# ALE RICCI-FLAT KÄHLER SURFACES AND WEIGHTED PROJECTIVE SPACES

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ABSTRACT. We show that the explicit ALE Ricci-flat Kähler metrics constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin and Kronheimer, and their free quotients are metrics obtained by Tian-Yau techniques. The proof relies on a construction of good compactifications of $\mathbb{Q}-$Gorenstein deformations of quotient surface singularities as log del Pezzo surfaces with only cyclic quotient singularities at infinity. We provide an intrinsic geometric description of the compactifications.

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## 1. INTRODUCTION

The classification of asymptotically locally Euclidean (ALE) Ricci-flat Kähler surfaces was accomplished by Kronheimer [13, 14] in the simply-connected case, and completed by the second author [18] in the non-simply connected case (see also [21]). More precisely, we have the following:

**Theorem 1.1** ([13, 14, 18]). *Let $(M, J, g, \omega_g)$ be a smooth ALE Ricci-flat Kähler surface asymptotic to $\mathbb{C}^2/G$, where $G$ is a finite subgroup of $U(2)$ acting freely on $\mathbb{C}^2 \setminus \{0\}$. Then, the complex manifold $(M, J)$ can be obtained as the minimal resolution of a fiber of a one-parameter $\mathbb{Q}-$Gorenstein deformation of the quotient*

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singularity $\mathbb{C}^2/G$. *Given the Kähler class* $\Omega = [\omega_g] \in H^2(M, \mathbb{R})$, *then* $g$ *is the unique ALE Ricci-flat Kähler metric* $g$ *in the class.*

*Moreover, any complex surface* $(M, J)$ *obtained by the above construction admits a unique ALE Ricci-flat Kähler metric in any Kähler class* $\Omega$.

The classification of ALE Ricci-flat Kähler surfaces [18] is in direct correspondence with the explicit description of quotient singularities which admit $\mathbb{Q}-$Gorenstein smoothings, which is due to Kollár and Shepherd-Barron [10]. Following their terminology, we call such singularities of class $T$. The possible singularities are either rational double points, i.e. singularities of type $A_k, D_k, E_6, E_7$ and $E_8$, or finite cyclic singularities of the type $\dfrac{1}{dn^2}(1, dnm - 1)$. The rational double points correspond to the case when the surfaces $M$ are simply connected, and then the metrics are hyperkähler. They are associated to Gorenstein smoothings and have trivial canonical line bundle. In the second case, the surfaces $M$ have finite cyclic fundamental group and the metrics are non-hyperkähler. They are $\mathbb{Q}-$Gorenstein smoothings, and have torsion canonical line bundle.

The ALE Ricci-flat Kähler metrics were explicitly constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin and Kronheimer [4, 6, 8, 13] in the simply connected case. In [18], the second author completes the list in the non-simply connected case by adding the free quotients of certain $A-$type manifolds. The ALE property was used in an essential way to obtain both the classification and the uniqueness of the ALE Ricci-flat Kähler metric in a given Kähler cohomology class. Both the hyperkähler manifolds of type $A_k$ and $D_k$ [7, 3], and the non-hyperkähler manifolds [18] admit asymptotically locally flat (ALF) Ricci-flat Kähler metrics, which have cubic volume growth. The ALE and the ALF metrics are given by explicit formulas [4, 6, 8, 7, 3].

Another method of constructing Kähler-Einstein metrics on non-compact manifolds is by solving the complex Monge-Ampère equations. The technique is due to Tian-Yau [19, 20] and Bando-Kobayashi [1]. In particular, they proved the existence of Ricci-flat Kähler metrics on the complement of a divisor in a complex orbifolds, under certain technical conditions. In general, these metrics are not ALE, and if one insist that the ambient space is a smooth surface, the examples are scarce [19]. As we are only interested in manifolds with ALE Ricci-flat Kähler metrics, the relevant results are in the context of the complement of a divisor in an **orbifold** surface, and are due to Tian and Yau in [20]. A more detailed description of their results is included in section 3. The Tian-Yau metrics are obtained by analytical methods, and the existence of the metric is given implicitly. We compare the two constructions, and we find:

**Theorem A.** *The explicit ALE hyperkähler metrics on complex surfaces constructed by Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer, and their finite free quotients can be obtained by Tian-Yau techniques.*

To prove this theorem we first need to construct a good compactification of a fiber of a deformation. The proof of the Theorem A follows from Theorem B below, and the uniqueness part of Theorem 1.1.

Starting from the classification of ALE Ricci-flat Kähler surfaces, we give a natural setting where their methods apply. Using Kollár and Shepherd-Barron's classification, we consider compactifications of a fiber of a $\mathbb{Q}-$Gorenstein deformation as a hypersurface in a weighted projective space. These compactifications obey the Tian-Yau conditions and yield ALE metrics on the complement of the divisor at infinity. The properties of the compactifications are summarized in the following theorem:

**Theorem B.** *Let $M$ be a fiber of a $\mathbb{Q}-$Gorenstein deformation of a singularity of class $T$. Then $M$ embeds into a log del Pezzo surface, $\overline{M}$, as the complement of a smooth, rational curve, which is a rational multiple of the anticanonical divisor. The singularities along the divisor at infinity are all isolated finite cyclic quotients. Moreover, if $M$ is associated to a finite cyclic quotient singularity then there are infinitely many minimal compactifications with the above properties.*

We say that a compactification is minimal if there is no rational component of the divisor at infinity of self-intersection $(-1)$ and passing only through smooth points of $\overline{M}$. We recall [9] that a normal complex surface $\overline{M}$ with at worst log terminal singularities, i.e. quotient singularities, is called a log del Pezzo surface if its anti-canonical divisor $-K_{\overline{M}}$ is ample. We should point out that our constructions verify stronger conditions: if we denote by $C$ the curve at infinity, then the $\mathbb{Q}-$Cartier divisors $C$ and $-(K_{\overline{M}} + C)$ are both ample.

Our construction produces infinitely many minimal compactifications with the properties stated in Theorem B when the manifold is associated to a cyclic singularity, but only one for singularities of type $D_k, E_6, E_7$ and $E_8$. It would be interesting to see if there are other minimal compactifications in these cases satisfying the same properties.

Given a singularity of type $T$, and a $\mathbb{Q}-$Gorenstein smoothing, the underlying smooth manifold of a generic fiber is the Milnor fiber of the singularity. An arbitrary fiber of a deformation might admit singularities, which are all rational double points. Hence, if we consider the associated minimal resolution we obtain a manifold diffeomorphic to the Milnor fiber. The general construction of considering a deformation followed by the minimal resolution of the rational double points, constructs a family of complex structures on the Milnor fiber. Throughout this paper we emphasize which particular complex structure we consider.

Compactifications of Milnor fibers were also constructed by Saito [17] for rational double points. More generally, Némethi and Popescu-Pampu [16] constructed (smooth) compactifications of the Milnor fiber for each irreducible component of the versal deformation space of finite cyclic quotient singularities. Our construction generalizes Saito's compactifications [17] in several aspects. The construction in Theorem B is quite different from the Némethi-Popescu-Pampu compactification, and it would be interesting to compare the two constructions.

It will also be interesting to check if the known ALF Ricci-flat Kähler metrics can also be recovered by the Tian-Yau methods.

The last part of the paper is devoted to giving an intrinsic geometric description of the compactification the Milnor fiber of a singularity of type $\frac{1}{dn^2}(1, dnm - 1)$.

**Theorem C.** *Let $M$ be the minimal resolution of a generic fiber of a one-parameter $\mathbb{Q}$−Gorenstein deformation of a singularity of type $\frac{1}{dn^2}(1, dnm - 1)$, where $n, m$ are relatively prime, and $a, c$ any positive integers such that $\gcd(c, n) = 1$ and $am = c \mod n$. Then $M$ is biholomorphic to the complement of the proper transforms of the coordinate axes $(x = 0)$ and $(w = 0)$ in the blow-up of $d$ smooth points along the line $(x = 0) \setminus (w = 0)$ in the weighted projective space $\mathbb{P}_{[x:z:w]}(a, c, n)$.*

In particular, we obtain a geometric description of the minimal resolution of an $A_{d-1}$ singularity as the complement of two divisors:

**Corollary 1.2.** *The minimal resolution of a singularity of type $A_{d-1}$ is obtained as the complement of the proper transforms of the coordinate axes $(x = 0)$ and $(w = 0)$ in the iterated blow-up of order $d$ of a smooth point on the line $(x = 0) \setminus (w = 0)$ in the weighted projective space $\mathbb{P}_{[x:z:w]}(a, c, 1)$, for any positive integers $a, c$.*

**Notation and conventions.** We always work over the field of complex numbers $\mathbb{C}$.

(1) Let $n$ be a positive integer, $\mu_n$ the multiplicative group of the $n^{th}$−roots of unity, and $\epsilon \in \mu_n$ a generator. We denote by the symbol $\frac{1}{n}(a_1, \ldots, a_m)$ the action of the group $\mu_n$ on $\mathbb{C}^m$ defined by

$$\epsilon(z_1, \ldots, z_m) = (\epsilon^{a_1} z_1, \ldots, \epsilon^{a_m} z_m),$$

where $(a_1, \ldots, a_m) \in \mathbb{Z}^m$. We refer to the corresponding quotient space as a singularity of the type $\frac{1}{n}(a_1, \ldots, a_m)$. Whenever necessary, we include the coordinates

$$\mathbb{C}^m_{\mathbf{z}} / \frac{1}{n}(a_1, \ldots, a_m),$$

where $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m$, when denoting this cyclic quotient singularity.

If $\rho$ is a different choice of the generator of the group $\mu_n$, there exists an integer $k$, $\gcd(k, n) = 1$, such that $\epsilon = \rho^k$, and we obtain an equivalent notation of the singularity of the form $\frac{1}{n}(b_1, \ldots, b_m)$, where $b_i = ka_i \mod n$ for $i = 1, \ldots, m$.

Let $f \in \mathbb{C}[z_1, \ldots, z_m]$. If $(f = 0) \subseteq \mathbb{C}^m$ is invariant under the above action of $\mu_n$, we will denote by $(f = 0)/\mu_n$ the induced quotient. If necessary, we explicitly include in the notation the action of $\mu_n$, as above.

(2) Let $(w_0, w_1, \ldots, w_m)$ be an $(m + 1)-$tuple of positive integers. The weighted projective space $\mathbb{P}(w_0, \ldots, w_m)$ is defined as the quotient of $\mathbb{C}^{m+1} \setminus \{0\}$ by the $\mathbb{C}^*-$action given by

$$\lambda(z_0, z_1, \ldots, z_m) = (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \ldots, \lambda^{w_m} z_m).$$

Following [5], we say that the weighted projective space $\mathbb{P}(w_0, \ldots, w_m)$ is *well-formed* if $\gcd(w_0, \ldots, \widehat{w_i}, \ldots, w_m) = 1$, for each $i = 0, \ldots, m$.

The weighted projective space is covered by the standard charts $U_{z_i} = (z_i \neq 0) \simeq \mathbb{C}^m_{\mathbf{Z_i}} / \frac{1}{w_i}(w_0, \ldots, \widehat{w_i}, \ldots, w_m)$ centered at $P_i = [0 : \ldots : 1 : \ldots : 0]$, $i = 0, \ldots, m$. The affine coordinates $\mathbf{Z_i} = (Z_{0i}, \ldots, \widehat{Z_{ii}}, \ldots, Z_{mi})$ satisfy

$$Z_{ji}^{w_i} = \frac{z_j^{w_i}}{z_i^{w_j}}, \; j = 0, \ldots, \widehat{i}, \ldots, m,$$

and are well-defined up to the corresponding action of $\mu_{w_i}$. Whenever the coordinates are relevant in the descriptions of the spaces involved, we indicate them as

$$\mathbb{P}_{[z_0:\cdots:z_n]}(w_0, \ldots, w_n).$$

However, to simplify the notations we omit the indices when it is clear from the context.

(3) All of the varieties discussed in this paper have only mild singularities. In particular, they are all $\mathbb{Q}-$ factorial [9]. We will not distinguish between their $\mathbb{Q}-$Cartier divisors and Weil divisors with rational coefficients.

## 2. A COMPACTIFICATION OF THE MILNOR FIBER

We begin by recalling the terminology and some general results contained in [10, 15].

**Definition 2.1.** *A normal variety $X$ is $\mathbb{Q}-$Gorenstein if it is Cohen-Macaulay and a multiple of the canonical divisor is Cartier.*

**Definition 2.2.** *A flat map $\pi : \mathcal{X} \to \Delta \subseteq \mathbb{C}$ is called a one-parameter $\mathbb{Q}-$Gorenstein smoothing of a normal singularity $(X, x)$ if $\pi^{-1}(0) = X$ and there exists $U \subseteq \Delta$ an open neighborhood of $0$ such that the following conditions are satisfied.*

  i) *$\mathcal{X}$ is $\mathbb{Q}-$Gorenstein,*
  ii) *The induced map $\mathcal{X} \to U$ is surjective,*
  iii) *$X_t = \pi^{-1}(t)$ is smooth for every $t \in U \setminus \{0\}$.*

The following result of Kollár and Shepherd-Barron [10] gives a complete description of the singularities admitting a one-parameter $\mathbb{Q}-$Gorenstein smoothing:

**Proposition 2.3** (Kollár, Shepherd-Barron [10])**.** *The quotient singularities admitting a one-parameter $\mathbb{Q}-$Gorenstein smoothing are the following:*

  1) *Rational double points;*
  2) *Cyclic singularities of the type $\frac{1}{dn^2}(1, dnm - 1)$, for $d > 0$, $n \geq 2$, and $(m, n) = 1$.*

For convenience, we recall that the rational double points are isolated quotients of $\mathbb{C}^2$ by finite subgroups of $SU(2)$. They are classified by their types $A, D$ or $E$. The singularities of type $A_{k-1}$ are cyclic quotient singularities of the type $\frac{1}{k}(1, -1)$, while the other rational double points are quotients of $\mathbb{C}^2$ under the action of the non-cyclic binary polyhedral groups. They also admit a description as hypersurface singularities

$$(f(x, y, z) = 0) \subseteq \mathbb{C}^3,$$

where

$$f(x, y, z) = \begin{cases} xy + z^k, \text{ for singularities of type } A_{k-1}, \ k \geq 2, \\ x^2y + y^{k-1} + z^2, \text{ for singularities of the type } D_k, \ k \geq 4, \\ x^4 + y^3 + z^2, \text{ for singularities of the type } E_6, \\ x^3y + y^3 + z^2, \text{ for singularities of the type } E_7, \\ x^5 + y^3 + z^2, \text{ for singularities of the type } E_8. \end{cases}$$

$$(2.1)$$

**Definition 2.4.** *A normal surface singularity is called of class $T$ if it is a rational double point or a cyclic quotient singularity the type $\frac{1}{dn^2}(1, dnm - 1)$, for $d > 0$, $n \geq 1$, and $(m, n) = 1$.*

Using the natural sequence of abelian groups:

$$1 \to \mu_{dn} \to \mu_{dn^2} \to \mu_n \to 1,$$

the second type of singularities can be described as the double quotient

$$\mathbb{C}^2 / \frac{1}{dn}(1, -1) / \mu_n,$$

i.e. it is a quotient of an $A_{dn-1}-$ singularity. Motivated by this, we consider the hypersurface $\mathcal{Y} = (xy - z^{dn} = Q(z^n)) \subseteq \mathbb{C}^3 \times \mathbb{C}^d$, where $Q(z) = \sum_{k=0}^{d-1} e_k z^k$. It is convenient to introduce the polynomial

$$P(z) = z^d + Q(z) = \prod_{j=1}^{l} (z^n - a_j)^{k_j},$$

where $a_1, \ldots, a_l \in \mathbb{C}$ are distinct, and the positive integers $k_j, j = 1, \ldots, l$, satisfy

$$\sum_{j=1}^{l} k_j = d.$$

We denote by $(x, y, z)$ and $\mathbf{e} = (e_0, \ldots, e_{d-1})$ the linear coordinates on $\mathbb{C}^3$ and $\mathbb{C}^d$, respectively. The action of the group $\mu_n$ on $\mathcal{Y}$ is generated by:

$$\zeta(x, y, z, e_0, \ldots, e_{d-1}) := (\zeta x, \zeta^{-1} y, \zeta^m z, e_0, \ldots, e_{d-1}),$$

where $\zeta$ is a generator of $\mu_n$. Let $\mathcal{X} = \mathcal{Y}/\mu_n$ and $\phi : \mathcal{X} \to \mathbb{C}^d$ the quotient of the projection $\mathcal{Y} \to \mathbb{C}^d$. Let $X_0$ be the fiber $\phi^{-1}(0)$. Then $(X_0, 0)$ is a singularity of the type $\dfrac{1}{dn^2}(1, dnm - 1)$ and we have:

**Proposition 2.5.** [15, 10] *The map $\phi : \mathcal{X} \to \mathbb{C}^d$ is a $\mathbb{Q}$−Gorenstein deformation of the cyclic singularity $(X_0, 0)$ of type $\dfrac{1}{dn^2}(1, dnm - 1)$. Moreover, every $\mathbb{Q}$−Gorenstein deformation $\mathcal{X} \to \mathbb{C}$ of a singularity $(X_0, 0)$ of type $\dfrac{1}{dn^2}(1, dnm - 1)$ is isomorphic to the pullback of $\phi$ for some germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$.*

As in [15], given $\mathbf{e} \in \mathbb{C}^d \setminus \{0\}$, we want the group $\mu_n$, $n \geq 2$, to act freely on the fiber $Y_{\mathbf{e}} \subseteq \mathbb{C}^3$ of the deformation $\mathcal{Y} \to \mathbb{C}^d$. This condition is equivalent to the fact that $\{0\} \in \mathbb{C}^3$ lies only on the central fiber $Y_0$, and it translates into

$$a_j \neq 0, \text{ for any } j = 1, \dots, l. \tag{2.2}$$

In the case $n = 1$, any fiber of the form $Y_{\mathbf{e}} = (xy = P(z))$, is biholomorphic to a fiber of a deformation satisfying the above condition after a change of coordinates. We impose the condition (2.2) throughout this paper for any $n$.

The variety $X_{\mathbf{e}} = \phi^{-1}(\mathbf{e})$ is the Milnor fiber of the $\mathbb{Q}$−Gorenstein deformation if it is smooth. This translates into $l = d$, and $k_j = 1$, $j = 1, \dots, d$.

**Remark 2.6.** *Notice that the Milnor fiber of the smoothing of a singularity of type $A_{d-1}$ is recovered in the above construction by setting $n = 1$.*

2.1. **Cyclic quotient singularities of class $T$.** In this section we construct a family of singular compactifications of a fiber of a $\mathbb{Q}$−Gorenstein deformation of a singularity of the type $\dfrac{1}{dn^2}(1, dnm - 1)$. The compactifications are presented as hypersurfaces in appropriate weighted projective spaces.

Let

$$P(z) = \prod_{j=1}^{l} (z - a_j)^{k_j}, \tag{2.3}$$

where $a_1, \dots, a_l \in \mathbb{C}^*$ are distinct, and $k_j$, $j = 1, \dots, l$, are positive integers with $\sum_{j=1}^{l} k_j = d$. The variety

$$M = (xy = P(z^n)) / \mu_n \subseteq \mathbb{C}^3 / \frac{1}{n}(1, -1, m)$$

is a fiber of a $\mathbb{Q}$−Gorenstein deformation of a $\dfrac{1}{dn^2}(1, dnm - 1)$−singularity.

We define

$$\overline{M} = \left( xy = w^{dk} P\left(\frac{z^n}{w^k}\right) = \prod_{j=1}^{l} (z^n - a_j w^k)^{k_j} \right) \subseteq \mathbb{P}(a, b, c, e),$$

where we denoted by $[x : y : z : w]$ the homogeneous coordinates in the weighted projective space $\mathbb{P}(a, b, c, e)$, and $k$ is a positive integer. The weights should satisfy the homogeneity conditions:

$$a + b = dnc = dke. \tag{2.4}$$

We identify next sufficient conditions on the weights such that $M$ embeds into $\overline{M}$ as $\overline{M} \cap U_w$.

The standard affine coordinate chart $U_w = (w \neq 0) \subseteq \mathbb{P}(a, b, c, e)$ is isomorphic to $\mathbb{C}^3_{(X,Y,Z)}/\frac{1}{e}(a, b, c)$. We require the action of $\mu_e$ to be equivalent to an action of the type $\frac{1}{n}(1, -1, m)$. This forces $e = n$, and from the homogeneity condition we see that $k = c$. Furthermore, there should exist $\rho \in \mu_n$ such that

$$\begin{cases} 1) & \rho = \xi^a; \\ 2) & \rho^{-1} = \xi^b; \\ 3) & \rho^m = \xi^c. \end{cases}$$

From the homogeneity condition we see that $a + b = 0 \mod n$, and so the conditions 1) and 2) are equivalent. Therefore, for any given $c \geq 0$, the conditions $1) - 3)$ are simultaneously satisfied if and only if

$$am = c \mod n. \tag{2.5}$$

Let $u \in \{1, \ldots, n - 1\}$ be the unique integer such that $mu = 1 \mod n$. Then (2.5) is equivalent to

$$a = cu \mod n.$$

Finally, a necessary condition we require is that the weighted projective space $\mathbb{P}(a, b, c, n)$ is well-formed. From (2.4) and (2.5), we can see that this is equivalent to requiring that $\gcd(n, c) = 1$, which implies that $\gcd(a, n) = \gcd(b, n) = 1$. If $\gcd(a, c) = p \neq 1$, then $\gcd(b, c) = p$, and we write $a = pa', b = pb'$, and $c = pc'$, where $\gcd(a', c') = \gcd(b', c') = 1$. Notice that (2.4) yields $a' + b' = dnc'$, while from (2.5) we see that $a' = c'u \mod n$. Moreover, we have an isomorphism $\mathbb{P}(a, b, c, n) \simeq \mathbb{P}(a', b', c', n)$ [5]. By replacing $(a, b, c)$ by $(a', b', c')$, we can therefore assume that $\gcd(a, c) = 1$. For convenience, we summarize our requirements as

$$\gcd(c, n) = 1 \text{ and } \gcd(a, c) = 1. \tag{2.6}$$

**Proposition 2.7.** *Let $a, b$ and $c$ satisfying the conditions (2.4), (2.5) and (2.6), and let*

$$\overline{M} = \left( xy = \prod_{j=1}^{l} (z^n - a_j w^c)^{k_j} \right) \subseteq \mathbb{P}(a, b, c, n).$$

*We have*

1) *The variety $M$ embeds as a Zariski open subset in $\overline{M}$.*
2) *The singular points of $M$ are at most rational double points of type $A_{k_j-1}$, for $j = 1, \ldots, l$.*

3) *The singular points of $\overline{M}$ lying on $\overline{M} \setminus M$ are singularities of the types $\frac{1}{a}(c, n)$ and $\frac{1}{b}(c, n)$ at the points $R_1 = [1 : 0 : 0 : 0]$ and $R_2 = [0 : 1 : 0 : 0]$, respectively.*

4) *The complement $C = \overline{M} \setminus M$ is a smooth rational curve. Moreover, $C$ is an ample $\mathbb{Q}-$Cartier divisor of $\overline{M}$.*

5) *The anti-canonical divisor of $\overline{M}$ is ample, and as $\mathbb{Q}-$Cartier divisors, we have*

$$K_{\overline{M}} = -\frac{c + n}{n} C. \tag{2.7}$$

*In particular, $\overline{M}$ is a log del Pezzo surface.*

6) *The variety $\overline{M}$ is simply connected and its second homology group has rank $d$.*

*Proof.* 1) Notice that the condition (2.6) implies that the weighted projective space is well-formed. The conditions (2.4) and (2.5) were defined to ensure that $M$ embeds into $\overline{M}$ as $\overline{M} \cap U_w$.

2) As discussed, the chart $U_w = (w \neq 0) \subseteq \mathbb{P}(a, b, c, n)$ is isomorphic to $\mathbb{C}^3_{(X,Y,Z)}/\frac{1}{n}(a, b, c)$. In these coordinates

$$\overline{M} \cap U_w \simeq (XY = P(Z^n)) \left/ \frac{1}{n}(a, b, c).\right.$$

From (2.6) we find that the fixed point $(0, 0, 0) \notin \overline{M} \cap U_w$ is the only point of non-trivial isotropy.

The singular points of the hypersurface $(XY = P(Z^n)) \subseteq \mathbb{C}^3$ occur when the polynomial $P$ has multiple roots. We find that $S_j = [0 : 0 : a_j^{\frac{1}{n}} : 1] \in \overline{M}$ are singular points of type $A_{k_j-1}$ of $\overline{M} \cap U_w$, for any $j = 1, \ldots, l$, such that $k_j \geq 2$.

3) We compute the singularities of $\overline{M}$ at infinity in the standard charts covering the weighted projective space $\mathbb{P}(a, b, c, n)$.

Let $U_x = (x \neq 0) \subseteq \mathbb{P}(a, b, c, n)$. Then $U_x \simeq \mathbb{C}^3_{(Y,Z,W)}/\frac{1}{a}(b, c, n)$. In these coordinates

$$\overline{M} \cap U_x \simeq \left( Y = \prod_{j=1}^{l} (Z^n - a_j W^c)^{k_j} \right) \left/ \frac{1}{a}(b, c, n).\right.$$

Since $a$ is relatively prime to $b, c$, and $n$, the only point of non-trivial isotropy is the fixed point of the action of $\mu_a$ on $\mathbb{C}^3$, $(0, 0, 0) \in \overline{M} \cap U_x$. As the hypersurface $\left( Y = \prod_{j=1}^{j} (Z^n - a_j W^c)^{k_j} \right) \subseteq \mathbb{C}^3$ is smooth, the only singular point of $\overline{M} \cap U_x$ is $R_1 = [1 : 0 : 0 : 0]$. Notice that the coordinates $(Z, W)$ parametrize $\overline{M} \cap U_x$, and so the point $R_1 \in \overline{M}$ is a cyclic quotient singularity of the type $\frac{1}{a}(c, n)$.

An analogous computation finds one more singular point of $\overline{M}$ in the chart $U_y = (y \neq 0) \subseteq \mathbb{P}(a, b, c, n)$. This point is $R_2 = [0 : 1 : 0 : 0]$, a cyclic quotient singularity of the type $\frac{1}{b}(c, n)$. Moreover, in the chart $U_z = (z \neq 0) \subseteq \mathbb{P}(a, b, c, n)$, as

in the proof of part 1), we find no singular points on $\overline{M} \setminus M$, and we recover the singular points $S_j = [0 : 0 : a_j^{\frac{1}{n}} : 1] \in \overline{M} \cap U_z$ above.

4) The curve at infinity $C = \overline{M} \setminus M$ is the hyperplane section $(w = 0)$, of weight $n$. As a $\mathbb{Q}$-Cartier divisor

$$\mathcal{O}_{\overline{M}}(C) = \mathcal{O}_{\overline{M}}(n),$$

where by $\mathcal{O}_{\overline{M}}(1)$ we denote the restriction of the tautological sheaf $\mathcal{O}_{\mathbb{P}(a,b,c,n)}(1)$ to $\overline{M}$. In particular $C$ is an ample $\mathbb{Q}-$Cartier divisor.

In the local charts $\overline{M} \cap U_x \simeq \mathbb{C}^2_{(Z,W)}/\frac{1}{a}(c,n)$ and $\overline{M} \cap U_y \simeq \mathbb{C}^2_{(Z,W)}/\frac{1}{b}(c,n)$, the curve at infinity $C$ is given by the $(W = 0)$. Hence $C \cap \overline{M} \cap U_x \simeq \mathbb{C}/\frac{1}{a}(c)$, and $C \cap \overline{M} \cap U_y \simeq \mathbb{C}/\frac{1}{b}(c)$, and so $C$ is smooth with the induced differential structure. Moreover, since in the local chart $\overline{M} \cap U_x$ the curve $C$ corresponds to the $Z-$axis, and $C$ is a one point compactification of $C \cap \overline{M} \cap U_x$, then it must be rational.

5) Since the weighted projective space $\mathbb{P}(a,b,c,n)$ is well-formed and the hypersurface $\overline{M}$ does not contain its singular lines (if any), then the adjunction formula holds. The canonical divisor of $\overline{M}$ is

$$K_{\overline{M}} = (K_{\mathbb{P}(a,b,c,n)} + \overline{M})|_{\overline{M}} = \mathcal{O}_{\overline{M}}(dnc - a - b - c - n)$$

Using the homogeneity condition (2.4) we obtain

$$K_{\overline{M}} = \mathcal{O}_{\overline{M}}(-c - n) = -\frac{c + n}{n}C.$$

In particular, we see that $\overline{M}$ is a log del Pezzo surface.

6) To study the topological properties of $\overline{M}$, we have to consider two cases. In the first case, when $n = 1$ the manifold $M$ is a smooth deformation of a singularity of type $A_{d-1}$. As a consequence, it is simply connected and the rank of its second homology is $d - 1$. The compactification is obtained by adding a rational curve at infinity, hence by Van Kampen's theorem, the manifold $\overline{M}$ is simply connected. Moreover, with respect to the decomposition $\overline{M} = M \cup Nbhd(C)$, the intersection $M \cap Nbhd(C)$ is homotopic to the lens space $L_d(1, -1)$ whose first Betti number is $b_1(L_d(1, -1)) = 0$. Thus, by the Mayer-Vietoris sequence the second Betti number is $b_2(\overline{M}) = d$.

In the second case, when $n \geq 2$, the manifold $M$ is obtained by taking the quotient of a deformation of the $A_{dn-1}$ singularity by a free $\mu_n-$action. Hence the fundamental group of $M$ is $\pi_1(M) = \mathbb{Z}/n\mathbb{Z}$ which embeds in the fundamental group of a neighborhood of infinity which is $\pi_1(L_{dn^2}(1, dnm - 1)) = \mathbb{Z}/dn^2\mathbb{Z}$. The Euler characteristic of $M$ is

$$\chi(M) = \frac{1}{n}\chi(A_{dn-1}) = \frac{1}{n}dn = d,$$

hence the second Betti number of $M$ equals $d - 1$. The Van Kampen's theorem and the Mayer-Vietoris sequence for $\overline{M} = M \cup Nbhd(C)$ imply that $\overline{M}$ is simply connected, and its second Betti number is $d$.                                    $\square$

**Remark 2.8.** *For a given integer $c \geq 0$, the conditions $1) - 3)$ will be satisfied by $d$ pairs $(a, b)$ of non-negative integers. As a consequence, when taking different values of $c$, the Milnor fiber $M$ can be embedded in infinitely many log del Pezzo surfaces.*

**Examples 2.9.** *The embeddings of $M$ in log del Pezzo surfaces in Theorem B are indexed by the set of weights $(a, b, c)$. We list below some interesting examples:*

1) *the natural weights induced by the action $\mu_n$ :*

$$(a, b, c) = (1 + knm, (d - k)nm - 1, m), \ k \in \{0, \ldots, d - 1\}.$$

2) *a normalization for $c = 1$ :*

$$(a, b, c) = (u + kn, (d - k)n - 1, 1), \ k \in \{0, \ldots, d - 1\}.$$

*The case $d = 1, \ k = 0$ also appears in [11].*

**Remark 2.10.** *When $n = 1$, we compactify the singularity $A_{d-1}$ and its deformations in $\mathbb{P}(a, dc - a, c, 1)$. A special case, when $c = 1$, $d$ even and $a = b$ appears in Saito [17]. We generalize Saito's compactification to an infinite family of compactifications.*

2.2. **Non-cyclic quotient singularities of class $T$.** In this section we construct a family of singular compactifications of a fiber of the universal deformation of singularities of the types $D_k, E_6, E_7$ and $E_8$. These are also hypersurface singularities, and this allows us to exhibit the compactifications as hypersurfaces in appropriate weighted projective spaces. The construction is essentially due to Saito [17]. We extend it to an arbitrary fiber of the universal deformation space of a rational double point.

Let $f$ be one of the polynomials (2.1), and let $g_i \in \mathbb{C}[x, y, z], i = 1 \ldots, k$ be monomials yielding a basis of

$$\mathbb{C}[x, y, z]/\langle f, \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle.$$

The fiber of the universal deformation of such a rational double point singularity is the affine variety

$$M = \left( f = \sum_{i=1}^{k} a_i g_i \right) \subseteq \mathbb{C}^3,$$

where $a_i \in \mathbb{C}$ are fixed. Notice that for $a_i's$ general enough, the surface $M$ is smooth. In general, it is known that $M$ has at most rational double points singularities. If $a_i = 0, \ i = 1, \ldots, k$, we recover the central fiber which is a singularity of the type $\mathbb{C}^2/G$, where $G \subseteq SU(2)$. For convenience, let $h = f - \sum_{i=1}^{k} a_i g_i \in \mathbb{C}[x, y, z]$.

As before we would like to embed $M$ in a weighted projective space $\mathbb{P}(a, b, c, e)$ as $\overline{M} \cap U_w$, where $U_w$ is the standard chart $(w \neq 0)$ and $\overline{M}$ is a hypersurface given by a suitable quasi-homogeneization of the polynomial $h$ :

$$\overline{M} = \left( w^N h \left( \frac{x}{w^a}, \frac{y}{w^b}, \frac{z}{w^c} \right) = 0 \right) \subseteq \mathbb{P}(a, b, c, e),$$

for some positive integers $a, b, c, e$ and $N$. Then the weight $e$ of $w$ must be equal to one. A brief inspection of the polynomials (2.1) yields weights defined uniquely up to a common factor. The ambiguity is eliminated by the divisibility condition $\gcd(a, b, c) = 1$, which ensures that the ambient weighted projective space is well formed, and we find

$$(a, b, c) = \begin{cases} (k - 2, 2, k - 1), \text{ for singularities of the type } D_k \\ (3, 4, 6), \text{ for singularities of the type } E_6 \\ (4, 6, 9), \text{ for singularities of the type } E_7 \\ (6, 10, 15), \text{ for singularities of the type } E_8. \end{cases} \quad (2.8)$$

For the rest of this section, we consider $h$ as a quasi-homogeneous polynomial as above, and $(a, b, c)$ the corresponding weights. With these choices, the weighted degree of $\overline{M}$ in $\mathbb{P}(a, b, c, 1)$ is $N = a + b + c - 1$.

**Proposition 2.11.** *Let*

$$\overline{M} = \left( w^N h \left( \frac{x}{w^a}, \frac{y}{w^b}, \frac{z}{w^c} \right) = 0 \right) \subseteq \mathbb{P}(a, b, c, 1),$$

*where $(a, b, c)$ as in (2.8). Then*

1) *$M$ is embeds as a Zariski open subset in $\overline{M}$.*
2) *The singular points of $\overline{M}$ lying in $\overline{M} \setminus M$ are as follows:*
    i) **Case $D_k$:** *two singularities of type $\frac{1}{2}(1, 1)$, and one of type $\frac{1}{k-2}(1, 1)$.*
    ii) **Case $E_6$:** *two singularities of type $\frac{1}{3}(1, 1)$, and one of the type $\frac{1}{2}(1, 1)$.*
    iii) **Case $E_7$:** *three singularities of type $\frac{1}{2}(1, 1)$, $\frac{1}{3}(1, 1)$, and $\frac{1}{4}(1, 1)$, respectively*
    iv) **Case $E_8$:** *three singularities of type $\frac{1}{2}(1, 1)$, $\frac{1}{3}(1, 1)$, and $\frac{1}{5}(1, 1)$, respectively.*
3) *The complement $C = \overline{M} \setminus M$ is a smooth rational curve. Moreover, $C$ is an ample $\mathbb{Q}-$Cartier divisor of $\overline{M}$.*
4) *The anti-canonical divisor of $\overline{M}$ is*

$$K_{\overline{M}} = -2C. \quad (2.9)$$

   *In particular, $\overline{M}$ is a log del Pezzo surface.*
5) *The variety $\overline{M}$ is simply connected and its second homology group has rank $k + 1$ for a singularity of type $D_k$, and $n + 1$ for a singularity of type $E_n$, $n = 6, 7, 8$.*

*Proof.* The proofs of 1), 2), and 3) for $h = f - 1$ can be found in [17]. One can easily check that the singularities at infinity remain the same for any deformation $h$. The canonical class of $\overline{M}$ follows again from the adjunction formula:

$$K_{\overline{M}} = (K_{\mathbb{P}(a,b,c,1)} + \overline{M})|_{\overline{M}} = \mathcal{O}_{\overline{M}}(N - a - b - c - 1) = \mathcal{O}_{\overline{M}}(-2) = -2C.$$

The proof of 5) is as in Proposition 2.7.6. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Notice that in particular we obtained a compactification of a rational double point singularity with the properties stated in Proposition 2.11.

2.3. **Conclusions.** If $M$ is a fiber of the $\mathbb{Q}-$Gorenstein deformation of a singularity of class $T$, the Propositions 2.7 and 2.11 provide compactifications with the properties summarized in Theorem B.

In general, $M$ might admit rational double points as singularities. In this case, we consider the minimal resolution $N$ of $M$, and this gives us a special complex structure on the Milnor fiber of the singularity. We have the following:

**Corollary 2.12.** *The minimal resolution $N$ of a fiber $M$ of the $\mathbb{Q}-$Gorenstein deformation of a singularity of class $T$ embeds into a variety $\overline{N}$ as the complement of a smooth, rational curve, which is a rational multiple of the anticanonical divisor. The singularities along the divisor at infinity are all isolated finite cyclic quotients. Moreover, if $M$ is associated to a finite cyclic quotient singularity then there are infinitely many minimal compactifications with the above properties.*

*Proof.* Let $p : N \to M$ be the minimal resolution of $M$. Correspondingly, let $p : \overline{N} \to \overline{M}$ denote its extension to the compactification. Since the singular points of $M$ are at most rational double points

$$K_{\overline{N}} = p^* K_{\overline{M}} = -\beta C, \tag{2.10}$$

for some $\beta > 1$. The singularities of $\overline{N}$ are only along the divisor at infinity as described in Propositions 2.7 and 2.11. $\qquad\square$

Notice that $\overline{N}$ is no longer a log del Pezzo surface, as it contains the $(-2)-$curves introduced when resolving the singularities of $M$. Moreover, the divisor at infinity $C$ is only almost ample, see Definition 3.2 below.

**Remark 2.13.** *Let $M_0 = \mathbb{C}^2 / \dfrac{1}{dn^2}(1, dnm - 1), n \neq 1$, and $X$ its minimal resolution. The variety $M_0$ admits a compactification $\overline{M}_0$ to a hypersurface in a weighted projective space, as well, and it has similar properties to an $A-$type singularity. The minimal resolution $X$ of $M_0$ has non-trivial canonical divisor, and this implies that $X$ does not admit Ricci-flat Kähler metrics. As expected, since $K_{\overline{X}}$ is not a multiple of the curve at infinity we can see that the conditions in the Theorem 3.2 below are not satisfied.*

## 3. THE RELATION WITH TIAN-YAU'S RICCI-FLAT KÄHLER METRICS

In this section we recall for convenience the relevant theorems of Tian-Yau [20] and Bando-Kasue-Nakajima [2]. Both Tian-Yau [20] and Bando-Kasue-Nakajima [2] proved more general results, but we are going to restrict ourselves to the ALE Ricci-flat case in complex dimension two. We conclude by proving Theorem A.

We first start by recalling the definition of ALE $4-$manifolds:

**Definition 3.1.** *Let $G$ be a finite subgroup of $SO(4)$ acting freely on $\mathbb{R}^4 \setminus \{0\}$, and let $h_0$ be the Euclidean metric on $\mathbb{R}^4/G$. We say that the manifold $(M^4, g)$ is an ALE manifold asymptotic to $\mathbb{R}^4/G$ if there exist a compact subset $K \subseteq M$ and a map $\pi : M \setminus K \to \mathbb{R}^4/G$ that is a diffeomorphism between $M \setminus K$ and the subset $\{z \in \mathbb{R}^4/G \mid r(z) > R\}$ for some fixed $R > 0$, such that*

$$\nabla^k(\pi_*(g) - h_0) = O(r^{-4-k}) \text{ for all } k \geq 0. \tag{3.1}$$

To emphasize that the condition (3.1) is satisfied, we say that the metric is ALE of order 4 [2].

If the metric is Kähler, then the group $G$ is a subgroup of $U(2)$, and the diffeomorphism $\pi$ identifies $M \setminus K$ with a subset $\mathbb{C}^2/G$. This identification is not a biholomorphism in general. In fact, in some cases it can be showed that the complex structure can be approximated by the canonical complex structure $J_0$ up to lower order terms $O(r^{-4})$.

Tian and Yau constructed [20] complete Ricci-flat Kähler metrics on the complements of divisors on compact Kähler orbifolds satisfying certain conditions:

**Definition 3.2.** *Let $D$ be a divisor in the Kähler orbifold $\overline{M}$ of complex dimension 2. Then*

(i) *$D$ is almost ample if there exists an integer $m > 0$ such that a basis of $H^0(\overline{M}, \mathcal{O}(mD))$ gives a morphism from $\overline{M}$ into some projective space $\mathbb{P}^N$ which is a biholomorphism in a neighborhood of $D$.*

(ii) *$D$ is admissible if $\operatorname{Sing}(\overline{M}) \subseteq D$, $D$ is smooth in $\overline{M} \setminus \operatorname{Sing}(\overline{M})$, and for any $x \in \operatorname{Sing}(\overline{M})$ if $\pi_x : \widetilde{U}_x \to U_x$ be its local uniformization with $\widetilde{U}_x \subseteq \mathbb{C}^2$, then $\pi_x^{-1}(D)$ is smooth in $\widetilde{U}_x$.*

As we are interested in ALE manifolds, the topology at infinity has finite fundamental group, which necessarily implies that the divisor at infinity is a smooth rational curve. In this situation, Tian-Yau proved:

**Theorem 3.3** ([20]). *Let $\overline{M}$ be a compact Kähler orbifold of complex dimension 2. Let $D \simeq \mathbb{P}^1$ be an almost ample admissible divisor in $\overline{M}$, such that*

$$- K_{\overline{M}} = \beta D, \; for \; some \; \beta > 1. \tag{3.2}$$

*Then $M = \overline{M} \setminus D$ admits a complete Ricci-flat Kähler metric $g$.*

*Moreover, if we denote by $\mathcal{R}(g)$ the curvature tensor of $g$ and by $r$ the distance function on $M$ from some fixed point with respect to $g$, then $\mathcal{R}(g)$ decays at the order of at least $r^{-3}$ with respect to the $g-$norm whenever*

$$\mathcal{O}_D(D) = \frac{2}{\beta - 1} \mathcal{O}_{\mathbb{P}^1}(1). \tag{3.3}$$

*Furthermore the metric $g$ has euclidean volume growth.*

The metrics constructed using the Tian-Yau result are not a priori ALE. This follows from a remarkable result of Bando-Kasue-Nakajima:

**Theorem 3.4** ([2]). *Let $(M, g)$ be a Ricci-flat Kähler surface with*

1) *$VolB(p; r) \geq Cr^4$ for some $p \in M, C > 0$,*

2) *$\displaystyle\int_M |\mathcal{R}(g)|^2 dV_g < \infty$.*

*Then $(M, g)$ is ALE of order $4$.*

Here $B(p; r) \subseteq M$ denotes the ball or radius $r$ in $(M, g)$ centered at the point $p \in M$.

*Proof of Theorem A.* Given an ALE Ricci-flat Kähler surface $(M, J, g)$, in [13, 14, 18] it is showed that $g$ is one of the metrics constructed explicitly by Eguchi-Hanson, Gibbons-Hawking, Hitchin, Kronheimer, and their free quotients. On the other hand, in Theorem B and Corollary 2.12 we proved that the complex surface $(M, J)$ admits a compactification to a variety $(\overline{N}, J)$ with at most three three finite cyclic quotient singular points along the divisor at infinity, and no other singular points. The complex structure of the divisor at infinity was studied in Proposition 2.7.4. We showed that $C$ is an almost ample, admissible, smooth rational curve. Moreover, the numerical conditions of the Tian-Yau construction (3.2), (3.3) are also satisfied, as it can be seen from (2.10) and the adjunction formula. Then the complement of the divisor at infinity, $\overline{M} \setminus C$, admits a complete Ricci-flat Kähler metric by the Tian-Yau theorem. This metric must be ALE by Theorem 3.4. Hence, by the uniqueness part of Theorem 1.1, the two constructions yield the same metrics. $\qquad\square$

## 4. A GEOMETRIC DESCRIPTION OF THE COMPACTIFICATIONS

In Section §2, we gave an extrinsic description of the compactification $\overline{M}$ as a hypersurface in a weighted complex projective space. We give next an intrinsic description of the compactification. This will prove Theorem C.

Let $\overline{M} = \left( xy = \prod_{j=1}^{l} (z^n - a_j w^c)^{k_j} = w^{dc} P \left( \frac{z^n}{w^c} \right) \right) \subseteq \mathbb{P}_{[x:y:z:w]}(a, b, c, n),$

where $a_j \in \mathbb{C}^*$ are distinct, be the variety constructed in Proposition 2.7. Recall that $\overline{M}$ has singularities of type $A_{k_j-1}$ at the points $S_j = [0 : 0 : z_j : w_j] \in \mathbb{P}(a, b, c, n)$ with $z_j^n = a_j w_j^c$, whenever $k_j > 1$.

Let $\pi : \overline{M} \dashrightarrow \mathbb{P}(a, c, n)$ be the projection

$$\pi \left( [x : y : z : w] \right) = [x : z : w]$$

from the singular point $R_2 = [0 : 1 : 0 : 0] \in \overline{M}$ to the weighted hyperplane $(y = 0) \simeq \mathbb{P}_{[x:z:w]}(a, c, n)$.

Since for any $[x : y : z : w] \in \overline{M}$ with $x \neq 0$, we can write $y = \frac{w^{dc}}{x} P \left( \frac{z^n}{w^c} \right)$, it follows that the map $\pi$ is birational. More precisely, it is a biholomorphism between $\overline{M} \setminus (x = 0)$ and $\mathbb{P}(a, c, n) \setminus A$, where $A = (x = 0) \subseteq \mathbb{P}(a, c, n)$. Along the divisor $(x = 0) \subseteq \overline{M}$, the map $\pi$ is not defined at $R_2$, and contracts the lines

$$L_j = \{ [0 : y : z_j : w_j] | z_j^n = a_j w_j^c \} \subseteq \overline{M}, \ j = 1, \ldots, l,$$

passing through $R_2$ to distinct smooth points $s_j = \pi(L_j) = [0 : z_j : w_j] \in A$.

We eliminate the indeterminacy point $R_2$ of the rational map $\pi$ by a single suitable weighted blow-up, following [12, page 88].

As discussed in Proposition 2.7, the point $R_2 = [0 : 1 : 0 : 0] \in \overline{M}$ is a singularity of the type $\frac{1}{b}(c, n)$. More explicitly, if we denote by $(X, Z, W)$ the local coordinates in the chart $U_y = (y \neq 0) \subseteq \mathbb{P}(a, b, c, n)$, then

$$\left( \overline{M} \cap U_y, R_2 \right) \simeq \left( \mathbb{C}^2_{(Z,W)} / \frac{1}{b}(c, n), 0 \right).$$

We consider the weighted blow-up of the point $R_2$ in two steps. First, let $U = (tZ^n = sW^c) \subseteq \mathbb{C}^2_{(Z,W)} \times \mathbb{P}^1_{[s:t]}$, where $[s:t]$ are the homogeneous coordinates on $\mathbb{P}^1$, and let $E = ((0,0),[s:t])$ denote the exceptional divisor. The action of $\mu_b$ on $\mathbb{C}^2$ extends trivially to an action on $\mathbb{C}^2 \times \mathbb{P}^1$ as

$$\alpha\left((Z,W),[s:t]\right) = \left((\alpha^c Z, \alpha^n W),[s:t]\right),$$

where $\alpha \in \mu_b$. The variety $U$ is invariant under this action and $E$ is its fixed locus. Let $\widehat{U} = U/\mu_b$, and denote by $\widehat{E} \subseteq \widehat{U}$ the exceptional divisor.

Let $U_s, U_t \subseteq \mathbb{C}^2 \times \mathbb{P}^1$ be the standard charts given by $(s \neq 0)$ and $(t \neq 0)$, respectively. We denote by $\widehat{U_s \cap U}$ and $\widehat{U_t \cap U}$ the corresponding covering of $\widehat{U}$. In the chart $U_s$, $U$ is given by the equation $(t'Z^n = W^c) \subseteq \mathbb{C}^3_{(Z,W,t')}$, where $t' = t/s$, and so the action of the group $\mu_b$ is of the type $\frac{1}{b}(c,n,0)$. In this chart, the variety $U$ has a singular locus along the exceptional divisor $E = (Z = 0)$. We consider new complex coordinates on $U$ in order to eliminate some of the singularities, as follows.

Let $f : \mathbb{C}^2 \to \mathbb{C}^3$ defined as

$$(u,v) \mapsto (Z = u^c, W = u^n v, t' = v^c). \tag{4.1}$$

We notice that $t'Z^n - W^c = v^c(u^c)^n - (u^n v)^c = 0$, and hence $U \cap U_s$ contains the imagine of $f$. Let now $\mu_c$ act on $\mathbb{C}^2$ by

$$\rho(u,v) = (\rho u, \rho^{-n} v). \tag{4.2}$$

The image of $f$ is invariant under the action of $\mu_c$, and we have an induced holomorphic map:

$$f' : \mathbb{C}^2_{(u,v)}/\frac{1}{c}(1,-n) \to U \cap U_s$$

To show that this map defines new complex coordinates we need to prove that $f'$ is a homeomorphism. It suffices to show that $f'$ is a bijective map.

Let $(Z,W,t') \in \mathbb{C}^3$ be a point in $U \cap U_s$, that is $t'Z^n = W^c$. If $(Z,W,t') = (u^c, u^n v, v^c)$, then $u$ is uniquely determined up to a $c^{th}$ root of unity $\rho$, $u = \rho Z^{\frac{1}{c}}$. If $Z \neq 0$, then $v = \rho^{-n} W(Z^{\frac{1}{c}})^{-n}$. If $Z = 0$, then $W = 0, u = 0$, and $v$ is determined up to a $c^{th}$ root of unity. More precisely, as $\gcd(n,c) = 1$, we can write $v = \rho^{-n} t'^{\frac{1}{c}}$ for some $\rho \in \mu_c$. In both cases, $(u,v)$ is uniquely determined, up to the action (4.2) of the group $\mu_c$. Hence, any $(Z,W,t')$ on $U \cap U_s$ can be obtained as the image of a point $(u,v)$ of the form $(Z^{\frac{1}{c}}, W(Z^{\frac{1}{c}})^{-n})$ or $(0, t'^{-\frac{1}{c}})$.

This allows us to define new complex coordinates on the quotient $\widehat{U \cap U_s}$:

**Lemma 4.1.** *There exists an induced holomorphic map $\widehat{f}_s : \mathbb{C}^2_{(u',v)}/\frac{1}{c}(b,-n) \to \widehat{U \cap U_s}$. Moreover, the map $\widehat{f}_s$ is a homeomorphism and hence induces new complex coordinates on $\widehat{U \cap U_s}$.*

*Proof.* The $\mu_b$ action of type $\frac{1}{b}(c, n, 0)$ on $\mathbb{C}^3_{(Z,W,t')}$ is compatible with the action of type $\frac{1}{b}(1, 0)$ on $\mathbb{C}^2_{(u,v)}$, as $(Z, W, t') = (u^c, u^n v, v^c)$. Therefore, $f'$ induces a map:

$$\widehat{f}_s : \left(\mathbb{C}^2_{(u,v)}/\frac{1}{c}(1, -n)\right)/\frac{1}{b}(1, 0) \to (U \cap U_s)/\mu_b \simeq \widehat{U \cap U_s}.$$

Notice now that

$$\left(\mathbb{C}^2_{(u,v)}/\frac{1}{c}(1, -n)\right)/\frac{1}{b}(1, 0) \simeq \left(\mathbb{C}^2_{(u,v)}/\frac{1}{b}(1, 0)\right)/\frac{1}{c}(1, -n) \simeq \mathbb{C}^2_{(u',v)}/\frac{1}{c}(b, -n),$$

after we use the commutativity of the two actions and the reparametrization $u' = u^b$. As the map $f'$ is holomorphic and a homeomorphism, so is the induced map $\widehat{f}_s$.

More explicitly, the map

$$\widehat{f}_s : \mathbb{C}^2_{(u',v)}/\frac{1}{c}(b, -n) \to \widehat{U \cap U_s} \subset (\mathbb{C}^2_{(Z,W)} \times \mathbb{P}^1_{[s:t]})/\frac{1}{b}(c, n, 0, 0)$$

is of the form

$$\widehat{f}_s(u', v) = (((u'^{\frac{1}{b}})^c, (u'^{\frac{1}{b}})^n v), [1 : v^c]).$$

$\square$

Notice that since $a + b = dnc$, the singularity of the type $\frac{1}{c}(b, -n)$ is equivalent to a type $\frac{1}{c}(-a, -n)$ singularity, or equivalently to a type $\frac{1}{c}(a, n)$ singularity.

Reverting the roles of $n$ and $c$, an analogous computation which we do not reproduce, shows that we can consider complex coordinates on $(\widehat{U \cap U_t}, (0, 0, [0 : 1]))$ which are modeled by $(\mathbb{C}^2_{(w',r)}/\frac{1}{n}(b, -c), 0)$. More precisely, the local coordinates are given by the map

$$\widehat{f}_t : \mathbb{C}^2_{(w',r)}/\frac{1}{n}(b, -c) \to \widehat{U \cap U_t} \subset (\mathbb{C}^2_{(Z,W)} \times \mathbb{P}^1_{[s,t]})/\frac{1}{b}(c, n, 0, 0),$$

where

$$\widehat{f}_t(w', r) = (((w'^{\frac{1}{b}})^c r, (w'^{\frac{1}{b}})^n), [r^n : 1]) = (Z, W, [s : t]).$$

The transition map on the overlap set $U \cap \widehat{U_s \cap U_t}$, corresponding to $s \neq 0$ and $t \neq 0$, or equivalently $v \neq 0$ and $r \neq 0$ is of the form:

$$\widehat{f}_t^{-1} \circ \widehat{f}_s(u', v) = (u'(v^{\frac{1}{n}})^b, (v^{\frac{1}{n}})^{-c}) = (w', r),$$

defined on

$$\widehat{f}_t^{-1} \circ \widehat{f}_s : \mathbb{C}^2_{(u',v)}/\frac{1}{c}(b, -n) \setminus (v = 0) \to \mathbb{C}^2_{(w',r)}/\frac{1}{n}(b, -c) \setminus (r = 0).$$

The above mapping is well-defined and biholomorphic. Hence, we have constructed the weighted blow-up of $(\mathbb{C}^2_{(Z,W)}/\frac{1}{b}(c, n), 0))$ as the complex manifold

$$\widehat{U} \subseteq (\mathbb{C}^2_{(Z,W)} \times \mathbb{P}^1_{[s,t]})/\frac{1}{b}(c, n, 0, 0)$$

with an endowed complex structure which has a singularity of the type $\frac{1}{c}(a, n)$ at $((0, 0), [1 : 0])$ and a singularity of the type $\frac{1}{n}(a, c)$ at $((0, 0), [0 : 1])$. The

exceptional divisor $\widehat{E}$ is a smooth rational curve containing the two singular points of $\widehat{U}$.

**Proposition 4.2.** *There exists a weighed blow-up* $b : \widehat{M} \to \overline{M}$ *of the manifold* $\overline{M}$ *at the singular point* $R_2$ *with exceptional divisor* $\widehat{E}$ *such that the complex manifold* $\widehat{M}$ *has two new singular points of type* $\frac{1}{c}(b, -n) \simeq \frac{1}{c}(a, n)$ *and* $\frac{1}{n}(b, -c) \simeq \frac{1}{n}(a, c)$, *respectively, both along* $\widehat{E}$.

*Proof.* We construct $\widehat{M}$ by cutting out a neighborhood of the point $R_2 \in \overline{M} \cap U_y$ and gluing in $\widehat{U}$. Let $b : \widehat{M} \to \overline{M}$ be the blow-up projection.                   $\square$

Let
$$\Pi : \widehat{M} \dashrightarrow \mathbb{P}(a, c, n)$$
be the composition $\Pi = \pi \circ b$.

We denote by $\widehat{L}_j$ the proper transforms in $\widehat{M}$ of the weighted projective lines $L_j \subseteq \overline{M}$, $j = 1, \ldots, l$.

**Proposition 4.3.** *The map* $\Pi : \widehat{M} \to \mathbb{P}(a, c, n)$ *is a globally defined holomorphic map of degree* 1, *contracting only the curves* $\widehat{L}_j$, $j = 1, \ldots, l$ *to distinct smooth points. The exceptional divisor* $\widehat{E}$ *is mapped to the coordinate line* $(x = 0) \subseteq \mathbb{P}_{[x:z:w]}(a, c, n)$.

*Proof.* We need to analyze the mapping $\Pi$ in the two coordinate charts $\widehat{U \cap U_s}$ and $\widehat{U \cap U_t} \subseteq \widehat{U} \subseteq \widehat{M}$. In the chart $\widehat{U \cap U_s}$, the mapping is defined on the complement of the exceptional divisor $\widehat{E}$ corresponding to the open subset

$$\mathbb{C}^2_{(u',v)}/\frac{1}{c}(b, -n) \setminus (u' = 0) \simeq \widehat{U \cap U_s} \setminus \widehat{E} \subseteq (\mathbb{C}^2_{(Z,W)} \times \mathbb{P}^1_{[s,t]})/\frac{1}{b}(c, n, 0, 0)$$

and it is of the form

$$
\begin{aligned}
\Pi \circ \widehat{f}_s(u', v) &= \Pi \left( ((u'^{\frac{1}{b}})^c, (u'^{\frac{1}{b}})^n v), [1 : v^c] \right) \\
&= \pi \left( \left[ (u'^{\frac{n}{b}}v)^{dc} P\left( \frac{u'^{\frac{nc}{b}}}{u'^{\frac{nc}{b}}v^c} \right) : 1 : (u'^{\frac{1}{b}})^c : (u'^{\frac{1}{b}})^n v \right] \right) \\
&= \left[ u'^{\frac{dnc}{b}} v^{dc} P\left( \frac{1}{v^c} \right) : (u'^{\frac{1}{b}})^c : (u'^{\frac{1}{b}})^n v \right] \\
&= \left[ u'^{\frac{a+b}{b}} v^{dc} P\left( \frac{1}{v^c} \right) : (u'^{\frac{1}{b}})^c : (u'^{\frac{1}{b}})^n v \right] \\
&= \left[ u' v^{dc} P\left( \frac{1}{v^c} \right) : 1 : v \right].
\end{aligned}
$$

In this computation, we used that $\overline{M} \cap U_y$ is given by the equation $X = W^{dc} P(\frac{Z^n}{W^c})$, and for the last equality, that the image of $\Pi \circ \widehat{f}_s$ lies in the weighted projective space $\mathbb{P}(a, c, n)$. As $P$ is a polynomial of degree $d$, the first term is a polynomial in

$v$, and the projection can be extended to the entire local chart $\widehat{U \cap U_s}$. Notice that the orbifold point

$$0 \in \mathbb{C}^2_{(u',v)}/\frac{1}{c}(b,-n) \simeq \mathbb{C}^2_{(u',v)}/\frac{1}{c}(a,n)$$

is mapped to the point $[0:1:0] \in \mathbb{P}(a,c,n)$, as expected considering that the mapping is holomorphic of degree 1. In a similar way we can show that $\Pi \circ \widehat{f}_t(w',r) = [w'P(r^n):r:1]$.

Hence the map $\Pi$ extends to a holomorphic map defined on the entire variety $\widehat{M}$. The rational map $\pi$ contracts the lines $L_j$ to the points $s_j = [0:z_j:w_j]$, and so does its extension $\Pi$. $\qquad\square$

**Corollary 4.4.** *A smooth fiber of the $\mathbb{Q}-$Gorenstein deformation of a singularity of type $\dfrac{1}{dn^2}(1, dnm-1)$ is biholomorphic to the complement of the proper transforms of the coordinate lines $(x=0)$ and $(w=0)$ in the blow-up of $d$ distinct smooth points along the line $(x=0) \setminus (w=0)$ in the weighted projective space $\mathbb{P}_{[x:z:w]}(a,c,n)$.*

*Proof.* The variety $\widehat{M}$ has two new singular points introduced by the weighted blow-up, none of them lying on the contracted lines $\widehat{L}_j$, $j=1,\ldots,d$. As these lines are collapsed to distinct smooth points $s_j$, then they must be rational curves of self-intersection $(-1)$. Hence, $\widehat{M}$ is biholomorphic to the blow-up of $d$ distinct smooth points along the line $(x=0) \setminus (w=0)$ in the weighted projective space $\mathbb{P}_{[x:z:w]}(a,c,n)$. $\qquad\square$

*Proof of Theorem C.* An arbitrary fiber $M$ of the $\mathbb{Q}-$Gorenstein deformation has singular points $S_j$, whenever $k_j > 1$. Let $\overline{N}$ be the minimal resolution of the singular points $S_j$ in the compactification $\overline{M}$, $\widehat{N}$ the corresponding weighted blow-up of $\overline{N}$ at the point $R_2$, and $\widehat{\Pi}$ the induced projection

$$\widehat{\Pi} : \widehat{N} \to \mathbb{P}(a,c,n).$$

If $k_j = 1$, then the proper transform of $\widehat{L}_j$ is a rational curve of self-intersection $(-1)$, which is collapsed to the smooth point $s_j \in \mathbb{P}(a,c,n)$. In the case when $k_j > 1$, both the exceptional divisor above $S_j$ and the proper transform of the line $\widehat{L}_j$ are contracted to the smooth point $s_j$. As the singular point $S_j$ is of type $A_{k_j-1}$ the exceptional divisor introduced by the minimal resolution is a chain of $(k_j-1)$ rational curves of self-intersection $(-2)$. Hence, as we know that $\widehat{\Pi}$ contracts this chain and the proper transform of the line $\widehat{L}_j$ to the smooth point $s_j$ and there are no singular points on any of these curves, then this configuration must be obtained as the $k_j-$times iterated blow-up of $s_j$. In conclusion, the variety $\widehat{N}$ is biholomorphic to the $k_j-$times iterated blow-up of the points $s_j$, $j=1,\ldots,l$. $\qquad\square$

In particular, if we consider $M = (xy = (z-a)^n) \simeq \mathbb{C}^2/\frac{1}{n}(1,-1)$, and its compactification $\overline{M}$, the above proof implies the Corollary 1.2.

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