

INTRODUCTION TO COMPLEX NUMBERS

The importance of polynomials in solving differential equations. We start with an example. We want to find $y(t)$ satisfying

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = 0.$$

The abstract theory tells us, among other things, that any differential equation of this type¹ has at least one solution of the form $y = e^{kt}$ for some constant k . To find k , compute as follows:

$$y = e^{kt}, \quad y' = ke^{kt}, \quad y'' = k^2e^{kt}, \quad y''' = k^3e^{kt}.$$

Substitute those into the differential equation; that yields the equation

$$k^3e^{kt} + 6k^2e^{kt} + 5ke^{kt} = 0.$$

Since e^{kt} is never 0, we can divide it out. That leaves the *polynomial* equation

$$k^3 + 6k^2 + 5k = 0.$$

That has solutions $k = 0$, $k = -1$, and $k = -5$. All the preceding steps are reversible, so three solutions of the differential equation are 1 , e^{-t} , e^{-5t} . In fact, using other parts of the general theory (not discussed here), it can be shown that the *general* solution of this particular differential equation is

$$y = c_1 + c_2e^{-t} + c_3e^{-5t}$$

where c_1, c_2, c_3 are arbitrary constants.

The procedure gets more complicated when we introduce any of these complications into the problem:

- The problem above is *homogeneous*². For a *nonhomogeneous* problem, such as

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} = \sin(t),$$

the procedure is more complicated.

- The problem above had three *distinct* roots: 0 , -1 , -5 . A more complicated procedure is needed for equations with *repeated* roots, such as $k^2 - 6k + 9 = 0$.
- The problem above had *real* roots. The procedure must be modified for *complex* roots, as in the case of $k^2 + 9 = 0$. But a similar procedure still works. Thus, by enlarging our collection of “numbers,” we enlarge the collection of polynomial equations for which we have solutions, and consequently we enlarge the collection of differential equations for which we can find solutions.

¹That is, a constant-coefficient homogeneous linear ordinary differential equation — but don’t worry about the definitions right now; we’ll deal with them later.

²This use of the word “homogeneous” is unrelated to the use of that word in Chapter 2.

What are complex numbers? In calculus we used the *real number system*, \mathbb{R} ; we describe it as the points on a line. However, for some purposes — e.g., polynomials and differential equations — we need more numbers. The *complex number system*, \mathbb{C} , can be described as the points on a *plane*.

Of course, you’ve worked with points on a plane before. The point $(3, 5)$ is 3 units to the right of the origin and 5 units up. But it is customary to change notation at this point: When we think of that point as a complex number, we write it instead as $3 + 5i$. This emphasizes that it is viewed as just *one* “number,” rather than a pair of real numbers. (Some engineers write j instead of i .)

Some abbreviations are possible: the number $0 + 5i$ (which lies on the vertical coordinate axis) may be written more briefly as $5i$, and the number $3 + 0i$ (which lies on the horizontal axis) may be written simply as 3. That is, the complex numbers on the horizontal axis are just ordinary real numbers; the real numbers are a subset of the complex numbers. That is, $\mathbb{R} \subseteq \mathbb{C}$, where $\mathbb{R} = \{\text{real numbers}\}$ and $\mathbb{C} = \{\text{complex numbers}\}$.

There is also some peculiar terminology: the numbers on the horizontal axis are said to be **real**, and the numbers on the vertical axis (such as $5i$) are said to be **imaginary**. In a number such as $3 + 5i$, we say 3 is the **real part** and 5 is the **imaginary part**. The historical reason for this terminology will be explained a few paragraphs from now.

You’ve worked with points in the plane before. What may be new to you are the *arithmetic rules* for the complex numbers.

How to add two complex numbers: It’s done coordinatewise, like the addition of two-dimensional vectors:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

For instance, $(3 + 5i) + (7 - 2i) = 10 + 3i$, and $(6 + \pi i) + (\sqrt{2} - 7i) = (6 + \sqrt{2}) + (\pi - 7)i$. That might look more familiar if we use the ordered pair notation:

$$(a, b) + (c, d) = (a + c, b + d); \quad (3, 5) + (7, -2) = (10, 3), \quad \text{etc.}$$

But we will use the “ i ” notation instead. — Subtraction is similar.

How to multiply two complex numbers: This is more complicated, and it involves a formula you’ll need to memorize:

$$\boxed{(a + bi)(c + di) = (ac - bd) + (ad + bc)i}.$$

I may refer to this as the **Cartesian** formula for multiplication, to contrast it with the *polar* formula which will be given later. The Cartesian formula is probably easier to learn first, but the polar formula — which gives the same results, ultimately — will be easier to motivate³ in terms of geometry.

³I.e., explain why someone would *want* to define things this way.

Here is an example: $(3 + 5i)(7 - 2i) = (3 \cdot 7 - 5 \cdot (-2)) + (3 \cdot (-2) + 5 \cdot 7)i = 31 + 29i$.
A more surprising example is $(0 + i)(0 + i) = (-1 + 0i)$. More briefly,

$$i^2 = -1.$$

And we also have $(-i)^2 = -1$. Thus, the number -1 has two square roots.

Now you can see where the term “imaginary” came from: It is a historical accident. Complex numbers were first investigated algebraically; the points-in-the-plane geometric model only came many decades later. The first mathematicians who worked with the number i were so surprised by its properties that they thought “there can’t *really* be a number with these properties.” They worked out the computational properties of a collection of numbers in whose existence they didn’t believe. They found these “numbers” to be a convenient fiction, a useful intermediate step in going from a problem involving only real numbers to an answer involving only real numbers. They called the number i “imaginary,” and unfortunately that name stuck.

Here’s a bit of philosophy: In some sense, *all* numbers are objects that exist only in the imagination. The physical world that we live in may contain three airplanes or three apples, but it does not contain “three”; that is an abstract notion that only exists in our minds. The abstract notion is useful because it enables us to understand the physical world better. Positive numbers were good for trading sheep and goats; negative numbers explained debts and cold temperatures; and complex numbers are particularly good for explaining rotations and stretching motions, as we shall see below.

Absolute values. The absolute value of a complex number $\alpha = p + qi$ is the number $|\alpha| = |p + qi| = \sqrt{p^2 + q^2}$; it is the distance from 0 to $p + qi$.

Exercises. Simplify — i.e., write the product in the form $p + qi$, where p and q are real numbers. Also, find the absolute value of the product.

(1) $(3 + 4i)(5 + 6i)$.

(2) $(5 - 3i)(2 + 0i)$.

(3) $(2 - \sqrt{5}i)(2 + \sqrt{5}i)$.

(*Optional.*) If you’re already familiar with matrix multiplication, then you might like this explanation for complex multiplication: Represent the complex number $x + yi$ with the 2-by-2 matrix of real numbers $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}$. Then it turns out that addition of complex numbers is the same as addition of 2-by-2 matrices, and multiplication of complex numbers is the same as multiplication of 2-by-2 matrices. For instance, our example $(3 + 5i)(7 - 2i) = 31 + 29i$ can be restated as

$$\begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 31 & 29 \\ -29 & 31 \end{bmatrix}.$$

However, the real motivation for complex multiplication — i.e., the *real* reason why we want to define complex multiplication this way — will be apparent when we discuss polar coordinates, a few paragraphs from now.

Basic rules of arithmetic. Many of these are the same for complex numbers as they ever were for rational or real numbers. For any complex numbers α, β, γ , we have

- Commutative laws: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$.
- Associative laws: $(\alpha + \beta) + \gamma = \gamma + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.
- Distributive law: $\alpha(\beta + \gamma) = (\alpha\beta) + (\alpha\gamma)$.
- Identity laws: $\alpha + 0 = \alpha$ and $\alpha \cdot 1 = \alpha$.
- Additive inverse: $(a + bi) + ((-a) + (-b)i) = 0$.
- Multiplicative inverse: If a and b are real numbers, not both zero — that is, if $a + bi$ is some complex number different from 0 — then

$$(a + bi) \left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i \right) = 1.$$

For instance, $(6 + 8i) \cdot (0.06 - 0.08i) = 1$. So we can write $\frac{1}{6 + 8i} = 0.06 - 0.08i$.

And so $\frac{3 - i}{6 + 8i} = (3 - i)(.06 - .08i) = 0.1 - 0.3i$.

The preceding rules — commutative, associative, existence of inverses, etc. — are the rules for a *field*, in abstract algebra. Thus the complex numbers are a field, just as the rational numbers and the real numbers are fields.

However, the rationals and reals are *ordered* fields, and the complex numbers are not. An ordered field has a relation $>$ satisfying several further rules. But those rules imply that $x^2 + 1 > 0$ and consequently that $x^2 \neq -1$. Thus, no matter clever or complicated a definition we might choose, it is *not possible* to define an ordering relation $>$ on \mathbb{C} that makes it an ordered field.

Exercises. Simplify; write each number in the form $a + bi$ where a and b are real numbers that are simplified as much as possible.

(4) $\frac{1}{2 + i}$

(5) $\frac{7 - 3i}{2 + i}$

Solving polynomial equations with complex numbers. When we have more numbers, we can solve more problems. Some problems have solutions that can only be expressed in terms of the new numbers. For instance, we've already mentioned that $i^2 = -1$, so i is a solution of the equation $z^2 = -1$, or $z^2 + 1 = 0$. This problem didn't have a solution in the real number system. But (in principle) complex numbers enable us to solve *any* polynomial equation. We have:

Fundamental Theorem of Algebra. Let n be a positive integer, and let $a_0, a_1, a_2, \dots, a_n$ be complex numbers, with $a_n \neq 0$. Then the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_2 z^2 + a_1 z + a_0$$

can be factored into n first-degree factors:

$$p(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_{n-1})(z - r_n).$$

Thus, any n th degree polynomial has n roots r_1, r_2, \dots, r_n (if we count each repeated root as many times as its multiplicity).

The theorem above doesn't tell us how to find the roots; it just tells us the roots *exist*. In some cases it is hard to find the roots. We can always use computer programs to produce very accurate approximations of the roots; that is the approach used for practical applications in real life (in physics, chemistry, economics, etc.). But for purposes of instruction in this course on differential equations, we'll just stick to polynomials for whose exact solutions are not hard to find. Those examples, though perhaps not typical of "real life," are adequate for learning techniques of solving differential equations; thus we won't need to bother with the computers.

For polynomials of degree two, finding the roots is easy: Just use the **quadratic formula**. The roots of $az^2 + bz + c = 0$ are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the same formula you learned in high school (and it has the same proof), but it has a new meaning now: We no longer require $b^2 - 4ac$ to be a positive number. There are now more possibilities than we had in high school.

- If $b^2 - 4ac$ is positive, then $\sqrt{b^2 - 4ac}$ is also positive, and we obtain two solutions, as in high school.
- If $b^2 - 4ac = 0$, then $\sqrt{b^2 - 4ac} = 0$, and the quadratic equation only has one solution (repeated roots), as in high school.
- Here's a new case: If $b^2 - 4ac < 0$, then $\sqrt{4ac - b^2}$ is a positive number, and $\pm\sqrt{b^2 - 4ac} = \pm\sqrt{(-1)(4ac - b^2)} = \pm i\sqrt{4ac - b^2}$, so we obtain two solutions to the quadratic equation.
- And there are more new cases, because $b^2 - 4ac$ might not be a real number at all. We allow a, b, c to be any complex numbers (as long as $a \neq 0$). The number $b^2 - 4ac$ could be *any* complex number; it will still have two square roots⁴. Finding those square roots is a little complicated; we'll look at the method a few pages from now.

⁴Except that zero has only one square root.

Instead of the quadratic formula, you could also solve quadratic equations by completing the square:

$$az^2 + bz + c = 0 \Rightarrow z^2 + \frac{b}{a}z + \frac{c}{a} = 0 \Rightarrow z^2 + \frac{b}{a}z + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} \Rightarrow$$

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \Rightarrow z + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In some problems, this procedure is easier than the quadratic formula, because it yields simplifications along the way.

Example. Solve $z^2 + 4z + 5 = 0$. *Solution.* Let $a = 1$, $b = 4$, $c = 5$. We get $z = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$. This can also be solved by completing the square: $z^2 + 4z + 4 + 1 = 0$, so $(z + 2)^2 - (-1) = 0$, so $(z + 2)^2 - (i)^2 = 0$, so $(z + 2 + i)(z + 2 - i) = 0$, so $z = -2 \pm i$.

Example: Solve $z^3 + 5z^2 + 6z - 12 = 0$. *Solution:* Since this is a “textbook problem” and you’re not expected to know how to solve all third degree equations, it must have at least one easy solution. Try a few small integers; you’ll find right away that $z = 1$ is a solution. Now divide out $z - 1$. You’ll find $z^3 + 5z^2 + 6z - 12 = (z - 1)(z^2 + 6z + 12)$. Thus, it remains for us to solve the quadratic equation $z^2 + 6z + 12$. By the quadratic formula or by completing the square, we find that that problem has solutions $-3 \pm \sqrt{3}i$. Thus, the three solutions are 1 , $-3 + \sqrt{3}i$, and $-3 - \sqrt{3}i$.

Exercises. Solve each equation. Write the answers in the form $a + bi$ where a and b are real numbers that are simplified as much as possible. In most cases, a quadratic equation should have two answers.

(6) $z^2 + 10z + 100 = 0$.

(7) $z^2 + 6iz - 9 = 0$.

(8) $z^2 + 10z + 25 = 0$.

(9) $z^2 + 10z + 16 = 0$.

(10) $z^2 - 8z + 36 = 0$.

(11) $z^3 + 10z^2 + 100z = 0$.

(12) $z^3 + 5z^2 + 15z + 11 = 0$.

Polar coordinates. The rule for multiplying complex numbers doesn’t look quite so complicated and arbitrary if we represent our points in the plane using polar coordinates.

For any real numbers a and b , we can find some real number θ and some nonnegative real number r such that

$$(a, b) = (r \cos \theta, r \sin \theta), \quad \text{or, in other notation,} \quad a + bi = r \cos \theta + ir \sin \theta.$$

This should be review for you. Actually, we have $r = |a + bi| = \sqrt{a^2 + b^2}$. We can also express θ in terms of a and b , but not quite so precisely: We have

$$\theta = \arctan\left(\frac{b}{a}\right) + (\text{some multiple of } \pi),$$

where the multiple is not uniquely determined, but must be chosen so that (a, b) and $(\cos \theta, \sin \theta)$ lie in the same quadrant. Some other facts that you will need to know are

$$\sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}, \quad \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \quad \sin \frac{\pi}{3} = \cos \frac{\pi}{2} = \frac{1}{2}\sqrt{3}.$$

Now for the **polar formula for complex multiplication**: It turns out that

to multiply complex numbers, we add the angles and multiply the radii.

In other words,

$$(r \cos \theta + ir \sin \theta) (s \cos \varphi + is \sin \varphi) = rs \cos(\theta + \varphi) + irs \sin(\theta + \varphi).$$

The proof is fairly easy. The expression $[rs \cos \theta \cos \varphi - rs \sin \theta \sin \varphi] + i [rs \cos \theta \sin \varphi + rs \sin \theta \cos \varphi]$ is equal to the left side of the equation above, by the Cartesian formula for complex multiplication; and it is equal to the right side of the equation above, by the trigonometric formulas for $\cos(\theta + \varphi)$ and $\sin(\theta + \varphi)$.

Here's an example of multiplication:

$$\left(3 \cos \frac{\pi}{2} + 3i \sin \frac{\pi}{2}\right) \left(5 \cos \frac{\pi}{3} + 5i \sin \frac{\pi}{3}\right) = 15 \cos \frac{5\pi}{6} + 15i \sin \frac{5\pi}{6}.$$

The left and right side of that equation simplify to:

$$3i \cdot \left(\frac{5}{2} + \frac{5}{2}\sqrt{3}i\right) = -\frac{15}{2}\sqrt{3} + \frac{15}{2}\sqrt{3}i.$$

The polar representation shows that, when we multiply a collection of complex numbers by some particular complex number $r \cos \theta + ir \sin \theta$, the effect is to rotate the plane through an angle of θ and stretch (or shrink) the plane by a magnification factor of r . This effect is of great importance in physics and engineering: there are many physical processes that involve rotation and/or stretching — e.g., the rotation of a motor, the rotation of a star. Such processes can be expressed and computed more easily if we use complex numbers in our representation. There is nothing “imaginary” about this; these processes are very real and concrete.

Exercise. Simplify — i.e., write in the form $p + qi$, where p and q are real numbers expressed without mention of sine or cosine.

$$(13) \left(3 \cos \frac{\pi}{4} + 3i \sin \frac{\pi}{4}\right) \left(5 \cos \frac{2\pi}{3} + 5i \sin \frac{2\pi}{3}\right).$$

Powers of complex numbers. What happens when we use the polar formula for multiplication, to multiply a complex number times itself one or more times? Again, we multiply the radii and add the angles. We get

$$(r \cos \theta + ir \sin \theta)^2 = r^2 \cos(2\theta) + ir^2 \sin(2\theta),$$

$$(r \cos \theta + ir \sin \theta)^3 = r^3 \cos(3\theta) + ir^3 \sin(3\theta),$$

$$(r \cos \theta + ir \sin \theta)^4 = r^4 \cos(4\theta) + ir^4 \sin(4\theta),$$

and in general (this is **De Moivre's formula**)

$$(r \cos \theta + ir \sin \theta)^n = r^n \cos(n\theta) + ir^n \sin(n\theta).$$

Here is an example: Find the number $(2 + 2i)^7$. The most obvious method is to use the Cartesian formula and just multiply everything out. The procedure is rather tedious; you'll get

$$2^7 [1 + 7i + 21i^2 + 35i^3 + 35i^4 + 21i^5 + 7i^6 + i^7]$$

and then you simplify that to get $1024 - 1024i$. An easier method is to first rewrite in polar form

$$2 + 2i = 2^{3/2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

(You find that using $\arctan(2/2) = \pi/4$ and $\sqrt{2^2 + 2^2} = 2^{3/2}$.) Then compute

$$(2 + 2i)^7 = (2^{3/2})^7 \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) = (1024\sqrt{2}) \left(\frac{1}{\sqrt{2}} + i \frac{-1}{\sqrt{2}} \right) = 1024 - 1024i.$$

Perhaps this method is only a little easier, but it can be much much easier in cases where the exponent is greater than 7. Also, it is of theoretical importance; it leads to some ideas developed in the next section.

Exercise. Simplify — i.e., write in the form $p + qi$, where p and q are real numbers represented in a form that does not mention sine or cosine.

$$(14) (\sqrt{3} + i)^6$$

$$(15) (-i)^9$$

Roots of complex numbers. We can reverse that last process, to find n th roots. If we're given some number $a + bi$ and we want to find its n th roots, we just look for some complex number $r \cos \theta + ir \sin \theta$ that will satisfy

$$r^n \cos(n\theta) + ir^n \sin(n\theta) = a + bi.$$

A general formula is given in a box below, but initially it may be a bit difficult to understand; some students will understand it better if they first look at the particular examples given after it.

Our first step is to represent the given number $a + bi$ in polar coordinates: We find some numbers s and ψ such that $a + bi = s \cos \psi + is \sin \psi$. As usual, ψ is not uniquely determined. For most applications of polar coordinates, we just pick any one convenient value for ψ , among the many possible values. But when we're taking n th roots, we need n different angles. We write

$$a + ib = s \cos \psi + is \sin \psi = s \cos(\psi + 2\pi) + is \sin(\psi + 2\pi) = \\ s \cos(\psi + 4\pi) + is \sin(\psi + 4\pi) = \cdots = s \cos(\psi + 2(n-1)\pi) + is \sin(\psi + 2(n-1)\pi).$$

Those angles all yield the same point in the plane. But when we divide those angles by n , we do *not* get n representations of the same point; we get n different points.

The n th roots of $s(\cos \psi + i \sin \psi)$ are

$$\sqrt[n]{s} \left(\cos \frac{\psi}{n} + i \sin \frac{\psi}{n} \right), \quad \sqrt[n]{s} \left(\cos \frac{\psi+2\pi}{n} + i \sin \frac{\psi+2\pi}{n} \right), \\ \sqrt[n]{s} \left(\cos \frac{\psi+4\pi}{n} + i \sin \frac{\psi+4\pi}{n} \right), \quad \sqrt[n]{s} \left(\cos \frac{\psi+6\pi}{n} + i \sin \frac{\psi+6\pi}{n} \right), \\ \cdots, \quad \sqrt[n]{s} \left(\cos \frac{\psi+2(n-1)\pi}{n} + i \sin \frac{\psi+2(n-1)\pi}{n} \right).$$

In summary,

If $a + bi$ has angle ψ , then the n th roots of $a + bi$ are

$$\sqrt[n]{|a + bi|} \cos \left(\frac{\psi + 2k\pi}{n} \right) + i \sqrt[n]{|a + bi|} \sin \left(\frac{\psi + 2k\pi}{n} \right) \quad (k = 0, 1, 2, \dots, n-1).$$

Those n points are equally spaced along a circle, centered at the origin, with radius $\sqrt[n]{|a + bi|}$.

(Actually, if you prefer you could use $k = 1, 2, 3, \dots, n$, or use any n consecutive integers for k . You could even use nonconsecutive integers; just be sure to use n integers that, when divided by n , yield different remainders.)

Here is an example: Let's find the four fourth roots of $-2 + 2\sqrt{3}i$. Here $a = -2$ and $b = 2\sqrt{3}$, so $|a + bi| = \sqrt{a^2 + b^2} = \sqrt{4 + 12} = 4$. Then $b/a = -\sqrt{3}$, which has arctangent equal to $-\pi/3$.

Note that the arctangent alone is *not* quite enough to tell us what angle we need for ψ ; we cannot jump to the conclusion that $\psi = -\pi/3$. Indeed, *both* of the points $2 - 2\sqrt{3}i$ and $-2 + 2\sqrt{3}i$ yield arctangent equal to $-\pi/3$, but those two points are in different quadrants; their angles differ by π . The point we're interested in is $-2 + 2\sqrt{3}i$, which is in the upper left quadrant of the plane, so we need $\frac{\pi}{2} \leq \psi \leq \pi$. To get that result, take the number $-\pi/3$ and add π ; that gives us $\psi = 2\pi/3$.

Thus, we have found that $-2 + 2\sqrt{3}i = 4 \cos\left(\frac{2\pi}{3}\right) + 4i \sin\left(\frac{2\pi}{3}\right)$. The formula now tells us immediately that the four fourth roots of $-2 + 2\sqrt{3}i$ are

$$\sqrt[4]{4} \cos\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right) + \sqrt[4]{4}i \sin\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right) \quad (k = 0, 1, 2, 3).$$

Let's simplify that a bit. Note that $\sqrt[4]{4} = \sqrt{2}$. Thus the four fourth roots are

$$\begin{aligned} \sqrt{2} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right), & \quad \sqrt{2} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right), \\ \sqrt{2} \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right), & \quad \sqrt{2} \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right). \end{aligned}$$

Now make use of these values of the trigonometric functions:

α	$\cos \alpha$	$\sin \alpha$
$\pi/6$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
$2\pi/3$	$\frac{-1}{2}$	$\frac{1}{2}\sqrt{3}$
$7\pi/6$	$\frac{-1}{2}\sqrt{3}$	$\frac{-1}{2}$
$5\pi/3$	$\frac{1}{2}$	$\frac{-1}{2}\sqrt{3}$

Simplifying, we find that the four fourth roots of $-2 + 2\sqrt{3}i$ are

$$\frac{1}{2}\sqrt{6} + \frac{1}{2}\sqrt{2}i, \quad \frac{-1}{2}\sqrt{2} + \frac{1}{2}\sqrt{6}i, \quad \frac{-1}{2}\sqrt{6} + \frac{-1}{2}\sqrt{2}i, \quad \frac{1}{2}\sqrt{2} + \frac{-1}{2}\sqrt{6}i.$$

We can check these answers using the Cartesian formula for multiplication.

In some problems, an easier method is available — we can just solve a polynomial equation. For instance, to find the three cube roots of 8, start with the polynomial equation $z^3 - 8 = 0$. We already know that one of the three cube roots is 2, so factor it out; we get $(z - 2)(z^2 + 2z + 4) = 0$. Now solve $z^2 + 2z + 4 = 0$, either by the quadratic formula or by completing the square. We obtain $z = -1 \pm \sqrt{3}i$. Thus the three cube roots of 8 are $\boxed{2, -1 + \sqrt{3}i, \text{ and } -1 - \sqrt{3}i}$. As a partial check of your answer, you might sketch all three of those points on one graph (keeping in mind that $\sqrt{3}$ is approximately 1.7); those three points should be equally spaced along a circle centered at the origin.

We could get the same answer by the trigonometric method. The number 8 has radius 8 and angle 0. Its three cube roots should have radius 2 and angles 0, $2\pi/3$, and $4\pi/3$. Thus the three cube roots are

$$2 \cos(0) + 2i \sin(0), \quad 2 \cos\left(\frac{2\pi}{3}\right) + 2i \sin\left(\frac{2\pi}{3}\right), \quad 2 \cos\left(\frac{4\pi}{3}\right) + 2i \sin\left(\frac{4\pi}{3}\right).$$

Simplify those three numbers; you get the three boxed numbers of the previous paragraph.

Exercises. Evaluate and simplify:

- (16) What are the two square roots of $2i$?

(17) What are the four fourth roots of 1 ?

(18) What are the three cube roots of -1 ?

Historical remarks: To a large extent, complex numbers were discovered in the 16th century, when mathematicians were trying to solve cubic and quartic equations. However, complex numbers were not really understood until the points-in-a-plane idea developed, a couple of hundred years later.

The quadratic formula was known, in a less abstract form, thousands of years ago. The cubic formula was found in the 16th century, but we generally don't teach it to our college students, because it is too complicated. Here it is: Divide out the coefficient of z^3 ; thus we can assume that the problem we're trying to solve is $z^3 + bz^2 + cz + d = 0$. Then the solution is

$$z = \sqrt[3]{-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2} + \sqrt{\left(-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2}\right)^2 + \left(\frac{c}{3} - \frac{b^2}{9}\right)^3}} \\ + \sqrt[3]{-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2} - \sqrt{\left(-\frac{b^3}{27} + \frac{bc}{6} - \frac{d}{2}\right)^2 + \left(\frac{c}{3} - \frac{b^2}{9}\right)^3}} - \frac{b}{3}.$$

That looks like six solutions, since there are 3 cube roots and 2 square roots, but actually there is some repetition involved, and we only end up with 3 solutions. (I will *not* require you to memorize this formula. I just thought you might like to see it once, for curiosity's sake.)

For polynomials of degree 4, there is also an analogous formula, but it is much much more complicated, so I won't write it down here. For polynomials of degree 5 or higher, there are five solutions (we know that by the Fundamental Theorem of Algebra); but they can't be given by an analogous formula. That surprising fact was proved by Abel in 1826. They can't be expressed using just $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$, and arithmetic operations; some additional functions are needed. One such additional function, for instance, is the inverse of the function $x^5 + x$.

Extending the basic transcendental functions. There are several ways to define the cosine function. You probably learned it first in terms of a unit circle. If you draw an angle θ from the horizontal axis, then $\cos \theta$ is the distance in the x -direction, etc. But here is another way to obtain that same cosine function:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \dots$$

This is an *infinite series*. Remember what that means: You don't really try to add up infinitely many terms; that would take more time than we have. But the later terms in the series are small, and so if we just add up the first few terms (e.g., the first million terms or so), we get a pretty accurate approximation of where the sum is going⁵.

⁵I'm glossing over some technicalities here, since we don't need them yet at this point in our study of differential equations; but some sticklers for accuracy will complain if I don't at least mention the technicalities. It's not quite enough for the individual terms to get small; their sums have to get small also. For instance, the individual terms in the series $1 + (1/2) + (1/3) + (1/4) + \dots$ do get small, but their partial sums do not converge to a finite number. In some sense, the problem is that the numbers $1/n$ do converge to 0, but rather slowly. In contrast, the terms in the series for $\cos z$ converge to 0 very fast. That will also be the case for most of the series we study for differential equations. Just how fast is "fast enough" is a technical matter that we don't need to discuss right now.

This whole discussion assumes you're using *radians* for your measurement of angles. If you use degrees, you'll have to replace the formulas given here with other, more complicated formulas. For instance,

$$\cos(z \text{ degrees}) = 1 - \left(\frac{\pi}{180}\right)^2 \frac{z^2}{2!} + \left(\frac{\pi}{180}\right)^4 \frac{z^4}{4!} - \left(\frac{\pi}{180}\right)^6 \frac{z^6}{6!} + \left(\frac{\pi}{180}\right)^8 \frac{z^8}{8!} - \dots$$

Radians yield simpler formulas; that's why radians are preferred by mathematicians and are used in math classes. In all math classes, you should assume that all angles are expressed in radians unless some other measurement is specified explicitly. And make sure that your calculator is set to "radians," not "degrees," when you use it on a math test.

The unit circle explanation only defines $\cos(z)$ when z is a real number. But the infinite series makes sense if z is any *complex* number! The two definitions give the same result when z is real, but the infinite series extends the definition.

And this approach also works for the functions $\sin z$ and e^z . Here are the formulas for those functions:

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \dots \\ e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots \end{aligned}$$

This gives us a *natural* way of *extending* these functions — they still take their familiar old values when z is a real number, but the functions now are defined for all z in the complex plane. And they preserve many of their familiar properties — e.g.,

$$\begin{aligned} \cos(\theta + \varphi) &= \cos \theta \cos \varphi - \sin \theta \sin \varphi \\ \sin(\theta + \varphi) &= \cos \theta \sin \varphi + \sin \theta \cos \varphi \\ e^{\theta + \varphi} &= e^\theta e^\varphi \end{aligned}$$

for any *complex* numbers θ and φ . Also,

$$\frac{d \sin z}{dz} = \cos z, \quad \frac{d \cos z}{dz} = -\sin z, \quad \frac{d e^z}{dz} = e^z, \quad \text{where} \quad \frac{df(z)}{dz} = \lim_{u \rightarrow z} \frac{f(z) - f(u)}{z - u}.$$

In that last limit, z approaches u from all directions, not just from the left or right.

We also get some new formulas that may be surprising:

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad e^{-i\theta} = \cos \theta - i \sin \theta, \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Because of those last few formulas, we can do all of our differential equations in terms of exponentials, without using sines and cosines; this turns out to be a big simplification in our reasoning.

Transforming complex solutions to real solutions. The theory of constant coefficient differential equations yields solutions involving terms such as e^{kt} , where the constant k is a *complex* number. However, it is a general principle that

any initial value problem that involves only real numbers in the problem, should involve only real numbers in the answer, and customarily we rewrite the answer so that it only mentions real numbers

even though complex numbers may be convenient as an intermediate step in getting to that answer. How can we rewrite our answer so that it will not involve anything but real numbers?

In applications the complex numbers generally⁶ occur in pairs, so that we end up with terms like $Ae^{(P+Qi)t} + Be^{(P-Qi)t}$ where P and Q are some particular constants that we find, and A and B are arbitrary constants that we don't find (or that we find by plugging in some given initial conditions). Keep in mind that A and B are *arbitrary* constants — they need not be real numbers. Now compute as follows:

$$\begin{aligned} Ae^{(P+Qi)t} + Be^{(P-Qi)t} &= e^{Pt} [Ae^{Qit} + Be^{-Qit}] \\ &= e^{Pt} [A(\cos Qt + i \sin Qt) + B(\cos Qt - i \sin Qt)] \\ &= e^{Pt} [(A+B) \cos Qt + (A-B)i \sin Qt] \\ &= e^{Pt} (\alpha \cos Qt + \beta \sin Qt) \end{aligned}$$

where α and β are new arbitrary constants. They are related to the old arbitrary constants:

$$\alpha = A + B, \quad \beta = (A - B)i \quad (*)$$

but in most cases we don't need to keep track of (*), because we won't need to go back to the original A and B anyway. In most cases, all we need to know is that

any function that can be expressed in the form $Ae^{(P+Qi)t} + Be^{(P-Qi)t}$ for some constants A and B , can also be expressed in the form $e^{Pt} (\alpha \cos Qt + \beta \sin Qt)$ for some constants α and β .

We can forget about the intermediate computation steps. And in applications, usually α and β turn out to be real numbers (though A and B might not be real).

Here is an example, taken from page 164 problem 11 of our textbook: We want to solve $y'' - 4y' + 5y = 0$. The associated polynomial equation is $k^2 - 4k + 5 = 0$. That has solutions $k = 2 \pm i$. Hence the general solution of the differential equation can be written as

$$y = Ae^{(2+i)t} + Be^{(2-i)t} = e^{2t}(Ae^{it} + Be^{-it})$$

with arbitrary constants A and B . But that equals $\boxed{y = e^{2t}(\alpha \cos t + \beta \sin t)}$ for some *other* arbitrary constants α and β . That's the answer we want. We can forget about the intermediate steps and the A and B ; we don't really need those and don't need to bother with the computations of how they are related to α and β .

By the way, if at some point you *do* need to go back to the exponential expressions, the transformation (*) is reversible. To see that, just multiply both sides of the second equation by i ; that yields

$$\alpha = A + B, \quad i\beta = -A + B.$$

Now adding or subtracting, and dividing by 2, we obtain

$$A = \frac{\alpha - i\beta}{2}, \quad B = \frac{\alpha + i\beta}{2}. \quad (**)$$

⁶And even if we don't get a pair, we could *make* a pair, by taking one of A, B to be 0.