A geometric approach to the conjugacy problem for semisimple Lie groups

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Conjugacy Length Function

\( G \) group with length function \( | \cdot | : G \to [0, \infty) \)
(e.g. word length if finitely generated).

Definition (Conjugacy length function)

\( \text{CLF}_{G} : [0, \infty) \to [0, \infty) \)
minimal function satisfying:

For \( x \geq 0, u, v \in G \) such that \( |u| + |v| \leq x \), then
\( u \) is conjugate to \( v \) \( \iff \exists g \in G \) such that
(i) \( gug^{-1} = v \) and
(ii) \( |g| \leq \text{CLF}_{G}(x) \).

Lemma
\( \Gamma \) finitely generated with solvable WP, \( | \cdot | \) word length. Then:
Conjugacy problem is solvable \( \iff \text{CLF}_{\Gamma} \) is recursive.

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Example: free groups

$F$ free group, finite generating set $X$.

$u, v$ reduced words on $X \cup X^{-1}$.

e.g. $u = aabbbaba^{-1}$
$v = babababba^{-1}b^{-1}$
Example: free groups

Let $F$ be a free group with finite generating set $X$. Let $u, v$ be reduced words on $X \cup X^{-1}$.

Algorithm to solve conjugacy problem

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The conjugator will be a product of subwords of $u$ and $v$. Hence

$$\text{CLF}_F(x) \leq x.$$

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$g = bababa^{-1}$
$v = gug^{-1}$
## Known results include:

<table>
<thead>
<tr>
<th>Class of groups</th>
<th>$\text{CLF}(x)$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Hyperbolic groups</td>
<td>linear</td>
<td>Bridson–Haefliger</td>
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<tr>
<td>CAT(0) and biautomatic groups</td>
<td>$\leq \exp(x)$</td>
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<td>RAAGs &amp; special subgroups</td>
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<td>Crisp–Godelle–Wiest</td>
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<td>2-Step Nilpotent</td>
<td>quadratic</td>
<td>Ji–Ogle–Ramsey</td>
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<tr>
<td>$\pi_1(M)$ where $M$ prime 3–manifold</td>
<td>$\leq x^2$</td>
<td>Behrstock–Druţu, S</td>
</tr>
<tr>
<td>Free solvable groups</td>
<td>$\leq x^3$</td>
<td>S</td>
</tr>
</tbody>
</table>

Plus:

- wreath products (S),
- group extensions (S),
- relatively hyperbolic groups (Ji–Ogle–Ramsey, Z. O’Conner, Bumagin).
Mapping class groups

$S$ connected, oriented surface of genus $g$ and $p$ punctures.

$$\text{Mod}(S) = \text{Homeo}^+(S)/\sim$$
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Theorem (Masur-Minsky '00; Behrstock-Druţu '11; J. Tao '13)

$$\text{CLF}_{\text{Mod}(S)}(x) \preceq x.$$
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Theorem (Masur-Minsky ’00; Behrstock-Druțu ’11; J. Tao ’13)

\[ \text{CLF}_{\text{Mod}(S)}(x) \leq x. \]

Question: What about for arithmetic groups? Or $\text{Out}(F_n)$?
Semisimple Lie groups

$G$ real semisimple Lie group, finite centre and no compact factors.

d$G$ left-invariant Riemannian metric.

$X = G/K$ associated symmetric space.

$\Gamma < G$ non-uniform lattice.

e.g. $\text{SL}_n(\mathbb{Z}) < \text{SL}_n(\mathbb{R})$ and $X = \text{SL}_n(\mathbb{R})/\text{SO}(n)$. 

Jordan decomposition:

Each $g \in G$ has unique decomposition as $g = su$ where:

- $s$ is semisimple (translates along an axis in $X$);
- $u$ is unipotent (fixes a point in the boundary of $X$),

and $s, u$ commute.
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Complete Jordan decomposition:

Each $g \in G$ has unique decomposition as

$$g = kau$$

where:

- $k$ is elliptic
- $a$ is real hyperbolic
- $u$ is unipotent

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Complete Jordan decomposition:

Each \( g \in G \) has unique decomposition as

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where:

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Slope

Let \( a \in G \) be real hyperbolic. The slope of \( a \) tells you the location of translated geodesics in Weyl chambers. (It lies in \( \partial X/G \)).
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Theorem (S ’14)

Fix slope $\xi$. Then there exists $d_\xi, \ell_\xi > 0$ such that for $a, b \in G$ real hyperbolic of slope $\xi$ and such that $|a|, |b| > d_\xi$.

Note: $|a| = d_G(1, g)$
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**Theorem (S ’14)**

*Fix slope $\xi$. Then there exists $d_\xi, \ell_\xi > 0$ such that for $a, b \in G$ real hyperbolic of slope $\xi$ and such that $|a|, |b| > d_\xi$.*

$a$ is conjugate to $b \iff \exists g \in G$ such that

(i) $ga = bg$

(ii) $|g| \leq \ell_\xi(|a| + |b|)$.

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Assume $G$ is higher rank and $\Gamma < G$ is an irreducible lattice.

**Corollary**

*Fix a slope $\xi$. Then there exists $\ell_\xi > 0$ such that $a, b \in \Gamma$, real hyperbolic of slope $\xi$, are conjugate if and only if there is a conjugator $g \in G$ such that*

$$|g| \leq \ell_\xi(|a|_\Gamma + |b|_\Gamma).$$

*Note: $|a|_\Gamma$ is word length.*
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If $Z_\Gamma(a)$ is virtually $\mathbb{Z}$, then $g$ can be “pushed” to a conjugator $\gamma$ in $\Gamma$, retaining the linear bound on its length.
Idea of proof

Theorem

\( a \) is conjugate to \( b \) \iff \exists g \in G \text{ such that } (i) \ ga = bg \text{ and } (ii) \ |g| \leq \ell_\xi(|a| + |b|).
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Assume slope \(\xi\) is regular. Then

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\text{Min}(a) := \left\{ x \in X \mid d(x, ax) = \inf_{y \in X} d(y, ay) \right\}
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and \(\text{Min}(b)\) are maximal flats.
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**Lemma**

- If \( ga = bg \) then \( g \text{ Min}(a) = \text{Min}(b) \);
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Lemma

- If \( ga = bg \) then \( g \text{Min}(a) = \text{Min}(b) \);
- if \( g \text{Min}(a) = \text{Min}(b) \) then \( \exists k \in G \) fixing a point in \( \text{Min}(a) \) such that
  \[ (gk)a = b(gk). \]
Idea of proof, continued

Minimal distance between the flats is important — corresponds to length of shortest conjugator.

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Thank you for your attention!