Name (please print):

Math 198 Test 1, 29 January 2010, 4 pages, 25 points, 50 minutes.

(6 points) $\dfrac{dy}{dx} = \dfrac{x-y}{2x-2y+1}$

Solution. Substitute $u = x - y$. Then

$$\frac{du}{dx} \;=\; 1 - \frac{dy}{dx} \;=\; 1 - \frac{u}{2u+1} \;=\; \frac{u+1}{2u+1}$$

Hence

$$x \;=\; \int dx \;=\; \int \frac{2u+1}{u+1} du$$

(3 points for getting that far)

$$=\; \int \left(2 - \frac{1}{u+1}\right) du$$

(now we're up to 4 points)

$$=\; 2u - \ln|u+1| + C$$

Return to the original variables;

$$x \;=\; 2x - 2y - \ln|x-y+1| + C$$

Simplify;

$$\boxed{2y = x - \ln|x-y+1| + C} \quad \text{or} \quad \boxed{x - 2y = \ln|x-y+1| - C}.$$

Raise e to the power on both sides to obtain still another form of the answer:

$$\boxed{ke^{x-2y} = x - y + 1}.$$

(This gets rid of the absolute value sign, since the constant k could be positive or negative.)

The problem can also be done using the substitution $u = 2x - 2y + 1$, which some students preferred. That yields $x - y = \frac{1}{2}u - \frac{1}{2}$, and

$$\frac{du}{dx} \;=\; 2 - 2\frac{du}{dx} \;=\; 2 - 2 \cdot \frac{\frac{1}{2}u - \frac{1}{2}}{u} \;=\; 2 - \frac{u-1}{u} \;=\; 2 - 1 + \frac{1}{u} \;=\; 1 + \frac{1}{u}.$$

Separating the variables yields

$$dx = \frac{1}{1+\frac{1}{u}}du$$

(3 points for getting that far)

$$= \frac{u}{u+1}du = \left(1 - \frac{1}{u+1}\right)du$$

(now we're up to 4 points). Integrating both sides,

$$x = u - \ln|u+1| + C = 2x - 2y + 1 - \ln|2x - 2y + 2| + C$$

which, upon simplifying, yields the same answer as in the previous method.

(7 points) Find the second order differential equation whose general solution is $y = ax^4 + bx^2$.

Solution. This can be done by eliminating a first or by eliminating b first. I'll work it both ways, so that you can compare your attempted solution with the way it should have been done.

Eliminating b first: Multiply the given equation through by x^{-2}:

$$x^{-2}y = ax^2 + b$$

Differentiate both sides.

$$-2x^{-3}y + x^{-2}y' = 2ax$$

(Getting that far was worth 4 points.) Now isolate a by multiplying through by x^{-1}:

$$-2x^{-4}y + x^{-3}y' = 2a$$

Differentiate both sides:

$$8x^{-5}y - 5x^{-4}y' + x^{-3}y'' = 0$$

Tidy up by multiplying through by x^5:

$$\boxed{8y - 5xy' + x^2y'' = 0}.$$

Eliminating a first: Multiply the given equation through by x^{-4}; thus

$$x^{-4}y \;=\; a + bx^{-2}.$$

Now differentiate both sides with respect to x:

$$-4x^{-5}y + x^{-4}y' \;=\; -2bx^{-3}.$$

(Getting that far was worth 4 points.) Next, multiply through by x^3, to isolate b:

$$-4x^{-2}y + x^{-1}y' \;=\; -2b.$$

Differentiate both sides:

$$8x^{-3}y - 5x^{-2}y' + x^{-1}y'' \;=\; 0.$$

Again, simplify:

$$\boxed{8y - 5xy' + x^2y'' = 0}.$$

That is the preferred form for the answer, but I gave full credit for some answers that were equivalent to that, even though they really should have been simplified a bit more. Among those, the most common answer was $8x^{-3}y - 5x^{-2}y' + x^{-1}y'' = 0$.

Checking:

$$\begin{array}{rrcrr} \text{If} & y & = & ax^4 & +bx^2 \\ \text{then} & y' & = & 4ax^3 & +2bx \\ \text{and} & y'' & = & 12ax^2 & +2b \end{array} \qquad \begin{array}{|rcrr} 8y & = & 8ax^4 & +8bx^2 \\ -5xy' & = & -20ax^4 & -10bx^2 \\ x^2y'' & = & 12ax^4 & +2bx^2 \\ \hline \text{sum} & = & 0 & \end{array}$$

(5 points) Solve explicitly for y:

$$\frac{dy}{dx} + \frac{(y+3)^2}{x-1} \;=\; 0$$

Solution. Separate the variables:

$$(y+3)^{-2}dy \;=\; -(x-1)^{-1}dx.$$

Integrate both sides:

$$-\int \frac{dy}{(y+3)^2} \;=\; \int \frac{dx}{x-1}$$

$$\frac{1}{y+3} = \ln|x-1| + C$$

Finally, solve algebraically for y:

$$y + 3 = \frac{1}{\ln|x-1| + C}$$

$$\boxed{y = \frac{1}{\ln|x-1| + C} - 3}.$$

(7 points) Solve explicitly for y:

$$\frac{dy}{dx} = y^{-2}x^2\cos(x^3 + e^x), \qquad y(0) = 1.$$

Your answer should be an expression involving an integral.

Solution. Separating the variables yields

$$y^2 dy = x^3\cos(x^3 + e^x)dx.$$

We can integrate the left side, but not the right side. So let's instead consider

$$y'(x) = [y(x)]^{-2}\, x^2\cos(x^3 + e^x)$$

$$y'(t) = [y(t)]^{-2}\, t^2\cos(t^3 + e^t)$$

$$[y(t)]^2\, y'(t) = t^2\cos(t^3 + e^t)$$

$$\frac{1}{3}\frac{d}{dt}\left\{[y(t)]^3\right\} = t^2\cos(t^3 + e^t)$$

Integrate both sides from $t = 0$ to $t = x$.

$$\frac{1}{3}\int_0^x \frac{d}{dt}\left\{[y(t)]^3\right\}\, dt = \int_0^x t^2\cos(t^3 + e^t)\, dt$$

$$\frac{1}{3}\left\{[y(x)]^3 - [y(0)]^3\right\} = \int_0^x t^2\cos(t^3 + e^t)\, dt$$

Plug in the given initial value:

$$\frac{1}{3}\left\{[y(x)]^3 - 1\right\} = \int_0^x t^2\cos(t^3 + e^t)\, dt.$$

Getting that far was worth 6 points. Several students got that far and then messed up their algebra in solving for $y(x)$:

$$[y(x)]^3 - 1 = 3\int_0^x t^2\cos(t^3+e^t)\,dt$$

$$[y(x)]^3 = 1 + 3\int_0^x t^2\cos(t^3+e^t)\,dt$$

$$\boxed{y = \sqrt[3]{1+3\int_0^x t^2\cos(t^3+e^t)\,dt}}\,.$$

Total number of points is 25.