Name: ANSWER KEY
Math 198 Quiz 2, 17 March 2010, 2 pages, 15 points, 20 minutes.

(4 points) Find the general solution of $y''' + 4y' = 0$.

Solution. The corresponding polynomial is $k^3 + 4k = k(k^2 + 4) = (k - 0)(k + 2i)(k - 2i)$, so the general solution is $y = a_1 e^{0x} + a_2 e^{2ix} + a_3 e^{-2ix}$, which gets rewritten as

$$\boxed{y = c_1 + c_2 \cos 2x + c_3 \sin 2x}\,.$$

(4 points) Find the general solution of $y'' + 4y' + 4y = 0$.

Solution. The corresponding polynomial is $k^2 + 4k + 4 = (k + 2)^2$, so the solution is

$$\boxed{y = (c_1 + c_2 x)e^{-2x}}\,.$$

(7 points) Find the general solution of $y'' - y' - 6y = 2 + e^{-2x}$.

Solution. The equation can be rewritten as $Ly = f$, where $L = D^2 - D - 6 = (D + 2)(D - 3)$, and where $f(x) = 2 + e^{-2x}$ has annihilator $M = D(D + 2)$.

The homogeneous problem $Ly = 0$ has solution $y_c = c_1 e^{-2x} + c_2 e^{3x}$. Getting that far is worth 2 points. Also, computing $ML = D(D + 2)^2(D - 3)$ is worth a point.

From the equation $Ly = f$ we obtain $MLy = Mf = 0$, and so the answer is of the form

$$y = a_0 + (a_1 + a_2 x)e^{-2x} + a_3 e^{3x}.$$

Now we're up to 4 points total so far. The equation can be rewritten as

$$y = a_0 + (c_1 + a_2 x)e^{-2x} + c_2 e^{3x}$$

to incorporate the terms that we've already found in y_c; thus we obtain

$$y_p = a_0 + a_2 x e^{-2x}$$

(now we're up to 5 points). We still need to find a_0 and a_2; each of them will be worth 1 point.

We can omit the terms $a_1 e^{-2x}$ and $a_3 e^{3x}$, since those can be absorbed into y_c. Thus we are seeking a solution of the form

$$\begin{array}{rclll|r} y_p & = & a_0 & +a_2 x e^{-2x} & & -6 \\ y_p' & = & & -2a_2 x e^{-2x} & +a_2 e^{-2x} & -1 \\ y_p'' & = & & 4a_2 x e^{-2x} & -4a_2 e^{-2x} & 1 \\ \hline y_p'' - y_p' - 6y_p & = & -6a_0 & +0a_2 x e^{-2x} & -5a_2 e^{-2x} & \\ & \stackrel{?}{=} & 2 & & +e^{-2x} & \end{array}$$

That requires $-6a_0 = 2$ and $-5a_2 = 1$, hence $a_0 = -1/3$ and $a_2 = -1/5$. Thus we obtain

$$\boxed{y = (c_1 - \frac{1}{5}x)e^{-2x} + c_2 e^{3x} - \frac{1}{3}}\,.$$