Name:
Math 198 Final exam, 5 May 2010, 8 pages, 50 points, 120 minutes.

(7 points) $y'' - 4y' + 4y = x + 1 + e^x$

Solution. Rewrite that as $Ly = f(x)$, where $L = (D-2)^2$, and where $f(x) = x + 1 + e^x$ has annihilator $M = D^2(D-1)$.

Then the complementary solution y_c is the solution of $y_c = 0$, which is $y_c = c_1 e^{2x} + c_2 x e^{2x}$.

We also have $D^2(D-1)(D-2)^2 = LMy = 0$, so

$$y \;=\; y_c + y_p \;=\; c_1 e^{2x} + c_2 e^{2x} + a_1 + a_2 x + a_3 e^x,$$

where we need to find a_1, a_2, a_3. Compute

$$\begin{array}{rrcccc}
[4] & y_p & = & a_1 & +a_2 x & +a_3 e^x \\
[-4] & y_p' & = & a_2 & & +a_3 e^x \\
[1] & y_p'' & = & & & a_3 e^x \\
\hline
\multicolumn{2}{r}{y_p'' - 4y_p' + 4y_p} & = & (4a_1 - 4a_2) & +4a_2 x & +a_3 e^x \\
 & & \stackrel{?}{=} & 1 & +x & +e^x
\end{array}$$

thus requiring $4a_1 - 4a_2 = 1$ and $4a_2 = 1$ and $a_3 = 1$. Hence $a_2 = 1/4$ and $a_1 = 1/2$, so we end up with

$$\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} + \frac{1}{4}x + e^x}.$$

Partial credit: The correct answer is the sum of five terms. For answers that are mostly correct, deduct one point for each erroneous term among those five.

(8 points) $4xy'' + 2y' + y = 0$

(The next page is intentionally blank, because you might need two pages for this problem.)

Solution.

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Getting that far was worth 2 points. Some students omitted the r, after which all further work was pointless — but I gave some partial credit anyway; see notes at the end of this solution.

$$\begin{aligned}
[2] \quad y' &= \textstyle\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\
y'' &= \textstyle\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} \\
[4] \quad xy'' &= \textstyle\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} \\
[1] \quad y &= \textstyle\sum_{n=0}^{\infty} c_{n-1} x^{n+r-1} \quad \text{(where } c_{-1} = 0\text{)} \\
\hline
4xy'' + 2y' + y &= \textstyle\sum_{n=0}^{\infty} \left\{c_{n-1} + 4(n+r)(n+r-1)c_n + 2(n+r)c_n\right\} x^{n+r}
\end{aligned}$$

Getting that far correctly is worth 3 points.

Thus $0 = c_{n-1} + (n+r)\left[4(n+r-1)+2\right]c_n$ for $n = 0, 1, 2, \ldots$, which can be rewritten as

$$0 = c_{n-1} + 2(n+r)(2n+2r-1)c_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

(now we're up to 4 points).

When $n = 0$, this yields $r(2r-1) = 0$, so $r = 0$ or $r = 1/2$; identifying those two roots correctly brings us up to 5 points.

For higher values of n, we get

$$c_n = \frac{-c_{n-1}}{(2n+2r-1)(2n+2r)} \quad \text{or} \quad c_{n+1} = \frac{-c_n}{(2n+2r+1)(2n+2r+2)}$$

which then yields these two columns of computation:

	$r = 0$	$r = 1/2$
	$c_n = \dfrac{-c_{n-1}}{(2n-1)(2n)}$	$c_n = \dfrac{-c_{n-1}}{(2n)(2n+1)}$

(Getting those right brings us to 6 points.*)

$(n=1)$	$c_1 = \dfrac{-c_0}{1\cdot 2}$	$c_1 = \dfrac{-c_0}{2\cdot 3}$
$(n=2)$	$c_2 = \dfrac{-c_1}{3\cdot 4} = \dfrac{c_0}{4!}$	$c_2 = \dfrac{-c_1}{4\cdot 5} = \dfrac{c_0}{5!}$
$(n=3)$	$c_3 = \dfrac{-c_2}{5\cdot 6} = \dfrac{-c_0}{6!}$	$c_3 = \dfrac{-c_2}{6\cdot 7} = \dfrac{-c_0}{7!}$

and so on.

*Those could also be written as

$c_{n+1} = \dfrac{-c_n}{(2n+1)(2n+2)}$	$c_{n+1} = \dfrac{-c_n}{(2n+2)(2n+3)}$

This results in the answer

$$\boxed{\begin{aligned} y \;=\; & A\left(\frac{1}{0!} - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \cdots\right) \\ + \; & B\left(\frac{1}{1!} - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \cdots\right) x^{1/2} \end{aligned}}$$

or

$$\boxed{\begin{aligned} y \;=\; & A\left(1 - \frac{1}{2}x + \frac{1}{24}x^2 - \frac{1}{720}x^3 + \cdots\right) \\ + \; & B\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + \cdots\right) x^{1/2} \end{aligned}}$$

Optional: That turns out to be the same thing as $y = A\cos\sqrt{x} + B\sin\sqrt{x}$.

Only 4 students got full credit on this one. Many other students had the right idea but made all sorts of computational errors along the way.

A few students began by looking for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ — that is, they took the exponent n instead of $n + r$. That's incorrect, but I decided to follow the computation for a few steps, to determine allow at least a little bit of partial credit. The computation begins the same as above, but with all the r's replaced by 0's. Thus, in the equation that I said earlier was worth 3 points, we now get

$$4xy'' + 2y' + y = \sum_{n=0}^{\infty} \left\{ c_{n-1} + 4n(n-1)c_n + 2nc_n \right\} x^n$$

(worth 2 points), and the equation that I said earlier was worth 4 points now becomes

$$0 = c_{n-1} + 2n(2n-1)c_n \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

(worth 3 points). That last equation can also be written as

$$c_n = \frac{-c_{n-1}}{(2n)(2n-1)} \quad \text{or} \quad c_{n+1} = \frac{-c_n}{(2n+2)(2n+1)}.$$

(6 points) Find the first-order differential equation whose general solution is

$$y \;=\; \sqrt[3]{\frac{4x}{3+cx^4}}.$$

Simplify as much as possible.

Solution. I thought this was an easy problem; I was surprised at how few students got it entirely right.

$$\begin{aligned} y^3 &= \frac{4x}{3+cx^4} \\ y^{-3} &= \frac{3+cx^4}{4x} \\ 4xy^{-3} &= 3+cx^4 \\ 4xy^{-3}-3 &= cx^4 \\ 4x^{-3}y^{-3}-3x^{-4} &= c \end{aligned}$$

or $c = \dfrac{4x-3y^3}{x^4y^3}$ or something equivalent to that; 4 points for getting this far correctly. Now differentiate both sides with respect to x.

$$\begin{aligned} -12(x^{-4}y^{-3}+x^{-3}y^{-4}y'-x^{-5}) &= 0 \\ x^{-4}y^{-3}+x^{-3}y^{-4}y'-x^{-5} &= 0 \end{aligned}$$

I will give only 5 points (out of 6) for the answers in either of those last two lines; after all, the instructions did say "simplify as much as possible."

Multiply both sides through by x^5y^4, to obtain $\boxed{xy+x^2y'=y^4}$; another preferred form for the answer is $\boxed{\dfrac{dy}{dx}=\dfrac{y^4}{x^2}-\dfrac{y}{x}}$. Full credit will be given to anything equivalent to those and not appreciably more complicated looking. (In case you're interested, the equation we've ended up with is a Bernoulli equation.)

A couple of students, after reaching the line that says $4x^{-3}y^{-3}-3x^{-4}=c$, rewrote it as $\dfrac{4}{x^3y^3}-\dfrac{3}{x^4}=c$ and then incorrectly took reciprocals on both sides, arriving at the incorrect equation $\dfrac{x^3y^3}{4}-\dfrac{x^4}{3}=c_1$ (where $c_1=1/c$). That is a very severe conceptual error. It is essentially saying that $\frac{1}{a}+\frac{1}{b}$ is equal to $\frac{1}{a+b}$ — a special case of the "everything is additive" error that I addressed at

http://www.math.vanderbilt.edu/~schectex/commerrs/#Additive

(7 points) Find the general solution of $x\dfrac{dy}{dx} + 2y = 1$.

Solution. This can be solved by either of two methods.

Method 1. It's linear. Rewrite it in standard form as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x}$$

(worth 1 point). Thus $P(x) = \frac{2}{x}$, so $\int P(x)dx = 2\ln x + C_1$, and we drop the C_1 for convenience. Then the integrating factor is $I(x) = e^{\int P(X)dx} = e^{2\ln x} = x^2$ (now we're up to 2 points). Multiply the standard form equation through by x^2, to obtain

$$x^2\frac{dy}{dx} + 2xy = x$$

(worth 3 points)

$$\left(x^2y\right)' = x$$

(worth 5 points)

$$x^2y = \int x\,dx$$

(worth 6 points)

$$x^2y = \frac{1}{2}x^2 + C$$

which is acceptable for full credit, but I really would prefer:

$$\boxed{y = \tfrac{1}{2} + Cx^{-2}}.$$

Method 2. It has variables separable. Rewrite the equation as

$$x\frac{dy}{dx} = 1 - 2y$$

$$\frac{dy}{1-2y} = \frac{dx}{x}$$

$$\frac{dy}{2y-1} = -\frac{dx}{x}$$

$$\frac{dy}{y-\frac{1}{2}} = -2\frac{dx}{x}$$

Integrate both sides:

$$\ln\left|y - \frac{1}{2}\right| = -2\ln|x| + k$$

(5 points for getting that far)

$$e^{\ln\left|y-\frac{1}{2}\right|} = e^k e^{-2\ln|x|}$$

$$\left|y - \frac{1}{2}\right| = e^k\left|x^{-2}\right|$$

$$y - \frac{1}{2} = \pm e^k x^{-2}$$

$$y - \frac{1}{2} = Cx^{-2}$$

$$\boxed{y = \tfrac{1}{2} + Cx^{-2}}.$$

(8 points) Solve $x^2dy = (y^2+2xy)dx$ with initial value $(x, y) = (1, -2)$, explicitly for y.

Solution. The problem is homogeneous, so substitute $y = ux$ (worth 1 point already) and eliminate y (or substitute $x = vy$ and eliminate x; see below). We get

$$\begin{aligned}
x^2(udx + xdu) &= (u^2x^2 + 2x^2u)dx \\
&\quad \text{(worth 2 points)} \\
udx + xdu &= (u^2 + 2u)dx \\
xdu &= (u^2 + u)dx \\
\frac{du}{u(u+1)} &= \frac{dx}{x} \\
&\quad \text{(worth 3 points)} \\
\int \left(\frac{1}{u} - \frac{1}{u+1}\right) du &= \int \frac{dx}{x} \\
&\quad \text{(worth 4 points)} \\
\ln \left|\frac{u}{u+1}\right| &= \ln|x| + c_1 \\
&\quad \text{(worth 5 points)} \\
\frac{u}{u+1} &= c_2x \\
&\quad \text{(worth 6 points)} \\
y &= c_2x(y + x)
\end{aligned}$$

A bit of algebra yields $y = \dfrac{c_2x^2}{1 - c_2x}$ or $y = \dfrac{x^2}{c_3 - x}$, either of which is worth 7 points. Plug in $(x, y) = (1, -2)$; that yields $c_2 = 2$, hence

$$\boxed{y = \frac{2x^2}{1 - 2x}}.$$

If we instead substitute $x = vy$ and eliminate x, here is the resulting computation:

$$\begin{aligned}
v^2y^2dy &= (y^2+2vy^2)(vdy+ydv) \\
v^2dy &= (1+2v)(vdy+ydv) \\
(v^2-v-2v^2)dy &= (1+2v)ydv \\
-(v^2+v)dy &= (1+2v)ydv \\
\frac{-dy}{y} &= \frac{1+2v}{v(v+1)}dv
\end{aligned}$$

Use the method of partial fractions to find A and B satisfying

$$\begin{aligned}
\frac{1+2v}{v(v+1)} &= \frac{A}{v}+\frac{B}{v+1} \\
1+2v &= A(v+1)+Bv \\
&= (A+B)v+A
\end{aligned}$$

That yields $A = B = 1$, so the computation continues

$$\begin{aligned}
\frac{-dy}{y} &= \left(\frac{1}{v}+\frac{1}{v+1}\right)dv \\
-\int\frac{dy}{y} &= \int\left(\frac{1}{v}+\frac{1}{v+1}\right)dv \\
-\ln|y| &= \ln|v|+\ln|v+1|+c_1 \\
\frac{1}{y} &= v(v+1)c_2 \\
\frac{1}{y} &= \frac{x}{y}\left(\frac{x}{y}+1\right)c_2 \\
y &= x(x+y)c_2
\end{aligned}$$

and the remaining computation is as explained earlier.

The equation $x^2dy = (y^2+2xy)dx$ can also be solved as a Bernoulli equation with $n = 2$. Rewrite it as $x^2y' - 2xy = y^2$. Solve it with the substitution $u = y^{1-n} = y^{-1}$; then $y = u^{-1}$ and $y' = -u^{-2}u'$. Thus the differential equation becomes

$$\begin{aligned} -u^{-2}x^2u' - 2xu^{-1} &= u^{-2} \\ x^2u' + 2xu &= -1 \\ u' + 2x^{-1}u &= -x^{-2} \end{aligned}$$

Use $P(x) = 2x^{-1}$, $\int P(x)dx = 2\ln x$, $I(x) = e^{\int P(x)dx} = x^2$.

$$\begin{aligned} x^2u' + 2xu &= -1 \\ (x^2u)' &= -1 \\ x^2u &= c - x \\ y^{-1} = u &= cx^{-2} - x^{-1} \\ y &= \frac{1}{cx^{-2} - x^{-1}} = \frac{x^2}{c - x} \end{aligned}$$

and continue as in the earlier solutions.

(7 points) Given that $y_1 = x$ is one solution of the differential equation

$$x^2y'' - (x^2 + 2x)y' + (x + 2)y = 0,$$

find the general solution.

Solution. We will use the method of reduction of order. Let $y = ux$, and eliminate y. Reasoning that far is worth 2 points already. Compute:

$x + 2$	y	$=$	ux		
$-(x^2 + 2x)$	y'	$=$	u	$+u'x$	
x^2	y''	$=$		$2u'$	$+u''x$
	Computing those correctly brings us up to 4 points. Then:				
Adding,					

$$\text{left side of diff eqn} = u\left\{\begin{array}{c} (x+2)x \\ -(x^2+2x) \end{array}\right\} + u'\left\{\begin{array}{c} -(x^2+2x)x \\ +2x^2 \end{array}\right\} + u''\left\{x^3\right\}$$

$$= \qquad u'\left\{-x^3\right\} \qquad +u''\{x^3\}$$

Divide out the x^3; this leaves $u'' - u' = 0$, or $D(D-1)u = 0$. Getting that far is worth 5 points. That has general solution $u = c_1 + c_2 e^x$ (worth 6 points), hence the original problem has solution $\boxed{y = c_1 x + c_2 x e^x}$.

(7 points) $x^2 y'' - 5xy' + 11y = 0$

Solution. Look for solutions of the form $y = x^k$. Then $y' = kx^{k-1}$ and $y'' = k(k-1)x^{k-2}$, so we get

$$\begin{aligned} k(k-1) - 5k + 11 &= 0 \\ k^2 - 6k + 11 &= 0 \\ k^2 - 6k + 9 &= -2 \\ (k-3)^2 &= -2 \\ k &= 3 \pm \sqrt{2}i \end{aligned}$$

and so

$$\begin{aligned} y &= c_1 x^{3+\sqrt{2}i} + c_2 x^{3-\sqrt{2}i} \\ y &= x^3 (c_1 x^{\sqrt{2}i} + c_2 x^{-\sqrt{2}i}) \end{aligned}$$

Recall that $x^{pi} = e^{ip\ln x} = \cos(p\ln x) + i\sin(p\ln x)$; apply that with $p = \pm\sqrt{2}$. Thus

$$\boxed{y = x^3 \left[A\cos(\sqrt{2}\ln x) + B\sin(\sqrt{2}\ln x)\right]}.$$

Partial credit: In an answer that resembles a correct one, deduct 5/3 points for each of the following ingredients that is missing or that is misrepresented:

x^3, A and B, cos and sin, $\sqrt{2}$, ln, x

Then round to the nearest integer. Thus,

number of erroneous ingredients	1	2	3	4
number of points scored on problem	5	4	2	0
