Name: **ANSWER KEY**.
Math 198 Quiz 1, 12 Feb 2010, 2 pages, 15 points, 20 minutes.

(7 points) $\left(2x\sin y + ye^{xy} + \frac{1}{x}\right) dx + \left(x^2\cos y + xe^{xy}\right) dy \;=\; 0$

Solution. Labeling

$$\underbrace{\left(2x\sin y + ye^{xy} + \frac{1}{x}\right)}_{M} dx + \underbrace{\left(x^2\cos y + xe^{xy}\right)}_{N} dy \;=\; 0$$

we obtain $\partial M/\partial y = 2x\cos y + (xy+1)e^{xy} = \partial N/\partial x$, and thus the problem is exact. We want to find a function f simultaneously satisfying $\partial f/\partial x = M$ and $\partial f/\partial y = N$.

The condition $\partial f/\partial x = M$ yields

$$\begin{aligned} f &= 2\sin y \int x dx + y\int e^{xy} dx + \int \tfrac{1}{x} dx \\ &= x^2\sin y + e^{xy} + \ln|x| + a(y) \end{aligned}$$

and the condition $\partial f/\partial y = N$ yields

$$\begin{aligned} f &= x^2 \int \cos y \, dy + x\int e^{xy} dy \\ &= x^2\sin y + e^{xy} + b(x). \end{aligned}$$

Those are simultaneously satisfied by taking $a(y) = \mathit{const}$, $b(x) = \ln|x| + \mathit{const}$, and

$$f \;=\; x^2\sin y + e^{xy} + \ln|x| + \mathit{const}$$

so the answer to the problem is

$$\boxed{x^2\sin y + e^{xy} + \ln|x| \;=\; C}.$$

Strictly speaking, the absolute value sign should not be omitted, but I'm going to permit its omission with no penalty.

(8 points) $(x^2 + xy + y^2)dx \;=\; xydy$

Solution. This is *not* exact — we can see that by rewriting it as

$$\underbrace{x^2 + xy + y^2}_{M} \;+\; \underbrace{-xy}_{N} dy \;=\; 0$$

which satisfies $\partial M/\partial y = x + 2y$ and $\partial N/\partial x = -y$; those are different. But this problem is homogeneous.

The easier solution is by substituting $y = ux$:

$$(x^2 + x^2u + x^2u^2)dx = x^2u(udx + xdu)$$

Divide out x^2.

$$(1 + u + u^2)dx = u(udx + xdu)$$

Subtract u^2dx from both sides.

$$(1 + u)dx = u(xdu)$$

Separate the variables.

$$\frac{dx}{x} = \frac{udu}{1+u} = \left(1 - \frac{1}{1+u}\right)du$$

Integrate both sides.

$$\ln|x| = u - \ln|1+u| + C$$

$$\boxed{\ln|x| = \tfrac{y}{x} - \ln|1 + \tfrac{y}{x}| + C}$$

$$\ln|x| + \ln|1 + \frac{y}{x}| = \frac{y}{x} + C$$

$$\boxed{\ln|x+y| = \tfrac{y}{x} + C}$$

$$\boxed{x\ln|x+y| = y + Cx}$$

$$\boxed{k(x+y) = e^{y/x}}$$

The problem could also be solved by the substitution $x = vy$, though that's harder:

$$(v^2y^2 + vy^2 + y^2)(vdy + ydv) = vy^2dy$$

Divide out y^2:

$$(v^2 + v + 1)(vdy + ydv) = vdy$$

$$(v^2 + v + 1)ydv = (-v^2 - v - 1)vdy + vdy$$

$$(v^2 + v + 1)ydv = -(v^2 + v)vdy$$

$$\frac{v^2 + v + 1}{v^2(v+1)}dv = -\frac{dy}{y}$$

The left side can be decomposed by partial fractions:

$$\frac{v^2 + v + 1}{v^2(v+1)} = \frac{A}{v} + \frac{B}{v^2} + \frac{C}{v+1}$$

$$\begin{aligned} v^2 + v + 1 &= Av(v+1) + B(v+1) + Cv^2 \\ v^2 + v + 1 &= (A+C)v^2 + (A+B)v + B \end{aligned}$$

$$1 = A + C, \qquad 1 = A + B, \qquad 1 = B$$

which yields the solution $B = 1$, $A = 0$, $C = 1$, so

$$\left(\frac{1}{v^2} + \frac{1}{v+1}\right) dv = -\frac{dy}{y}.$$

Integrate both sides;

$$-v^{-1} + \ln|v+1| = -\ln|y| + C$$

$$\boxed{-\tfrac{y}{x} + \ln\left|\tfrac{x}{y} + 1\right| = -\ln|y| + C}$$

$$\ln\left|\frac{x}{y} + 1\right| + \ln|y| = \frac{y}{x} + C$$

$$\boxed{\ln|x+y| = \tfrac{y}{x} + C}$$

$$\boxed{k(x+y) = e^{y/x}}$$