Name:
Math 198 Quiz 3, 19 April 2010, 1 page, 15 points, 20 minutes.

Find the general solution of $(1+2x^2)y'' + 8xy' + 4y = 0$.

Solution.

$$\begin{aligned}
[4] \qquad y &= \textstyle\sum_{n=0}^{\infty} c_n x^n \\
y' &= \textstyle\sum_{n=0}^{\infty} n c_n x^{n-1} \\
y'' &= \textstyle\sum_{n=0}^{\infty} n(n-1) c_n x^{n-2} \\
[1] \qquad &= \textstyle\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \\
[8] \qquad xy' &= \textstyle\sum_{n=0}^{\infty} n c_n x^n \\
[2] \qquad x^2 y'' &= \textstyle\sum_{n=0}^{\infty} n(n-1) c_n x^n \\
(1+2x^2)y'' + 8xy' + 4y &= \textstyle\sum_{n=0}^{\infty} \{4c_n + (n+2)(n+1)c_{n+2} \\
&\qquad + 8nc_n + 2n(n-1)c_n\} x^n
\end{aligned}$$

$$4c_n + (n+2)(n+1)c_{n+2} + 8nc_n + 2n(n-1)c_n = 0 \qquad (n = 0, 1, 2, \ldots)$$

$$(n+2)(n+1)c_{n+2} + 2(n+2)(n+1)c_n = 0 \qquad (n = 0, 1, 2, \ldots)$$

$$c_{n+2} = -2c_n \qquad (n = 0, 1, 2, \ldots)$$

c_0, c_1 are arbitrary, and

$$\begin{aligned}
c_2 &= -2c_0 & c_3 &= -2c_1 \\
c_4 &= 4c_0 & c_5 &= 4c_1 \\
c_6 &= -8c_0 & c_7 &= -8c_1 \\
&\cdots & &\cdots
\end{aligned}$$

And so

$$\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots \\
&= c_0 + c_1 x - 2c_0 x^2 - 2c_1 x^3 + 4c_0 x^4 + 4c_1 x^5 - 8c_0 x^6 - 8c_1 x^7 + \cdots
\end{aligned}$$

which yields

$$\boxed{\begin{aligned} y = \; & c_0\left(1 - 2x^2 + 4x^4 - 8x^6 + 16x^8 - \cdots\right) \\ & + c_1\left(x - 2x^3 + 4x^5 - 8x^7 + 16x^9 - \cdots\right) \end{aligned}}$$

or

$$\boxed{y = (c_0 + c_1 x)\left(1 - 2x^2 + 4x^4 - 8x^6 + 16x^8 - \cdots\right)}$$

or

$$\boxed{y = (c_0 + c_1 x) \sum_{n=0}^{\infty} \left(-2x^2\right)^n}$$

or (using the formula for the sum of a geometric series)

$$\boxed{y = \frac{c_0 + c_1 x}{1 + 2x^2}}.$$

Check: If $(1+2x^2)y = c_0 + c_1 x$, then differentiating twice yields $(1+2x^2)y'' + 2(4x)y' + 4y = 0$. $\checkmark$