Introduction to Linear Operators

A *function* takes numbers to numbers — for instance, cos and ln are functions.

An *operator* takes functions to functions. For instance,

$$\frac{d}{dx} \qquad \text{and} \qquad \int_0^x (\ )dt$$

are operators.

An operator L is *linear* if it satisfies

$$L(af + bg) \;=\; aL(f) + bL(g)$$

for all constants a, b and functions f, g. For instance, differentiation and integration are linear operators, whereas squaring is not a linear operator.

A *linear equation* or *linear problem* generally means

$$Ly = f$$

where L is a given linear operator, f is a given function, and y is the function we're trying to find. The problem is *homogeneous* if $f = 0$, or *nonhomogeneous* if $f \neq 0$. Here are some examples:

$$x^3\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + (7 - x)\frac{dy}{dx} + 3xy = 17\cos x$$

is a third order nonhomogeneous linear ordinary differential equation with nonconstant coefficients. The *associated homogeneous equation* is

$$x^3\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + (7 - x)\frac{dy}{dx} + 3xy = 0.$$

And

$$7\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 2x^2$$

is a third order nonhomogeneous linear ordinary differential equation with *constant* coefficients, and its associated homogeneous problem is

$$7\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0.$$

We devote a great deal of attention to linear problems because (i) usually they're easier to solve than nonlinear ones, and (ii) many real-world problems can be modeled by linear equations. Moreover, some real-world problems that are inherently nonlinear can nevertheless be approximated by nearby linear equations — i.e., their *linearizations.* Likewise, we devote extra attention to constant-coefficient problems, because they're easier than the ones with nonconstant coefficients.

Let's consider any third order nonhomogeneous linear ODE:

$$a_3(x)y'''(x) + a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$$

where a_3, a_2, a_1, a_0, f are given functions (and a_3 is not zero) and we're trying to find y. (I'm using third order just for an illustration, but the same analysis will apply to nth order for any n.) In general, the solution of such an ODE is of the form

$$y \quad = \quad y_p \; + \; \underbrace{c_1y_1 + c_2y_2 + c_3y_3}_{y_c}$$

where y_1, y_2, y_3 and y_p are functions that we must find, and c_1, c_2, c_3 are arbitrary constants — i.e., the three parameters for the three-parameter family of curves that we expect to find as the solution to a third order differential equation (see our discussion from earlier this semester). The fact that the several parts of the solution are simply *added* — rather than connected in some more complicated fashion, e.g., multiplied together or placed under a square root sign or something — is because we're dealing with a *linear* problem.

The function y_p is any one *particular* solution of the nonhomogeneous problem. We just choose one, and it doesn't matter which one, and the choice is not unique (though usually we prefer the simplest one). The function y_c is called the *complementary function*, because it complements y_p (that's different from "compliments"; look it up). The function y_p is the general solution of the associated homogeneous problem. Again, the functions y_1, y_2, y_3 are not uniquely determined, but usually we prefer them as simple as possible.

Here is an example: It turns out that the general solution of

$$y''' - y' = e^{3x}$$

is the infinite family of functions represented by this three-parameter formula:

$$y \quad = \quad \underbrace{\frac{1}{24}e^{3x}}_{y_p} \; + \; \underbrace{c_1e^{-x} + c_2 + c_3e^{x}}_{y_c}.$$

And that's probably the simplest way to express it, for most purposes. But the same family of functions can also be expressed this way:

$$y \quad = \quad \underbrace{\frac{1}{24}e^{3x}}_{y_p} + \underbrace{b_1 \cosh x + b_2 + b_3 \sinh x}_{y_c}$$

and for some purposes that representation might be more convenient. And the same family of functions can be represented in many other, less convenient ways — for instance,

$$y \quad = \quad \underbrace{\frac{1}{24}e^{3x} + 7e^x}_{y_p} + \underbrace{k_1(e^{-x} + 2e^x) + k_2(1 - e^x) + k_3 e^x}_{y_c}.$$

Note that the three functions $e^{-x} + 2e^x$, $1 - e^x$, and e^x must be *different.* Actually, we can make a stronger statement about them: They must be *linearly independent.* This means that no one of them can be written as a sum of constants times the others. For instance, the set of functions

$$y_1 = e^x - 1, \qquad y_2 = e^{-x} - e^x, \qquad y_3 = 1 - e^{-x}$$

is linearly dependent, since $y_1 = -y_2 - y_3$. The three parameter family of functions

$$y \quad = \quad \underbrace{\frac{1}{24}e^{3x} + 7e^x}_{y_p} + \underbrace{m_1(e^x - 1) + m_2(e^{-x} - e^x) + m_3(1 - e^{-x})}_{y_c}$$

is not a correct solution to the differential equation, because it misses many of the solution functions that are represented by the previous solutiosn that I've listed. In fact, it's not really a three-parameter family — it just looks like one. When you're looking for the general solution to a homogeneous nth order linear equation, you know you're done when you've found n linearly indepedent solutions (not just n different solutions). Judging linear independence is much easier when n is 2: Then it just means that neither of the two functions is a constant times the other.

Variation of Parameters

Finding the particular solution y_p and finding the complementary function y_c are two different kinds of problems, but they are related. In fact, if you've found y_c, then there is actually a *formula* for y_p, but it's very complicated. I will show it to you in the case of second-order equations, but even in that case it's a bit complicated. This is taken from pages 157-158 of your textbook, but I've changed the notation a little (take $v_j = u_j'$ if you want to match up my explanation with that in the textbook).

Say you want to solve

$$y'' + P(x)y' + Q(x)y = f(x)$$

where P, Q, f are given functions. And suppose that you've already found y_1 and y_2, the two ingredients in $y_c = c_1y_1 + c_2y_2$. Then we just need to find y_p, and it turns out that y_p can be expressed in terms of y_1, y_2, and f, by this admittedly complicated procedure:

Let v_1 and v_2 be the solution to this pair of two linear equations in two unknowns:

$$\begin{array}{rcrcl} y_1v_1 & + & y_2v_2 & = & 0, \\ y_1'v_1 & + & y_2'v_2 & = & f. \end{array}$$

(You can solve that by Cramer's rule with determinants, if you like.) Then

$$y_p \quad = \quad y_1 \int v_1(x)dx \; + \; y_2 \int v_2(x)dx$$

turns out to be a particular solution of the nonhomogeneous problem.