctly identifying the auxiliary polynomial equation $k(k - 1) - 4k - 14 = 0$.
 - 2 points for getting correctly from the polynomial equation (whatever you thought it should have been) to its two solutions.
 - 2 points for getting from the solutions of the polynomial equation, to a correct expression of the answer.
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(18 points)

$$\begin{array}{ll} (1) & (D - 2)x + Dy = e^{2t} \\ (2) & Dx - y = e^{-3t} \end{array}$$

Use additional sheets of paper as needed.

Solution. By one method or another, we must eliminate one of the two variables x or y , leaving a differential equation involving just x or just y (as a function of t). That will be one of the two equations

$$(a) \quad \begin{cases} (D+2)(D-1)x = e^{2t} - 3e^{-3t} \\ (D+2)(D-1)y = 2e^{2t} + 5e^{-3t} \end{cases} \quad \mathbf{OR}$$

or it could be one of those equations times another operator – e.g., it could be something like

$$(a') \quad D(D+2)(D-1)x = 2e^{2t} + 9e^{-3t}.$$

Getting to one of the equations in (a), or getting to an equation like that in (a'), is worth **5 points**, if done correctly.

[One way to get to step (a) is as follows: Equation (2) can be solved for y , as a function of x and t :

$$(*) \quad y = Dx - e^{-3t}.$$

Plug that formula for y into equation (1), to obtain an equation involving just x and t ; that would be the first equation I've listed in (a). Alternatively, if you subtract equation (2) from equation (1), you'll obtain a differential equation involving $2x$ but no Dx :

$$(*') \quad 2x = (D+1)y + e^{-3t} - e^{2t};$$

plug that into either equation (1) or equation (2) to obtain an equation that involves only y and t ; that will be the second equation I've listed in (a). There are still other methods. For instance, multiply equation (2) by D and add it to equation (1), to eliminate y ; or multiply equation (1) by D and multiply equation (2) by $-(D-2)$ and add the results, to eliminate x .]

The next two stages amount to solving a differential equation by the method of annihilators, but it's convenient to break that method into two stages. Once an equation of the type described above has been obtained, the next step is to write down the general form of its solution. Getting correctly to that stage, from whatever you had for step (a), is worth another **4 points**. In the case of the two

answers that I've given in step (a), the resulting answers would be either of

$$(b) \quad \begin{cases} x = c_1e^{-2t} + c_2e^t + a_1e^{2t} + a_2e^{-3t} \\ y = b_1e^{-2t} + b_2e^t + a_1e^{2t} + a_2e^{-3t} \end{cases} \quad \text{OR}$$

where c_1, c_2, b_1, b_2 will remain arbitrary in the final answer (i.e., they are part of x_c or y_c), while a_1 and a_2 are numbers we must find in the next step of the problem (i.e., they are part of x_p or y_p).

Next, we take the x_p or y_p result from step (b); use it to calculate x'_p and x''_p or to calculate y'_p and y''_p ; and plug those results back into the equation obtained in step (a). In this fashion we find a_1 and a_2 , and so we obtain one of the following two results:

$$(c) \quad \begin{cases} x = c_1e^{-2t} + c_2e^t + \frac{1}{4}e^{2t} - \frac{3}{4}e^{-3t} \\ y = b_1e^{-2t} + b_2e^t + \frac{1}{2}e^{2t} + \frac{5}{4}e^{-3t} \end{cases} \quad \text{OR}$$

Getting to this stage correctly, from whatever you had for steps (a) and (b), is worth **4 points**.

Finally, having found one of x or y , we must now find the other one; that's worth the remaining **5 points**. The methods for doing this will vary, depending on what you've already done so far. The easiest cases would be:

- if you used (*) to eliminate y and find x , you can now use (*) again to find y ; or
- if you used (*') to eliminate x and find y , you can now use (*') again to find x .

We end up with an answer that looks like one of the following:

$$\begin{array}{l} x = -\frac{1}{2}b_1e^{-2t} + b_2e^t + \frac{1}{4}e^{2t} - \frac{3}{4}e^{-3t} \\ y = b_1e^{-2t} + b_2e^t + \frac{1}{2}e^{2t} + \frac{5}{4}e^{-3t} \end{array}$$

OR

$$\begin{array}{l} x = c_1e^{-2t} + c_2e^t + \frac{1}{4}e^{2t} - \frac{3}{4}e^{-3t} \\ y = -2c_1e^{-2t} + c_2e^t + \frac{1}{2}e^{2t} + \frac{5}{4}e^{-3t} \end{array}$$

If you're unsure of your answers, you can use them to also compute Dx and Dy , and then plug those formulas back into the original equations (1) and (2), to see if they are satisfied. That actually doesn't take terribly long, in this example.