

REMARKS ON ASTHENO-KÄHLER MANIFOLDS, BOTT-CHERN AND AEPPLI COHOMOLOGY GROUPS

IONUȚ CHIOSE AND RAREȘ RĂSDEACONU

Dedicated to the memory of Professor Jean-Pierre Demailly.

ABSTRACT. We provide a new cohomological obstruction to the existence of astheno-Kähler metrics, and study relevant examples.

1. INTRODUCTION

Let (M, J, g) be a Hermitian manifold of real dimension $2n$ and $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ its fundamental 2-form. If ω is d -closed, then the metric g is called Kähler, and a complex manifold carrying such a metric is called a Kähler manifold. Kähler manifolds exist in abundance and satisfy well-documented remarkable cohomological properties. For example, they are formal and the $\partial\bar{\partial}$ -lemma holds [DGMS], the Hodge symmetry is satisfied, and the Hodge-Frölicher spectral sequence degenerates at the first page. Furthermore, a result of Harvey-Lawson [HL] states that a compact complex manifold carries a Kähler metric if and only if it carries no positive $(1, 1)$ -components of boundaries. Nevertheless, the failure of such cohomological properties obstructs the existence of Kähler metrics, and many examples of non-Kähler manifolds are present in the literature. It is only natural to impose weaker conditions on the fundamental 2-form ω . Such conditions have been oftentimes considered and studied, and many of them involve the closure with respect to the d or $\partial\bar{\partial}$ operators of the (k, k) -form ω^k , for some integer $k \geq 1$. However, the presence of special classes of Hermitian metrics hardly ever imposes good cohomological behavior.

A Hermitian metric satisfying

$$\partial\bar{\partial}\omega^{n-2} = 0$$

is called an *astheno-Kähler metric*. Such metrics were introduced by Jost and Yau in their study of the Hermitian harmonic maps, and used to prove an extension of Siu's Rigidity Theorem to non-Kähler manifolds [JY, Theorem 6]. Later, Li, Yau and Zheng found other interesting applications, such as a generalization to higher dimension of Bogomolov's Theorem on class VII_0 surfaces [LYZ, Corollary 3], while Carlson and Toledo used them in [CT] to obtain results on the fundamental groups of class VII surfaces. Several construction methods of astheno-Kähler metrics are currently known (see Section 2). However, only few obstructions to their existence are known. The obstructions are derived from an observation of Jost and Yau

Date: November 8, 2022.

2020 Mathematics Subject Classification. Primary: 53C55; Secondary: 32J18.

Key words and phrases. Complex manifolds, astheno-Kähler metrics, Bott-Chern cohomology, Aeppli cohomology.

[JY], who noticed that on a manifold carrying an astheno-Kähler metric every holomorphic 1-form is closed. This remark was generalized by Li, Yau and Zheng [LYZ] who found a Harvey-Lawson type criterion for astheno-Kähler metrics: if a compact complex manifold admits a weakly positive, $\partial\bar{\partial}$ -exact, non-vanishing $(2, 2)$ -current, then it cannot carry an astheno-Kähler metric (see also [FGV] for more details).

The aim of this article is to exhibit and study a new relation between the Bott-Chern and Aeppli cohomologies of a compact complex manifold which appears at the level of $(0, 1)$ -forms in the presence of an astheno-Kähler metric. In Section 4 the following result is proved.

Theorem A. *On a compact astheno-Kähler manifold M , the following inequalities hold:*

$$h_{BC}^{0,1}(M) \leq h_A^{0,1}(M) \leq h_{BC}^{0,1}(M) + 1. \quad (1.1)$$

Such a result yields an obstruction to the existence of astheno-Kähler metrics on a given complex manifold. In Section 5 we test this obstruction against two classes of non-Kähler manifolds, the Nakamura and the Oeljeklaus-Toma manifolds, confirming that they cannot carry astheno-Kähler metrics.

Several applications are obtained by appealing to two results of independent interest regarding the Bott-Chern and Aeppli cohomologies of compact complex manifolds which are proved in Section 3. The first one is a very weak form of a Künneth formula. It should be pointed out that while a Künneth formula for the Dolbeault cohomology is available [GH], a similar formula is not known for the Bott-Chern and Aeppli cohomology theories¹. However, a weak form holds for $(0, 1)$ -forms, which suffices for the applications considered here.

Theorem B. *If X and Y are compact complex manifolds, then*

$$\begin{aligned} h_{BC}^{0,1}(X \times Y) &= h_{BC}^{0,1}(X) + h_{BC}^{0,1}(Y) \\ h_A^{0,1}(X \times Y) &\geq h_A^{0,1}(X) + h_A^{0,1}(Y). \end{aligned} \quad (1.2)$$

This result is used in combination with Theorem A in Section 5 to study the existence of astheno-Kähler metrics on Cartesian products. As a particular case, the following result is obtained:

Theorem C. *A Cartesian product of two compact complex surfaces admits an astheno-Kähler metric if and only if at least one of the surfaces admits a Kähler metric.*

It is noticed in Remark 5.9 that to prove that a Cartesian of two non-Kähler surfaces does not carry an astheno-Kähler metric, the Jost-Yau obstruction provides no relevant information.

A second result of independent interest obtained in Section 3 concerns the Bott-Chern and Aeppli cohomologies for nilmanifolds with nilpotent complex structure:

Theorem D. *Let M be non-Kähler nilmanifold equipped with a nilpotent complex structure. If $h_A^{0,1}(M) = h_{BC}^{0,1}(M)$, then M is a compact complex torus.*

As an immediate application of Theorems A and D we obtain:

¹After the preprint version of this article appeared, J. Stelzig informed the authors that he is able to prove a more general Künneth formula [St3].

Corollary E. *Let M be non-Kähler nilmanifold equipped with a nilpotent complex structure. If M admits an astheno-Kähler metric then $h_A^{0,1}(M) = h_{BC}^{0,1}(M) + 1$.*

In small dimensions, there are known examples of nilmanifolds with nilpotent complex structure saturating the upper bound and carrying invariant astheno-Kähler metric [FT2, FGV, RT, ST]. In Section 2 it is noticed that there exist astheno-Kähler metrics on nilmanifolds with nilpotent complex structure of arbitrary dimension (see Theorem 2.5).

Conversely, one may ask if a compact complex manifold saturating either one of the bounds in (1.1) carries an astheno-Kähler metric. For the upper bound, it is already known that in dimension six the answer is negative. For example, the Iwasawa manifold has $h_{BC}^{0,1} = 2$ and $h_A^{0,1} = 3$ [An], and yet it does not carry astheno-Kähler metrics, as it supports non-closed holomorphic 1-forms, contradicting the Jost-Yau criterion. More examples of nilmanifolds satisfying similar properties can be found in dimension six from the classification of the SKT structures in [FPS] and the Bott-Chern cohomology computations in [LUV]. For the upper bound, in Section 6 one more class of relevant examples is indicated.

Theorem F. *Any compact Vaisman manifold of dimension at least three satisfies $h_A^{0,1} = h_{BC}^{0,1} + 1$ and carries no astheno-Kähler metric.*

As a consequence, an old conjecture of Li, Yau and Zheng [LYZ, page 108] is confirmed:

Corollary G. *There exists no astheno-Kähler metrics on similarity Hopf manifolds of dimension at least three.*

For the lower bound, in Section 6 two examples are exhibited, a solvmanifold of complex dimension $2n$, $n \geq 3$ and the Fujiki class \mathcal{C} non-Kähler manifolds of dimension three, which do not carry astheno-Kähler metrics while saturating the lower bound in (1.1). In particular, one can see that the class of astheno-Kähler manifolds is not invariant under modifications (see Corollary 6.8). The authors are not aware of any example of a non-Kähler, compact, complex, astheno-Kähler manifold satisfying $h_{BC}^{0,1} = h_A^{0,1}$.

2. EXISTENCE AND OBSTRUCTIONS

Definition 2.1. Let X be an n -dimensional complex manifold. X is called p -pluriclosed if it admits a Hermitian metric g whose fundamental form ω satisfies

$$i\partial\bar{\partial}\omega^{n-p} = 0.$$

Gauduchon showed that 1-Gauduchon metrics always exist [Ga], a result with widespread implications, and such metrics are known as *Gauduchon metrics*. The 2-Gauduchon metrics appeared for the first time in the work of Jost and Yau [JY] under the name of *astheno-Kähler metrics*, and used in a variety of applications [CT, JY, LYZ]. The $(n-1)$ -pluriclosed metrics were introduced by Bismut [Bi] and have received a lot of attention in the recent years under different names, such as *strongly Kähler with torsion* (SKT) or *pluriclosed metrics*.

2.1. Existence of astheno-Kähler metrics on nilmanifolds. There are currently a few general methods to construct non-Kähler astheno-Kähler metrics. The most powerful results are obtained on complex nilmanifolds.

Definition 2.2. A complex nilmanifold $\Gamma \backslash G$ is a quotient of a simply-connected, connected nilpotent Lie group G endowed with a left invariant integrable almost complex structure by a lattice $\Gamma \subset G$ of maximal rank.

A nilmanifold $\Gamma \backslash G$ inherits its complex structure from that of G by passing to the quotient.

In complex dimension three, where the notions of SKT and astheno-Kähler metrics coincide, we have the following result of Fino, Parton and Salamon:

Theorem 2.1 (Theorem 1.2 [FPS]). *Let $M = \Gamma \backslash G$, be a (real) six-dimensional nilmanifold with an invariant complex structure J . Then the SKT condition is satisfied if and only if M has a basis $\alpha_i, i = 1, 2, 3$ of $(1, 0)$ -forms such that:*

$$\begin{cases} d\alpha_1 = 0 \\ d\alpha_2 = 0 \\ d\alpha_3 = A\bar{\alpha}_1 \wedge \alpha_2 + B\bar{\alpha}_2 \wedge \alpha_2 + C\alpha_1 \wedge \bar{\alpha}_1 + D\alpha_1 \wedge \bar{\alpha}_2 + E\alpha_1 \wedge \alpha_2, \end{cases} \quad (2.1)$$

where A, B, C, D, E are complex numbers such that

$$|A|^2 + |D|^2 + |E|^2 + 2\Re(\bar{B}C) = 0.$$

Remark 2.2. There are 18 isomorphism classes of the underlying nilpotent Lie algebras [Sa], out of which one is the compact complex three-dimensional torus, the only Kähler example, and only four of them are non-Kähler and admit SKT metrics [FPS, Theorem 3.2]. The interested reader is also referred to [Ug] for a different proof of these result.

To construct examples of left invariant astheno-Kähler metrics on complex nilmanifolds in arbitrary dimension it is useful to impose suitable conditions on the structure equations. Generalizing Theorem 2.1, examples of astheno-Kähler metrics with interesting properties were found in the recent years on nilmanifolds with nilpotent complex structure [FT2, FGV, RT, ST, LU].

Let G be a nilpotent group of dimension $2n$, with Lie algebra \mathfrak{g} . Suppose G carries a left invariant integrable almost complex structure J . An ascending series $\{\mathfrak{a}_l; l \geq 0\}$ compatible with J is defined inductively by

$$\mathfrak{a}_0 = 0, \quad \mathfrak{a}_l = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1} \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}\}, \quad l \geq 1.$$

It is easy to verify that the \mathfrak{a}_l is an ideal of \mathfrak{g} and a complex subspace of \mathfrak{g} . Moreover, $\mathfrak{a}_l \subseteq \mathfrak{a}_{l+1}$, for each $l \geq 0$, and if $\mathfrak{a}_l = \mathfrak{a}_{l+1}$ for some $l \geq 0$, then $\mathfrak{a}_r = \mathfrak{a}_l$ for all $r \geq l$ [CFGU].

Definition 2.3 (cf., [CFGU]). Let G be a $2n$ -dimensional simply-connected connected nilpotent Lie group with Lie algebra \mathfrak{g} , and equipped with a left invariant integrable almost complex structure J .

- 1) We shall say that J is a nilpotent complex structure if $\mathfrak{a}_k = \mathfrak{g}$ for some $k > 0$.
- 2) Furthermore, if J is a nilpotent, left-invariant complex structure on G , and Γ is a co-compact lattice of G , we shall say that the compact nilmanifold $\Gamma \backslash G$ with the complex structure induced by J has a nilpotent complex structure.

We recall next the following characterization of the nilpotent complex structure:

Theorem 2.3 (Theorems 12 and 13, [CFGU]). *Let G be a $2n$ -dimensional simply-connected nilpotent Lie group with Lie algebra \mathfrak{g} , and equipped with a left invariant integrable almost complex structure J . The complex structure J is nilpotent if and only if there exists a (complex) basis $\{\alpha_i, i \leq n\}$ of left invariant forms of type $(1, 0)$ such that the structure equations of G are of the form*

$$d\alpha_i = \sum_{j < k < i} A_{j,k}^i \alpha_j \wedge \alpha_k + \sum_{j,k < i} B_{j,k}^i \alpha_j \wedge \bar{\alpha}_k, \quad i = 1, \dots, n, \quad (2.2)$$

where $A_{j,k}^i$ and $B_{j,k}^i$ are constants. Conversely, the structure equations (2.2) define a simply-connected connected nilpotent Lie group G with nilpotent left invariant complex structure.

Remark 2.4. The complex parallelizable nilmanifolds are precisely those nilmanifolds with nilpotent complex structure for which the coefficients $B_{j,k}^i$ in (2.2) vanish, in which case all of the forms $\alpha_i, i = 1, \dots, n$ are holomorphic.

The following result exhibits astheno-Kähler metrics on compact nilmanifolds equipped with a nilpotent complex structure, generalizing the construction in [FT2, Theorem 2.7] to arbitrary dimension.

Theorem 2.5. *Let G be the simply-connected nilpotent Lie group with nilpotent complex structure given by the $(1, 0)$ -forms $\alpha_i, i = 1, \dots, n$, satisfying the structure equations:*

$$\begin{cases} d\alpha_i = 0, i = 1, \dots, n-1, \\ d\alpha_n = \sum_{i < j < n} A_{i,j} \alpha_i \wedge \alpha_j + \sum_{k,l < n} B_{k,l} \alpha_k \wedge \bar{\alpha}_l, \end{cases}$$

where $A_{i,j}, B_{k,l} \in \mathbb{C}$. Then G carries an invariant astheno-Kähler metric if

$$\sum_{i < j < n} |A_{i,j}|^2 + \sum_{i,j < n, i \neq j} |B_{i,j}|^2 + 2\Re\left(\sum_{i < j < n} B_{ii} \bar{B}_{jj}\right) = 0.$$

Furthermore, for every $n \geq 3$, there exist compact nilmanifolds with nilpotent complex structures of complex dimension n carrying an invariant astheno-Kähler metric.

Proof. As in [FT2, Theorem 2.7], it is enough to find sufficient conditions satisfied by the coefficients on the structure equations (2.2) such that the diagonal metric

$$g = \frac{1}{2} \sum_{i=1}^n \alpha_i \otimes \bar{\alpha}_i + \bar{\alpha}_i \otimes \alpha_i$$

is astheno-Kähler. In other words, we will require that $\partial\bar{\partial}\omega^{n-2} = 0$, where

$$\omega = \frac{i}{2} \sum_{i=1}^n \alpha_i \wedge \bar{\alpha}_i$$

is the fundamental form of the metric g . To adapt to the structure equations, it is convenient to write ω as

$$\omega = \frac{i}{2} (\omega_0 + \alpha_n \wedge \bar{\alpha}_n),$$

where $\omega_0 = \sum_{i=1}^{n-1} \alpha_i \wedge \bar{\alpha}_i$. An immediate computation shows that

$$\omega^{n-2} = \left(\frac{i}{2}\right)^{n-2} (\omega_0^{n-2} + (n-2)\omega_0^{n-3} \wedge \alpha_n \wedge \bar{\alpha}_n), \quad (2.3)$$

while, for every $p > 0$

$$\omega_0^p = \binom{n-1}{p} \sum_{i_1 < \dots < i_p < n} \alpha_{i_1 \bar{i}_1 \dots i_p \bar{i}_p},$$

where $\alpha_{i_1 \bar{i}_1 \dots i_p \bar{i}_p}$ is a short notation for $\alpha_{i_1} \wedge \bar{\alpha}_{i_1} \wedge \dots \wedge \alpha_{i_p} \wedge \bar{\alpha}_{i_p}$, a notation which we will adopt henceforth.

Notice now from the structure equations (2.5) that $\partial\alpha_i = \bar{\partial}\alpha_i = \partial\bar{\alpha}_i = \bar{\partial}\bar{\alpha}_i = 0$ for every $i = 1, \dots, n-1$, while

$$\partial\alpha_n = \sum_{i < j < n} A_{ij} \alpha_{ij} \quad \text{and} \quad \bar{\partial}\alpha_n = \sum_{i, j < n} B_{ij} \alpha_{i\bar{j}}. \quad (2.4)$$

Hence, we have $\partial\bar{\partial}\alpha_n = \bar{\partial}\bar{\partial}\alpha_n = 0$, and $\partial\omega_0^k = \bar{\partial}\omega_0^k = 0$ for every $k > 0$. In particular, from (2.3), we see now that

$$\begin{aligned} \partial\bar{\partial}\omega^{n-2} &= \left(\frac{i}{2}\right)^{n-2} (n-2)\omega_0^{n-3} \wedge \partial\bar{\partial}\alpha_n \bar{\alpha}_n \\ &= c_n \left(\sum_{i_1 < \dots < i_{n-3} < n} \alpha_{i_1 \bar{i}_1 \dots i_{n-3} \bar{i}_{n-3}} \right) \wedge (\bar{\partial}\alpha_n \wedge \partial\bar{\alpha}_n - \partial\alpha_n \wedge \bar{\partial}\bar{\alpha}_n) \\ &= c_n \left(\sum_{i < j < n} |A_{ij}|^2 + \sum_{i, j < n, i \neq j} |B_{ij}|^2 + 2\Re\left(\sum_{i < j < n} B_{ii} \bar{B}_{jj}\right) \right) \alpha_{1\bar{1} \dots (n-1)\bar{(n-1)}}. \end{aligned}$$

where c_n is a non-zero constant depending on n .

Imposing now the condition $\partial\bar{\partial}\omega^{n-2} = 0$ and using (2.4), we find that the metric g is astheno-Kähler if and only if

$$\sum_{i < j < n} |A_{ij}|^2 + \sum_{i, j < n, i \neq j} |B_{ij}|^2 + 2\Re\left(\sum_{i < j < n} B_{ii} \bar{B}_{jj}\right) = 0. \quad (2.5)$$

The conclusion of the theorem follows by noticing that (2.5) admits solutions with $A_{ij}, B_{ij} \in \mathbb{Q}[i]$, for example one can take $A_{ij} = 1$, for all $1 \leq i < j < n$, $B_{ij} = \frac{1}{2}$ for all $1 \leq i \neq j < n$, and $B_{ii} = i$. In this case, it is guaranteed by Malčev's theorem [Mal] that we can find a maximal rank lattice $\Gamma \subset G$ such that $M = \Gamma \backslash G$ is a compact nilmanifold with nilpotent complex structure. The induced metric will automatically be astheno-Kähler. \square

2.2. Other constructions of astheno-Kähler metrics. For convenience, we will mention a few other sources of astheno-Kähler manifolds. Products of Sasakian manifolds carry astheno-Kähler metrics, as observed by Matsuo [Mat]. In particular, the Calabi-Eckmann manifolds carry such metrics. This method was generalized by Fino, Grantcharov and Vezzoni. In [FGV], who constructed astheno-Kähler metrics on torus bundles over a Kähler base. This technique recovers some of the known astheno-Kähler metrics on nilmanifolds with nilpotent complex structure, Matsuo's metrics, and also produces non-homogeneous ones. We will not add the

details, as we will only briefly mention the Calabi-Eckmann case. For $n > 3$ other examples of astheno-Kähler manifolds have been found by Fino and Tomassini [FT2] via the twist construction [Sw].

Finally, according to Fino and Tomassini [FT1, FT2], starting from previously known examples of SKT and astheno-Kähler manifolds one can construct new ones:

Theorem 2.6 (Fino, Tomassini). *Let M be a complex manifold, $Y \subset M$ is a compact complex submanifold, and \widetilde{M} the blowing-up of M along Y .*

- 1) *If M admits a pluriclosed metric then \widetilde{M} admits a pluriclosed metric.*
- 2) *If M admits an astheno-Kähler metric ω such that*

$$\partial\bar{\partial}\omega = 0 \quad \text{and} \quad \partial\bar{\partial}\omega^2 = 0, \quad (2.6)$$

then \widetilde{M} admits an astheno-Kähler metric satisfying (2.6), too.

2.3. Known obstructions to the existence of astheno-Kähler metrics. The presence of astheno-Kähler metric was noticed to force strong conditions on holomorphic 1-forms by Jost and Yau.

Lemma 2.7 (Jost, Yau [JY]). *Let X be a compact astheno-Kähler manifold. Then every holomorphic 1-form on X is closed.*

Proof. Let ω satisfy the conditions of the definition. Let ϕ be a holomorphic 1-form, i.e., $\bar{\partial}\phi = 0$. Then

$$0 \leq \int \partial\phi \wedge \bar{\partial}\bar{\phi} \wedge \omega^{n-2} = \int \phi \wedge \bar{\phi} \wedge \partial\bar{\partial}\omega^{n-2} = 0.$$

and this implies $\partial\phi = 0$. □

A more general obstruction to the existence of astheno-Kähler metrics first noticed in [LYZ] (see also [FGV, Corollary 2.2]) is given by the Harvey-Lawson type criterion:

Theorem 2.8. *If a compact complex manifold M admits a weakly positive and $\partial\bar{\partial}$ -exact and non-vanishing $(2, 2)$ -current, then it does not admit an astheno-Kähler metric.*

Proof. Suppose $i\partial\bar{\partial}T$ is weakly positive, where we consider T as a form with distribution coefficients. Then for an astheno-Kähler metric ω we have by integration by parts

$$0 < \int_M i\partial\bar{\partial}T \wedge \omega^{n-2} = \int_M T \wedge i\partial\bar{\partial}\omega^{n-2} = 0,$$

which gives a contradiction. □

Remark 2.9. Note that for a holomorphic one-form α , the form $i(\partial\alpha \wedge \bar{\partial}\bar{\alpha}) = i\partial\bar{\partial}(\alpha \wedge \bar{\alpha})$ is weakly positive. This observation implies that the obstruction of Jost and Yau in Lemma 2.7 is a consequence of Theorem 2.8.

Remark 2.10. Furthermore, we remark that the statement of Corollary 2.8 cannot be reversed, since in general not every positive $(n-2, n-2)$ -form arises as $(n-2)$ -power of a positive $(1, 1)$ -form.

3. BOTT-CHERN AND AEPPLI COHOMOLOGIES

Let X be a compact complex manifold of dimension n , and denote by $\mathcal{A}^{p,q}(X)$ its space of smooth (p, q) -forms. The Bott-Chern cohomology groups are

$$H_{BC}^{p,q}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) \mid d\alpha = 0\}}{\{i\partial\bar{\partial}\beta \mid \beta \in \mathcal{A}^{p-1,q-1}(X)\}},$$

while the Aeppli cohomology groups are

$$H_A^{p,q}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) \mid i\partial\bar{\partial}\alpha = 0\}}{\{\partial\beta + \bar{\partial}\gamma \mid \beta \in \mathcal{A}^{p-1,q}(X), \gamma \in \mathcal{A}^{p,q-1}(X)\}}.$$

The Bott-Chern and Aeppli cohomologies satisfy a remarkable symmetry property [Sc]:

$$H_{\#}^{p,q}(X) = \overline{H_{\#}^{q,p}(X)}, \quad (3.1)$$

where $\# \in \{BC, A\}$. Furthermore, the groups $H_{BC}^{p,q}(X, \mathbb{C})$ and $H_A^{n-p,n-q}(X, \mathbb{C})$ are dual via the pairing

$$H_{BC}^{p,q}(X, \mathbb{C}) \times H_A^{n-p,n-q}(X, \mathbb{C}) \rightarrow \mathbb{C}, ([\alpha], \{\beta\}) \rightarrow \int_X \alpha \wedge \beta. \quad (3.2)$$

We will denote by $[\eta]$ the class of a d -closed (p, q) -form η in $H_{BC}^{p,q}(X)$ and by $\{\zeta\}$ the class of a $i\partial\bar{\partial}$ -closed (p, q) -form ζ in $H_A^{p,q}(X)$. Notice that the elements in $H_{BC}^{0,1}(X)$ are the d -closed $(0, 1)$ -forms on X .

3.1. A (very) weak Künneth formula. Let X and Y two compact complex manifolds of dimensions $\dim X = n$ and $\dim Y = m$. Let

$$p : X \times Y \rightarrow X \quad \text{and} \quad q : X \times Y \rightarrow Y$$

be two natural projections.

Proof of Theorem B. For the statement regarding the Bott-Chern cohomology, it suffices to show that the natural map

$$\mathfrak{s}^{0,1} : H_{BC}^{0,1}(X) \oplus H_{BC}^{0,1}(Y) \rightarrow H_{BC}^{0,1}(X \times Y), \quad (\alpha, \beta) \mapsto p^*\alpha + q^*\beta$$

is an isomorphism.

Let $\alpha \in H_{BC}^{0,1}(X)$ and $\beta \in H_{BC}^{0,1}(Y)$ such that $(\alpha, \beta) \in \ker(\mathfrak{s}^{0,1})$. That means $\alpha \in \mathcal{A}^{0,1}(X)$ and $\beta \in \mathcal{A}^{0,1}(Y)$ are d -closed forms in X and Y , respectively, and $p^*\alpha + q^*\beta = 0$ in $\mathcal{A}^{0,1}(X \times Y)$. The latter trivially implies $\alpha = 0$ and $\beta = 0$, and the injectivity of $\mathfrak{s}^{0,1}$ is proven. To show its surjectivity, let $\eta \in H_{BC}^{0,1}(X \times Y)$. We have $\partial\eta = \bar{\partial}\eta = 0$, which imply $\bar{\partial}\bar{\eta} = 0$. Consider now the class $[\bar{\eta}] \in H_{\bar{\partial}}^{1,0}(X \times Y)$ in the Dolbeault cohomology of $X \times Y$. From the Künneth formula for the Dolbeault cohomology [GH, page 105], it follows that there exist $\alpha \in \mathcal{A}^{0,1}(X)$ and $\beta \in \mathcal{A}^{0,1}(Y)$ such that $\partial\alpha = 0$, $\partial\beta = 0$ and such that

$$\eta = p^*\alpha + q^*\beta. \quad (3.3)$$

Since η is d -closed, it follows that $\bar{\partial}\eta = p^*\bar{\partial}\alpha + q^*\bar{\partial}\beta$ and if we consider local coordinates on X and Y , the above equality implies that $\bar{\partial}\alpha = 0$ and $\bar{\partial}\beta = 0$. Therefore $d\alpha = 0$ and $d\beta = 0$, which together with (3.3) implies that the map $\mathfrak{s}^{0,1}$ is surjective, as well.

For the statement regarding the Aeppli cohomology of $X \times Y$, fix Hermitian metrics ω_X and ω_Y on X and Y , respectively and let

$$\mathcal{H}_A^{0,1}(X) = \{\alpha \in \mathcal{A}^{0,1}(X) \mid \partial\bar{\partial}\alpha = 0, \bar{\partial}^*\alpha = 0\},$$

where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ is considered with respect to ω_X . It is known from [Sc] that $\mathcal{H}_A^{0,1}(X)$ is a finite-dimensional vector space, isomorphic to $H_A^{0,1}(X)$. We also consider similarly defined sets $\mathcal{H}_A^{0,1}(Y)$ and $\mathcal{H}_A^{0,1}(X \times Y)$, where we equip $X \times Y$ with the product Hermitian metric $p^*\omega_X + q^*\omega_Y$. We define the map

$$\mathfrak{t}^{0,1} : \mathcal{H}_A^{0,1}(X) \oplus \mathcal{H}_A^{0,1}(Y) \rightarrow \mathcal{A}^{0,1}(X \times Y), \quad (\alpha, \beta) \mapsto p^*\alpha + q^*\beta.$$

This map is clearly injective. In order to prove the inequality

$$h_A^{0,1}(X) + h_A^{0,1}(Y) \leq h_A^{0,1}(X \times Y),$$

it is enough to show that the range of \mathfrak{t}_A is included in $\mathcal{H}_A^{0,1}(X \times Y)$. Let $\eta = p^*\alpha + q^*\beta$, where $\alpha \in \mathcal{H}_A^{0,1}(X)$ and $\beta \in \mathcal{H}_A^{0,1}(Y)$. To prove that $\eta \in \mathcal{H}_A^{0,1}(X \times Y)$, one can immediately see that $\partial\bar{\partial}\eta = 0$. To show that $\bar{\partial}^*\eta = 0$, we recall next a few standard results in complex geometry.

Let \star be the Hodge operator acting on the space of $(0,1)$ -forms on a compact complex manifold of complex dimension p , equipped with a Hermitian metric with fundamental for ω . Then the adjoint operator $\bar{\partial}^*$ acting on $(0,1)$ -forms is given by $\bar{\partial}^* = -\star\partial\star$. Also, according to [Vo, Proposition 6.29, p.150], for $(0,1)$ -forms we have

$$\star\gamma = \frac{i}{(p-1)!}\omega^{p-1} \wedge \gamma. \quad (3.4)$$

Therefore, a $(0,1)$ -form γ satisfies $\bar{\partial}^*\gamma = 0$ if and only if $\partial(\gamma \wedge \omega^{p-1}) = 0$.

We will prove now that $\bar{\partial}^*p^*\alpha = 0$. By the above consideration, it suffices to show that

$$\partial(p^*\alpha \wedge (p^*\omega_X + q^*\omega_Y)^{m+n-1}) = 0,$$

which for degree reasons is equivalent to proving

$$\partial(p^*\alpha \wedge p^*\omega_X^{n-1} \wedge q^*\omega_Y^m) = 0,$$

As $\bar{\partial}^*\alpha = 0$, we know that $\partial(\alpha \wedge \omega_X^{n-1}) = 0$, and compute:

$$\begin{aligned} \partial(p^*\alpha \wedge p^*\omega_X^{n-1} \wedge q^*\omega_Y^m) &= \partial(p^*\alpha \wedge p^*\omega_X^{n-1}) \wedge q^*\omega_Y^m \\ &= \partial p^*(\alpha \wedge \omega_X^{n-1}) \wedge q^*\omega_Y^m \\ &= p^*(\partial(\alpha \wedge \omega_X^{n-1})) \wedge q^*\omega_Y^m \\ &= 0. \end{aligned}$$

Hence $\bar{\partial}^*p^*\alpha = 0$ if $\bar{\partial}^*\alpha = 0$, and a similar proof shows that $\bar{\partial}^*q^*\beta = 0$ if $\bar{\partial}^*\beta = 0$, implying that $\bar{\partial}^*\eta = 0$. Therefore $\mathcal{H}_A^{0,1}(X \times Y)$ contains the range of \mathfrak{t}_A , and the inequality

$$h_A^{0,1}(X) + h_A^{0,1}(Y) \leq h_A^{0,1}(X \times Y)$$

follows easily. \square

3.2. The Bott-Chern cohomology of the blow-ups. A formula computing the Bott-Chern cohomology of the blow-up was conjectured by S. Rao, S. Yang and X. Yang in [RYY], and proved by Stelzig [St1]. We recall its statement for convenience:

Theorem 3.1. *Let Y be a compact complex manifold of complex dimension $n \geq 2$, $Z \subset Y$ be a closed complex submanifold of codimension $c \geq 2$, and $f : X \rightarrow Y$ be the blow-up of Y with center Z . Then, there exists an isomorphism*

$$H_{BC}^{p,q}(X) \simeq H_{BC}^{p,q}(Y) \oplus \left(\bigoplus_{i=1}^{c-1} H_{BC}^{p-i,q-i}(Z) \right) \quad (3.5)$$

for $p, q \leq n$.

As an immediate consequence we notice the following:

Corollary 3.2. *Let Y be a compact complex manifold of complex dimension $n \geq 2$, $Z \subset Y$ be a closed complex submanifold of codimension $c \geq 2$, and $f : X \rightarrow Y$ be the blow-up of Y with center Z . Then*

- 1) $h_{BC}^{0,p}(X) = h_{BC}^{0,p}(Y)$ for every $p \geq 0$.
- 2) $h_A^{0,1}(X) = h_A^{0,1}(Y)$.

Proof. Item 1) is the content of [YY1, Theorem 1.2.(ii)]. To prove item 2), by duality we have

$$h_A^{0,1}(X) = h_{BC}^{n,n-1}(X) = h_{BC}^{n,n-1}(Y) + \sum_{i=1}^{c-1} h_{BC}^{n-i,n-1-i}(Z).$$

However, as $\dim_{\mathbb{C}} Z = n - c$, for dimension reasons, the Bott-Chern numbers $h_{BC}^{n-i,n-1-i}(Z)$ vanish for every $1 \leq i \leq c - 1$, and so $h_A^{0,1}(X) = h_{BC}^{n,n-1}(Y) = h_A^{0,1}(Y)$. \square

Remark 3.3. Item 2) in Corollary 3.2 does not explicitly appear in [St1], [YY1, YY2] or [RYY], and it seems to be a new observation.

Remark 3.4. As a consequence of the weak factorization theorem [AKMW], and the duality between the Bott-Chern and Aeppli cohomologies, we find that $h_{BC}^{0,p}$, $p \geq 0$ and $h_A^{0,1}$ are bimeromorphic invariants.

3.3. The Bott-Chern cohomology of complex nilmanifolds. Let $M = \Gamma \backslash G$ be a compact nilmanifold equipped with a left-invariant integrable almost complex structure J , that is J comes from a (left invariant) complex structure, also denoted by J , on the Lie algebra \mathfrak{g} of G . According to Nomizu's Theorem [No], the de Rham cohomology of a compact nilmanifold can be computed by means of the cohomology of the Lie algebra of the corresponding nilpotent Lie group, where the differential in the complex $\wedge^{\bullet} \mathfrak{g}^*$ is the Chevalley-Cartan differential. This result was refined to compute the Dolbeault and Bott-Chern cohomology of complex nilmanifolds with nilpotent complex structures in [CFGU] and [An], respectively. For convenience, we recall the main result of [CFGU], which provides a useful tool for computing the cohomology of compact nilmanifolds with nilpotent complex structures.

Theorem 3.5 (Main Theorem, [CFGU]). *Let $\Gamma \backslash G$ be a compact nilmanifold with a nilpotent complex structure, and let \mathfrak{g} be the Lie algebra of G . Then there is a quasi-isomorphism of complexes*

$$(\wedge^{p,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \bar{\partial}) \hookrightarrow (\wedge^{p,\bullet}(\Gamma \backslash G), \bar{\partial}).$$

with respect to the operator $\bar{\partial}$ in the canonical decomposition $d = \partial + \bar{\partial}$ of the Chevalley-Eilenberg differential in $\wedge^\bullet(\mathfrak{g}_{\mathbb{C}}^*)$.

As a corollary of Theorem 3.5, Angella proved in [An, Theorem 3.8] that the inclusions of the corresponding subcomplexes yield isomorphisms

$$H_{\#}^{p,q}(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\cong} H_{\#}^{p,q}(M),$$

where $\# \in \{BC, A\}$. We will conclude this section by using Angella's result to prove Theorem D.

Proof of Theorem D. By the symmetry of Bott-Chern and Aeppli cohomologies, we can assume $h_{BC}^{1,0}(M) = h_A^{1,0}(M)$. In particular, we have

$$\begin{aligned} H_{BC}^{1,0}(M) &\simeq H_{BC}^{1,0}(\mathfrak{g}_{\mathbb{C}}) = \{\alpha \in \mathfrak{g}^{1,0} \mid d\alpha = 0\}, \\ H_A^{1,0}(M) &\simeq H_A^{1,0}(\mathfrak{g}_{\mathbb{C}}) = \{\eta \in \mathfrak{g}^{1,0} \mid \partial\bar{\partial}\eta = 0\}. \end{aligned}$$

Accordingly, it suffice to show that if $h_{BC}^{1,0}(\mathfrak{g}_{\mathbb{C}}) = h_A^{1,0}(\mathfrak{g}_{\mathbb{C}})$, then \mathfrak{g} is abelian. Since $M = \Gamma \backslash G$ is equipped with a nilpotent complex structure, by Theorem 2.3, there exists a basis $\{\alpha_i, 1 \leq i \leq n\}$ of the i -eigenspace $\mathfrak{g}^{1,0}$ of the extension of J to $\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ such that the structure equations of G are of the form (2.2).

Notice first from (2.2) that we have $d\alpha_1 = 0$. Suppose now $d\alpha_\ell = 0$ for every $1 \leq \ell < s$. Since $d\alpha_\ell = 0$, we have $\partial\alpha_\ell = \bar{\partial}\alpha_\ell = \partial\bar{\alpha}_\ell = \bar{\partial}\bar{\alpha}_\ell = 0$, $1 \leq \ell < s$. From the structure equations (2.2) we can immediately see that

$$\partial\alpha_s = \sum_{j < k < s} A_{j,k}^s \alpha_j \wedge \alpha_k \quad \text{and} \quad \bar{\partial}\alpha_s = \sum_{j,k < i} B_{j,k}^s \alpha_j \wedge \bar{\alpha}_k.$$

Then $\partial\bar{\partial}\alpha_s = 0$, which means $\alpha_s \in H_A^{1,0}(\mathfrak{g}_{\mathbb{C}}) = H_{BC}^{1,0}(\mathfrak{g}_{\mathbb{C}})$. Therefore $d\alpha_s = 0$. By induction, it follows that $d\alpha_i = 0$, for every $i = 1, \dots, n$, which is equivalent to \mathfrak{g} being abelian. \square

4. A NEW OBSTRUCTION

Let (M, ω) be compact an astheno-Kähler manifold of complex dimension n , and let f be a \mathcal{C}^∞ real function on M such that $e^{(n-1)f}\omega^{n-1}$ is $i\partial\bar{\partial}$ -closed. The existence of f follows from Gauduchon's Theorem [Ga]. We define now a linear map $L : H_A^{0,1}(M) \rightarrow \mathbb{C}$ as follows:

$$L(\{\alpha\}) = \int_M \partial\alpha \wedge e^{(n-1)f}\omega^{n-1}. \quad (4.1)$$

Note that L is well-defined since $e^{(n-1)f}\omega^{n-1}$ is $\partial\bar{\partial}$ -closed.

Proof of Theorem A. The natural morphism induced by the identity

$$i_{0,1} : H_{BC}^{0,1}(M) \rightarrow H_A^{0,1}(M)$$

is always injective, and $L(i_{0,1}([\phi])) = 0$ for all $[\phi] \in H_{BC}^{0,1}(M)$. Therefore we have a complex

$$0 \rightarrow H_{BC}^{0,1}(M) \xrightarrow{i_{0,1}} H_A^{0,1}(M) \xrightarrow{L} \mathbb{C}. \quad (4.2)$$

To prove Theorem A, it suffices to prove that the sequence (4.2) is exact, that is to show that $\ker(L) \subseteq H_{BC}^{0,1}(M)$.

Let $\{\alpha\} \in \ker(L)$, i.e., $\int_M \partial\alpha \wedge e^{(n-1)f}\omega^{n-1} = 0$. Consider the elliptic differential operator

$$P : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \quad P(h) = \Lambda(\partial\bar{\partial}h)$$

where Λ is the adjoint of

$$\mathcal{C}^\infty(M) \ni g \rightarrow ge^f\omega \in \mathcal{A}^{1,1}(M)$$

with respect to the metric induced by $e^f\omega$. It is easy to see that the formal adjoint of P is

$$P^*(h) = \bar{\partial}^*\partial^*(he^f\omega).$$

Another way of writing P^* is

$$P^*(h) = -\star\partial\bar{\partial}\left(h\frac{e^{(n-1)f}\omega^{n-1}}{(n-1)!}\right)$$

and it is well known [Ga] that, since $\partial\bar{\partial}\left(e^{(n-1)f}\omega^{n-1}\right) = 0$, the kernel of P^* is 1-dimensional. Namely, it consists of the constant functions on M .

Since P is elliptic, we have the following orthogonal decomposition:

$$\mathcal{C}^\infty(M) = P(\mathcal{C}^\infty(M)) \oplus \ker(P^*)$$

Therefore $\Lambda(\partial\alpha) \in P(\mathcal{C}^\infty(M))$ if and only if $\Lambda(\partial\alpha)$ is orthogonal to $\ker(P^*)$, that is, if and only if $((\Lambda(\partial\alpha), 1)) = 0$. But

$$((\Lambda(\partial\alpha), 1)) = ((\partial\alpha, e^f\omega)) = \int_M \partial\alpha \wedge \star e^f\omega = \int_M \partial\alpha \wedge \frac{e^{(n-1)f}\omega^{n-1}}{(n-1)!} = 0$$

since $\{\alpha\}$ is in the kernel of L and

$$\star e^f\omega = \frac{e^{(n-1)f}\omega^{n-1}}{(n-1)!}$$

where the Hodge star operator is with respect to $e^f\omega$. Therefore $\Lambda(\partial\alpha)$ is in the range of P , which means that $\{\alpha\}$ admits a representative, also denoted by α , such that $\Lambda(\partial\alpha) = 0$. This means that $\partial\alpha$ is primitive with respect to $e^f\omega$. It is easy to see that $\partial\alpha$ is also primitive with respect to ω , and therefore

$$\star\partial\alpha = -\frac{1}{(n-2)!}\partial\alpha \wedge \omega^{n-2}$$

[Vo, Proposition 6.29, p.150], where the Hodge star operator here is considered with respect to ω . Therefore the L^2 norm of $\partial\alpha$ with respect to ω is

$$\|\partial\alpha\|^2 = \int_M \partial\alpha \wedge \star\bar{\partial}\alpha = -\frac{1}{(n-2)!} \int_M \partial\alpha \wedge \bar{\partial}\alpha \wedge \omega^{n-2}$$

Since $\bar{\partial}\alpha$ is of type $(0,2)$, it is also primitive with respect to ω , and so

$$\star\bar{\partial}\alpha = \frac{1}{(n-2)!}\bar{\partial}\alpha \wedge \omega^{n-2}$$

[Vo, Proposition 6.29, p.150] and the L^2 norm of $\bar{\partial}\alpha$ with respect to ω is

$$\|\bar{\partial}\alpha\|^2 = \frac{1}{(n-2)!} \int_M \bar{\partial}\alpha \wedge \partial\bar{\alpha} \wedge \omega^{n-2}.$$

As the metric ω is astheno-Kähler, $\partial\bar{\partial}\omega^{n-2} = 0$, and it follows that

$$0 = \int_M \partial\bar{\partial}(\alpha \wedge \bar{\alpha}) \wedge \omega^{n-2} = \int_M (\bar{\partial}\alpha \wedge \partial\bar{\alpha} - \partial\alpha \wedge \bar{\partial}\bar{\alpha}) \wedge \omega^{n-2} = (n-2)! (|\bar{\partial}\alpha|^2 + |\partial\alpha|^2)$$

therefore $\partial\alpha = \bar{\partial}\alpha = 0$, which means that α is in $H_{BC}^{0,1}(M)$.

Since the sequence

$$0 \rightarrow H_{BC}^{0,1}(M) \rightarrow H_A^{0,1}(M) \rightarrow \mathbb{C}$$

is exact, Theorem A follows. \square

Corollary 4.1. *Every SKT compact complex manifold of dimension three satisfies the inequalities (1.1).*

Remark 4.2. According to the Jost-Yau's Lemma 2.7, on an astheno-Kähler manifold, every $\bar{\partial}$ -closed 1-form (i.e., a holomorphic 1-form) is d -closed. It follows that the identity map on $(1,0)$ -forms induces a morphism

$$H_{\bar{\partial}}^{1,0}(M) \rightarrow H_{BC}^{1,0}(M)$$

inverse to the map

$$H_{BC}^{1,0}(M) \rightarrow H_{\bar{\partial}}^{1,0}(M),$$

also induced by the identity. Hence this map is an isomorphism, and so $h_{BC}^{1,0} = h_{\bar{\partial}}^{1,0}$. Therefore, on an astheno-Kähler manifold we have

$$h_A^{1,0} - h_{\bar{\partial}}^{1,0} \in \{0, 1\}.$$

Remark 4.3. The cohomology group $H_{\bar{\partial}}^{0,1}(M)$ lies between $H_A^{0,1}(M)$ and $H_{BC}^{0,1}(M)$, i.e., the natural morphisms $H_{\bar{\partial}}^{0,1}(M) \rightarrow H_A^{0,1}(M)$ and $H_{BC}^{0,1}(M) \rightarrow H_{\bar{\partial}}^{0,1}(M)$ are injective, so it follows that, on an astheno-Kähler manifold, at least one of the morphisms

$$H_{\bar{\partial}}^{0,1}(M) \rightarrow H_A^{0,1}(M) \quad \text{and} \quad H_{BC}^{0,1}(M) \rightarrow H_{\bar{\partial}}^{0,1}(M)$$

is an isomorphism.

Remark 4.4. Theorem A generalizes in arbitrary dimension classical results on compact complex surfaces. Any compact complex surface is trivially astheno-Kähler, and if the surface also admits a Kähler metric, then $h_A^{0,1} = h_{BC}^{0,1}$ by the $\partial\bar{\partial}$ -lemma. If the surface is non-Kähler, one has $h_A^{0,1} = h_{BC}^{0,1} + 1$. Indeed, from [Ko, Theorem 3], it is known that, on a non-Kähler surface, $h_{\bar{\partial}}^{0,1} = h_{\bar{\partial}}^{1,0} + 1$. On the other hand, on compact complex surfaces, $h_{BC}^{0,1} = h_{BC}^{1,0} = h_{\bar{\partial}}^{1,0}$ and $h_{\bar{\partial}}^{0,1} = h_A^{0,1}$.

Remark 4.5. From the above proof, it follows that on an astheno-Kähler manifold one has $h_{BC}^{0,1} = h_A^{0,1}$ if and only if $\partial(e^{(n-1)f}\omega^{n-1})$ is $\partial\bar{\partial}$ -exact. Indeed, if $\partial(e^{(n-1)f}\omega^{n-1})$ is $\partial\bar{\partial}$ -exact, the application L defined above is zero, hence $h_{BC}^{0,1} = h_A^{0,1}$. Conversely, if $h_{BC}^{0,1} = h_A^{0,1}$, the natural morphism

$$H_{BC}^{n-1,n-1}(M, \mathbb{C}) \rightarrow H_A^{n-1,n-1}(M, \mathbb{C})$$

is onto (see Proposition 4.6 below), implying that $\partial(e^{(n-1)f}\omega^{n-1})$ is $\partial\bar{\partial}$ -exact since $e^{(n-1)f}\omega^{n-1}$ is $\partial\bar{\partial}$ -closed.

Proposition 4.6. *Let M be a compact complex astheno-Kähler n -fold, and let*

$$i_{p,q} : H_{BC}^{p,q}(M) \rightarrow H_A^{p,q}(M)$$

be the map induced by the identity. Then $i_{n-1,n-1}$ is surjective if and only if $i_{0,1}$ is injective.

Proof. Before proceeding with the proof, notice, by duality, that $i_{n-1,n-1}$ is surjective if and only if $i_{1,1}$ is injective. That means $i_{n-1,n-1}$ is surjective if and only if for every d -closed $(1,1)$ -form ϕ such that $\phi = \partial\alpha + \bar{\partial}\beta$, where α and β are smooth forms of type $(0,1)$ and $(1,0)$, respectively, there exists a smooth function $f : M \rightarrow \mathbb{C}$ such that $\phi = \partial\bar{\partial}f$.

Suppose the map $i_{0,1}$ is onto. That means for every $\partial\bar{\partial}$ -closed $(0,1)$ -form α , there exists a smooth function $g : M \rightarrow \mathbb{C}$ such that $\alpha + \bar{\partial}g$ is d -closed, i.e., $\bar{\partial}\alpha = 0$, and $\partial\alpha = -\partial\bar{\partial}g$. By the symmetry of the Bott-Chern cohomology, it also follows that for every $\partial\bar{\partial}$ -closed $(1,0)$ -form β , there exists a smooth function $h : M \rightarrow \mathbb{C}$ such that $\partial\beta = 0$, and $\bar{\partial}\beta = -\partial\bar{\partial}h$. Let now ϕ be a d -closed $(1,1)$ -form such that $\phi = \partial\alpha + \bar{\partial}\beta$, where α and β are $(0,1)$ and $(1,0)$ smooth forms, respectively. Since ϕ is d -closed, then $\partial\bar{\partial}\alpha = \partial\bar{\partial}\beta = 0$. It follows that $\phi = -\partial\bar{\partial}(g+h)$.

Conversely, let α be a $(0,1)$ -form such that $\partial\bar{\partial}\alpha = 0$. Let $\phi = \partial\alpha$. Then ϕ is a real d -closed $(1,1)$ -form in the kernel of $i_{1,1}$. Since $i_{1,1}$ is by assumption injective, there exists a smooth function $f : M \rightarrow \mathbb{C}$ such that $\phi = \partial\bar{\partial}f$, which means $\partial\alpha = \partial\bar{\partial}f$. Therefore $\bar{\partial}(\bar{\alpha} - \bar{\partial}f) = 0$, and from the Jost-Yau result, it follows that $\alpha - \bar{\partial}f$ is d -closed. That implies $\bar{\partial}\alpha = 0$ which concludes the proof of the proposition. \square

5. APPLICATIONS

We proceed by giving two examples of solvmanifolds, where Theorem A prohibits the existence of astheno-Kähler metrics.

5.1. The complex parallelizable Nakamura manifold. An example where our cohomological obstruction in Theorem A applies directly is the complex parallelizable Nakamura manifold.

Let G be semidirect product $\mathbb{C} \rtimes_{\phi} \mathbb{C}^2$ defined by

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Then there exist $a+bi, c+di \in \mathbb{C}$ such that $\mathbb{Z}(a+bi) + \mathbb{Z}(c+di)$ is a lattice in \mathbb{C} and $\phi(a+bi)$ and $\phi(c+di)$ are conjugate to elements of $SL(4; \mathbb{Z})$, where we regard $SL(2; \mathbb{C}) \subset SL(4; \mathbb{R})$. Hence we have a lattice $\Gamma := (\mathbb{Z}(a+bi) + \mathbb{Z}(c+di)) \rtimes_{\phi} \Gamma''$ of G such that Γ'' is a lattice of \mathbb{C}^2 . The quotient manifold $\Gamma \backslash G$ is the compact complex parallelizable manifold constructed by Nakamura [Na, §2].

The Bott-Chern cohomology of the Nakamura manifold was computed by Angella and Kasuya [AK]. They found that $h_{BC}^{0,1} = 1$, and if $b, d \in \mathbb{Z}\pi$, then $h_A^{0,1} = 5$, while $h_A^{0,1} = 3$ if $b \notin \mathbb{Z}\pi$ or $d \notin \mathbb{Z}\pi$. As a direct consequence of Theorem 1.1, we have:

Corollary 5.1. *The complex parallelizable Nakamura manifold does not admit astheno-Kähler metrics.*

Remark 5.2. Biswas proved in [Bis] that any compact complex parallelizable manifold admitting an astheno-Kähler metric is a compact complex torus. While Biswas'

proof relies on the Jost and Yau's obstruction in Lemma 2.7, in the case of the complex parallelizable Nakamura manifold, our proof relies on the new obstruction in Theorem A.

5.2. The Oeljeklaus-Toma manifolds. The Oeljeklaus-Toma manifolds, introduced in [OT] are interesting examples of compact, complex, non-Kähler manifolds, generalizing the Inoue surfaces [In].

Let s and t be two positive integers and consider $K \simeq \mathbb{Q}[X]/(f)$ an algebraic number field, where $f \in \mathbb{Q}[X]$ is a monic irreducible polynomial of degree $n = [K : \mathbb{Q}]$ with s real roots and $2t$ complex roots ([OT, Remark 1.1]). The field K admits s real embeddings and $2t$ complex embeddings:

$$\sigma_1, \dots, \sigma_s : K \rightarrow \mathbb{R}$$

$$\sigma_{s+1}, \dots, \sigma_{s+2t} : K \rightarrow \mathbb{C}, \text{ where } \sigma_{s+t+j} = \bar{\sigma}_{s+j}, j = 1, \dots, t.$$

The ring \mathcal{O}_K of algebraic integers of K is a finitely-generated free Abelian group of rank n . By the Dirichlet unit theorem, the multiplicative group \mathcal{O}_K^* of units of \mathcal{O}_K is a finitely-generated free Abelian group of rank $s + t - 1$. Furthermore, let

$$\mathcal{O}_K^{*,+} = \{a \in \mathcal{O}_K^* \mid \sigma_i(a) > 0, \text{ for all } i = 1, \dots, s\}.$$

be the subgroup totally positive units. It is a finite index subgroup of \mathcal{O}_K^* .

Denote now by $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ the upper complex half-plane. There exists an action

$$\mathcal{O}_K \times \mathcal{O}_K^{*,+} \curvearrowright \mathbb{H}^s \times \mathbb{C}^t,$$

induced by the translation $T : \mathcal{O}_K \curvearrowright \mathbb{H}^s \times \mathbb{C}^t$ given by

$$T_a(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) = (w_1 + \sigma_1(a), \dots, w_s + \sigma_s(a), z_{s+1} + \sigma_{s+1}(a), \dots, z_{s+t} + \sigma_{s+t}(a)),$$

and by the multiplication $R : \mathcal{O}_K^{*,+} \curvearrowright \mathbb{H}^s \times \mathbb{C}^t$ given by

$$R_u(w_1, \dots, w_s, z_{s+1}, \dots, z_{s+t}) = (w_1 \cdot \sigma_1(u), \dots, w_s \cdot \sigma_s(u), z_{s+1} \cdot \sigma_{s+1}(u), \dots, z_{s+t} \cdot \sigma_{s+t}(u)).$$

According to [OT, page 162], one can always choose a rank s subgroup $U \subset \mathcal{O}_K^{*,+}$ such that the induced action $\mathcal{O}_K \times U \curvearrowright \mathbb{H}^s \times \mathbb{C}^t$ is fixed-point-free, properly discontinuous and co-compact. One defines the Oeljeklaus-Toma manifold of type (s, t) associated to the algebraic number field K and to the admissible subgroup U of $\mathcal{O}_K^{*,+}$ as

$$X(K, U) = (\mathbb{H} \times \mathbb{C}^t) / \mathcal{O}_K \times U.$$

A result of Kasuya [Ka1] shows that the Oeljeklaus-Toma manifolds are solv-manifolds, a result with important consequences. In particular, one can use it to obtain information about the Dolbeault and Bott-Chern cohomology of such manifolds (see [ADOS] and the references therein). In the following lemma, we compute the cohomology groups $h_{\#}^{0,1}$, for $\# \in \{BC, A\}$ ².

Lemma 5.3. *Let X be an Oeljeklaus-Toma manifold of type (s, t) . Then*

$$h_{BC}^{0,1}(X) = 0 \quad \text{and} \quad h_A^{0,1}(X) = s.$$

²The authors would like to thank A. Otiman for kindly communicating to us the proof of this result.

Proof. By [ADOS, Corollary 11], we obtain

$$h_{BC}^{0,1}(X) = h_{\bar{\partial}}^{1,0}(X) = 0,$$

where the last equality was proved in [OT, Proposition 2.4].

By duality, from the same [ADOS, Corollary 11], we also have

$$h_A^{0,1}(X) = h_{BC}^{n,n-1}(X) = h_{\bar{\partial}}^{n,n-1}(X) + h_{\bar{\partial}}^{n-2,n}(X).$$

We will show that $h_{\bar{\partial}}^{n,n-1}(X) = s$ and $h_{\bar{\partial}}^{n-2,n}(X) = 0$.

Let

$$\rho_{p,m} := \#\{I \subseteq \{1, \dots, s+t\}, J \subseteq \{s+1, \dots, s+t\} \mid |I| = p, |J| = m, \sigma_I \bar{\sigma}_J = 1\},$$

where for a subset $K \subseteq \{1, \dots, s+t\}$, $\sigma_K := \prod_{i \in K} \sigma_i$.

By [ADOS, (18)] we have

$$h_{\bar{\partial}}^{n,n-1}(X) = \sum_{l=0}^{n-1} \binom{s}{l} \cdot \rho_{n,n-1-l}. \quad (5.1)$$

Notice now that $n = s+t$ and $\rho_{i,j} = 0$ if $j > t$ while $\rho_{n,t-1} = 0$, and so

$$h_{\bar{\partial}}^{n,n-1}(X) = \binom{s}{s-1} \cdot \rho_{n,t} = s,$$

as $\rho_{n,t} = 1$. Similarly, from (5.1), we find

$$h_{\bar{\partial}}^{n-2,n}(X) = \binom{s}{s} \cdot \rho_{n-2,t} = \rho_{2,0}.$$

However, by [APV, page 2419], we have $\rho_{2,0} = 0$, and so

$$h_A^{0,1}(X) = h_{BC}^{n,n-1}(X) = s.$$

□

As a consequence of Theorem A and Lemma 5.3, for $s \geq 2$ we obtain a new proof of a recent result of Angella, Dubickas, Otiman and Stelzig [ADOS, Corollary 5].

Corollary 5.4. *The Oeljeklaus-Toma manifolds of type (s, t) with $s \geq 2$ do not admit astheno-Kähler metrics.*

Remark 5.5. The Oeljeklaus-Toma manifolds with $s = 1$ do not admit astheno-Kähler, as well [ADOS, Corollary 5].

5.3. Cartesian products. A direct application of our obstruction and of the weak Künneth formula is the following observation:

Proposition 5.6. *Let X and Y be two astheno-Kähler manifolds saturating the upper bound in (1.1). Then $X \times Y$ does not admit astheno-Kähler metrics.*

Proof. As a consequence of Theorem B we find

$$\begin{aligned} h_A^{0,1}(X \times Y) &\geq h_A^{0,1}(X) + h_A^{0,1}(Y) \\ &= h_{BC}^{0,1}(X) + h_{BC}^{0,1}(Y) + 2 \\ &= h_{BC}^{0,1}(X \times Y) + 2. \end{aligned}$$

According to Theorem A, this prohibits the existence of an astheno-Kähler metric on $X \times Y$. □

Corollary 5.7. *Let S_1 and S_2 be two non-Kähler surfaces. Then $S_1 \times S_2$ does not admit astheno-Kähler metrics.*

However, more is true. In [LYZ], Li, Yau and Zheng noticed that a source of astheno-Kähler manifolds is provided by the products of the form $C \times S$, where C is a compact Riemann surface and S is a compact, complex, non-Kähler surface. We generalize here their method of constructing astheno-Kähler manifolds. The examples produced will shed a light on the sharpness of the bounds in (1.1).

Proposition 5.8. *Let S be a compact complex surface, X a compact Kähler manifold of complex dimension n . If Y denotes the blow-up of $X \times S$ along a smooth submanifold, then Y is an astheno-Kähler manifold and*

- 1) $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y) + 1$ if and only if S is non-Kähler.
- 2) $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y)$ if and only if S is Kähler.

Proof. By Theorem 2.6, to prove that Y carries an astheno-Kähler metric it suffices to show that the product manifolds $S \times X$ carries an astheno-Kähler metric satisfying the condition (2.6). Let ω_X be a Kähler metric on X and ω_S be a Gauduchon metric on S . Notice that such a metric ω_S exists on arbitrary compact complex manifolds [Ga]. Consider now the Hermitian metric $\omega = \omega_X + \omega_S$ on $X \times S$, where by abusing the notation, we omit the pull-back maps from the two factors. We compute

$$i\partial\bar{\partial}\omega^n = i\partial\bar{\partial}(\omega_X + \omega_S)^n = i\partial\bar{\partial}\omega_X^n + ni\partial\bar{\partial}(\omega_X^{n-1} \wedge \omega_S) + \binom{n}{2}i\partial\bar{\partial}(\omega_X^{n-2} \wedge \omega_S^2). \quad (5.2)$$

Since ω_X is Kähler, we have $\partial\omega_X = \bar{\partial}\omega_X = 0$, which implies

$$i\partial\bar{\partial}(\omega_X^{n-1} \wedge \omega_S) = -\omega_X^{n-1} \wedge (i\partial\bar{\partial}\omega_S) = 0,$$

since ω_S is Gauduchon. Similarly,

$$\partial\bar{\partial}(\omega_X^{n-2} \wedge \omega_S^2) = -\omega_X^{n-2} \wedge (i\partial\bar{\partial}\omega_S^2) = 0,$$

since $i\partial\bar{\partial}\omega_S^2 = 0$ is automatically satisfied on surfaces. Finally, the term $i\partial\bar{\partial}\omega_X^n$ vanishes for degree reasons. In conclusion, from (5.2) we see that $i\partial\bar{\partial}\omega^n = 0$, which means ω is an astheno-Kähler metric. Also, the same arguments can be easily adapted to show $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$, which shows that the metric ω satisfies the condition (2.6).

To prove claims 1) and 2), from Corollary 3.2 and Theorem B, we find:

$$\begin{aligned} h_{BC}^{0,1}(Y) &= h_{BC}^{0,1}(X \times S) = h_{BC}^{0,1}(X) + h_{BC}^{0,1}(S) = q + h_{BC}^{0,1}(S) \\ h_A^{0,1}(Y) &= h_A^{0,1}(X \times S) \geq h_A^{0,1}(X) + h_A^{0,1}(S) = q + h_A^{0,1}(S) \end{aligned} \quad (5.3)$$

where $q = h_{\bar{\partial}}^{0,1}(X) = h_{BC}^{0,1}(X) = h_A^{0,1}(X)$ is the irregularity of the Kähler manifold X . Therefore, if $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y)$ then $h_A^{0,1}(S) = h_{BC}^{0,1}(S)$. As in Remark 4.4, the latter is equivalent to S being a Kähler surface. Notice now that if S is Kähler, then $X \times S$ is Kähler, and so Y is Kähler. That implies $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y)$, proving claim 1). By contraposition, this also proves one direction of claim 2), that is if $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y) + 1$ then S is necessarily non-Kähler. Conversely, if $h_A^{0,1}(S) = h_{BC}^{0,1}(S) + 1$, by (5.3) we find $h_A^{0,1}(Y) \geq h_{BC}^{0,1}(Y) + 1$. However, since Y carries an astheno-Kähler metric, by Theorem A, we must have $h_A^{0,1}(Y) = h_{BC}^{0,1}(Y) + 1$. \square

Proof of Theorem C. The result is a direct consequence of Proposition 5.8 and Corollary 5.7. \square

Remark 5.9. We notice here that the Jost-Yau obstruction criterion does not provide any information in proving that a Cartesian product of two non-Kähler surfaces does not carry astheno-Kähler metrics, emphasizing the role played by Theorem A in the proof of Theorem C. Indeed, let S_1 and S_2 be two non-Kähler surfaces and ϕ a holomorphic 1-form on $S_1 \times S_2$. Since $\partial\alpha = 0$, then α is an element of the Dolbeault cohomology group $H_{\bar{\partial}}^{1,0}(X \times Y)$. Using the Künneth formula [GH, page 105], there exist holomorphic 1-forms α_1 and α_2 on S_1 and S_2 , respectively, such that $\phi = \alpha_1 + \alpha_2$. However, on surfaces, every holomorphic 1-form is closed as it can be seen from the Jost-Yau criterion. Therefore $d\alpha_1 = 0$ and $d\alpha_2 = 0$, which imply $d\phi = d(\alpha_1 + \alpha_2) = d\alpha_1 + d\alpha_2 = 0$.

Remark 5.10. It is interesting to notice that the product of either two Kodaira surfaces, two Inoue surfaces, or a Kodaira surface and a Inoue surface, while it does not carry any astheno-Kähler metric, it does admit SKT metrics [ST, Theorem 7.5].

6. SATURATING THE INEQUALITY (1.1)

The results presented earlier show that the upper bound in (1.1) is sharp: the Cartesian product of a compact, complex, non-Kähler surface and a compact Kähler manifold admits astheno-Kähler metrics and saturates the upper bound in (1.1). We will exhibit next a class of non-Kähler manifolds which do not carry astheno-Kähler metrics, while saturating the upper bound in (1.1), the class of Vaisman manifolds. The diagonal Hopf manifolds are examples of Vaisman manifolds.

6.1. Vaisman manifolds. Locally conformally Kähler (LCK) manifolds and, in particular, the Vaisman manifolds were introduced in the 70's by I. Vaisman, and have been given a lot of attention in the recent years. We will recall next some of the relevant results.

Let M be a complex manifold. A Hermitian metric on M is a locally conformally Kähler (LCK) metric if its fundamental form ω satisfies the condition $d\omega = \theta \wedge \omega$ for some non-zero 1-form θ . The 1-form θ is called the Lee form, and the pair (M, ω) is called a LCK manifold. If $\dim M \geq 3$, the Lee form is closed. We will decompose the 1-form θ as $\theta = \alpha + \bar{\alpha}$ where α is a $(1, 0)$ -form. Since $d\theta = 0$, we find $\partial\alpha = 0$ and $\bar{\partial}\alpha = -\partial\bar{\alpha}$.

Definition 6.1. A LCK manifold (M, ω) is Vaisman if $\nabla\theta = 0$, where ∇ denotes the Levi-Civita connection associated to ω , and θ is the Lee form.

We will recall next several results on Vaisman manifolds:

Theorem 6.1. *Let (M, ω) be a compact Vaisman manifold.*

- 1) (Vaisman, [Va]) *There exists a unique metric in its conformal class such that $|\theta|_{\omega} = 1$.*
- 2) (Verbitsky, [Ve1, Section 6]) *If the Lee form θ satisfies $|\theta|_{\omega} = 1$, then*

$$\omega = 2i\bar{\partial}\alpha + 2i\alpha \wedge \bar{\alpha}, \quad (6.1)$$

and all eigenvalues of the $(1,1)$ -form $\bar{\partial}\alpha$ are positive, except one which is equal to zero³.

Proof of Theorem F. Let M be a Vaisman manifold, with $\dim_{\mathbb{C}}M = n \geq 3$.

The Bott-Chern cohomology of Vaisman manifolds was recently computed by Istrati and Otiman [IO]. In particular, from Theorem 4.2 in [IO], and the duality between the Bott-Chern and Aeppli cohomologies $h_A^{0,1} = h_{BC}^{n,n-1}$ one can see that

$$h_A^{0,1}(M) = h_{BC}^{0,1}(M) + 1.$$

To prove that M does not carry astheno-Kähler metrics, let ω be a LCK metric on M , whose Lee form satisfies $|\theta|_{\omega} = 1$. Notice from Theorem 6.1.2 that

$$i\partial\bar{\partial}\omega = i\partial\bar{\partial}(2i\bar{\partial}\alpha + 2i\alpha \wedge \bar{\alpha}) = -2\bar{\partial}\alpha \wedge \partial\bar{\alpha} = -2(i\bar{\partial}\alpha) \wedge (i\partial\alpha) \leq 0. \quad (6.2)$$

Suppose now there exists an astheno-Kähler metric η on M . Using Stokes' theorem, we find:

$$0 \geq \int_M \eta^{n-2} \wedge i\partial\bar{\partial}\omega = \int_M i\partial\bar{\partial}\eta^{n-2} \wedge \omega = 0$$

hence, $\partial\bar{\partial}\omega = 0$, which means ω is also a SKT metric. However, it was noticed by Alexandrov and Ivanov in [AI, Remark 1] (see also [IP, Theorem 1.3]) that a non-Kähler LCK metric of dimension at least 3 cannot be SKT, which leads to a contradiction. \square

Remark 6.2. The authors are grateful to Alexandra Otiman for pointing out to them the relation between the Bott-Chern and Aeppli cohomology for $(0,1)$ -forms in the case of Vaisman manifolds, and that Theorem F was also independently noticed by Angella, Otiman and Stanciu (unpublished).

Remark 6.3. The key observation (6.2) in the above proof has also been recently noticed by Angella, Guedj and Lu [AGL, Proposition 3.10] within the framework of LCK metrics with potential.

Example 6.2. Let $A \in GL(n, \mathbb{C})$ be a linear operator acting on \mathbb{C}^n with all eigenvalues λ_i satisfying $|\lambda_i| > 1$. Denote by $\langle A \rangle \subseteq GL(n, \mathbb{C})$ the cyclic group generated by A . The quotient $M = (\mathbb{C}^n \setminus \{0\}) / \langle A \rangle$ is called a linear Hopf manifold. If A is diagonalizable, then M is called a diagonal Hopf manifold. In [KO], Kamishima and Ornea proved that a diagonal Hopf manifold is Vaisman (see also [OV]). As a consequence of Theorem F, no diagonal Hopf manifold M carries an astheno-Kähler metric. This result can be used to confirm a conjecture of Li, Yau and Zheng regarding the class of similarity Hopf manifolds.

Definition 6.3. A compact complex manifold X is called similarity Hopf manifold if and only if it is a finite undercover of a Hopf manifold $M = (\mathbb{C}^n \setminus \{0\}) / \langle \phi \rangle$, where $\phi(z) = azA$, $A \in U(n)$, $\mathbf{z} = (z_1, \dots, z_n)$, and $a > 1$.

Li, Yau and Zheng conjectured in [LYZ, page 108] that similarity Hopf manifolds cannot carry astheno-Kähler metrics. We obtain:

Corollary 6.4. *There exists no astheno-Kähler metrics on similarity Hopf manifolds of dimension at least three.*

³The eigenvalues of a $(1,1)$ -form ϕ are the eigenvalues of the symmetric operator $G(\phi)$ defined by the equation $\phi(v, Jv) = g(G(\phi)v, w)$, where J is the ambient complex structure, and g is a background Hermitian metric.

Proof. Suppose there exists a similarity Hopf manifold X , $\dim_{\mathbb{C}} X = n \geq 3$, equipped with an astheno-Kähler metric ω . Let $\pi : M \rightarrow X$ be an unramified finite covering, where $M = (\mathbb{C}^n \setminus \{0\}) / \langle \phi \rangle$, $\phi(\mathbf{z}) = a\mathbf{z}A$, $A \in U(n)$, $\mathbf{z} = (z_1, \dots, z_n)$, and $a > 1$. Notice that we have

$$\partial\bar{\partial}(\pi^*\omega)^{n-2} = \partial\bar{\partial}\pi^*\omega^{n-2} = \pi^*\partial\bar{\partial}\omega^{n-2} = 0.$$

That means $\pi^*\omega$ is an astheno-Kähler metric on the manifold M . However, since the matrix A is unitary, the matrix aA is diagonalizable and for every eigenvalue λ we have $|\lambda| = a > 1$. Therefore M is a diagonal Hopf manifold, and such manifolds cannot carry astheno-Kähler metrics. \square

Remark 6.5. The standard Hopf manifold $M = (\mathbb{C}^n \setminus \{0\}) / \langle \phi \rangle$, where $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a homothety $\phi(\mathbf{z}) = a\mathbf{z}$, with $a \in \mathbb{C}^*$, $|a| \neq 1$, is an elliptic fiber bundle over $\mathbb{C}\mathbb{P}^{n-1}$ diffeomorphic to $S^1 \times S^{2n-1}$. As an example of a Vaisman manifold it does not carry astheno-Kähler metrics. It is interesting to notice that the closely related Calabi-Eckmann manifold, which is also an elliptic fiber bundle over $\mathbb{C}\mathbb{P}^{n-1} \times \mathbb{C}\mathbb{P}^{m-1}$ diffeomorphic to $S^{2n-1} \times S^{2m-1}$ admits astheno-Kähler metrics [Mat], while satisfying $h_A^{0,1} = 1$ and $h_{BC}^{0,1} = 0$ for $1 < n < m$ [St2].

We will conclude this paper by discussing the sharpness of the lower bound in (1.1).

6.2. A higher dimensional generalization of the completely solvable Nakamura manifold. Our next example was first studied by Kasuya [Ka2]. It generalizes I. Nakamura's example [Na, page 90] of a non-Kähler solvmanifold satisfying the $\partial\bar{\partial}$ -lemma.

Let $G = \mathbb{C} \ltimes_{\phi} \mathbb{C}^{2n}$, where

$$\phi(x + iy)(w_1, \dots, w_{2n}) = (e^{a_1 x} w_1, e^{-a_1 x} w_2, \dots, e^{a_n x} w_{2n-1}, e^{-a_n x} w_{2n}),$$

where $a_i \neq 0$ are integers. Notice we can write $G = \mathbb{R} \times (\mathbb{R} \ltimes_{\phi} \mathbb{C}^{2n})$. The group G admits a co-compact lattice $\Gamma = t\mathbb{Z} \times \Delta$, where Δ is a lattice in $\mathbb{R} \ltimes_{\phi} \mathbb{C}^{2n}$ for $t > 0$. For $t \neq r\pi$ for some $r \in \mathbb{Q}$, the complex manifold satisfies the Hodge symmetry and decomposition, but $X = \Gamma \backslash G$ does not admit a Kähler metric [Ka2]. In particular, we have $h_A^{0,1}(X) = h_{BC}^{0,1}(X)$. The interested reader may follow up the arguments in [AK] to find that $h_A^{0,1}(X) = h_{BC}^{0,1}(X) = 1$.

Consider now the $(1, 1)$ -form

$$\eta = \sqrt{-1} \left(dz \wedge d\bar{z} + \sum_{i=1}^n (e^{-2a_i x} dw_{2i-1} \wedge d\bar{w}_{2i-1} + e^{2a_i x} dw_{2i} \wedge d\bar{w}_{2i}) \right),$$

on X induced from $\mathbb{C} \ltimes_{\phi} \mathbb{C}^{2n}$. Then one can see that $i\partial\bar{\partial}\eta$ is a weakly positive, $\partial\bar{\partial}$ -exact non-vanishing $(2, 2)$ -current. By Theorem 2.8, we obtain:

Corollary 6.6. *The Kasuya-Nakamura solvmanifolds X does not carry any astheno-Kähler metric.*

Remark 6.7. In [FKV], Fino, Kasuya and Vezzoni proved that the Kasuya-Nakamura example cannot carry SKT metrics, as well.

6.3. Fujiki class \mathcal{C} manifolds of complex dimension three. Manifolds satisfying the $\partial\bar{\partial}$ -lemma automatically satisfy $h_A^{0,1} = h_{BC}^{0,1}$. In particular, manifolds of Fujiki class \mathcal{C} , i.e., manifolds bimeromorphic to Kähler manifolds, satisfy such condition. In [Ch], the first author proved that a Fujiki class \mathcal{C} manifold admits a SKT metric if and only if it is of Kähler type. In particular, any Fujiki class \mathcal{C} manifold of complex dimension three which admits an astheno-Kähler metric is of Kähler type. As a consequence, we notice:

Corollary 6.8. *The class of astheno-Kähler manifolds is not invariant under modifications.*

Proof. Let M be the 3 dimensional manifold constructed by Hironaka [Hi] which is a proper modification of the projective space $\mathbb{C}\mathbb{P}^3$ and which contains a positive linear combination of curves which is homologous to 0. In particular, M is a non-Kähler Fujiki class \mathcal{C} manifold, which, unlike $\mathbb{C}\mathbb{P}^3$, cannot carry an astheno-Kähler metric. \square

Remark 6.9. The authors are not aware of an example of a non-Kähler manifold admitting an astheno-Kähler metric and satisfying $h_A^{0,1}(M) = h_{BC}^{0,1}(M)$.

Acknowledgements. The second author was partially supported by a Professional Travel Grant from Vanderbilt University. The authors would like to thank Gueo Grantcharov for suggestions and encouragements in the early stages of this work, Alexandra Otiman for helping them understand the cohomology of Oeljeklaus-Toma and Vaisman manifolds and for useful comments, Victor Vuletescu for clarifications regarding locally Kähler manifolds, and J. Stelzig for sharing with us his unpublished notes on the Künneth formula in the Bott-Chern cohomology.

REFERENCES

- [AKMW] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), no. 3, 531–572.
- [AI] B. Alexandrov, S. Ivanov, *Vanishing theorems on Hermitian manifolds*. Differential Geom. Appl. **14** (2001), no. 3, 251–265.
- [An] D. Angella, *The cohomologies of the Iwasawa manifold and of its small deformations*. J. Geom. Anal. **23** (2013), no. 3, 1355–1378.
- [ADOS] D. Angella, A. Dubickas, A. Otiman, J. Stelzig, *On metric and cohomological properties of Oeljeklaus-Toma manifolds*. [arXiv:2201.06377](https://arxiv.org/abs/2201.06377) [math.DG].
- [AGL] D. Angella, V. Guedj, C. C. Lu *Plurisigned hermitian metrics*. [arXiv:2207.04705v1](https://arxiv.org/abs/2207.04705v1) [math.CV].
- [AK] D. Angella, H. Kasuya, *Bott-Chern cohomology of solvmanifolds*. Ann. Global Anal. Geom. **52** (2017), no. 4, 363–411.
- [APV] D. Angella, M. Parton, V. Vuletescu, *Rigidity of Oeljeklaus-Toma manifolds*,. Ann. Inst. Fourier **70** (2020), no. 6, 2409–2423.
- [Bi] J. M. Bismut, *A local index theorem of non-Kähler manifolds*, Math. Ann. **284** (1989) 681– 699.
- [Bis] I. Biswas, *On Hermitian structure on G/Γ* . Bull. Sci. Math. **137** (2013), no. 6, 716–717.
- [CT] J. A. Carlson, D. Toledo, *On fundamental groups of class VII surfaces*. Bull. London Math. Soc. **29** (1997): 98–102.
- [Ch] I. Chiose, *Obstruction to the existence of Kähler structures on compact complex manifolds*. Proc. Amer. Math. Soc. **142**, 3561–3568 (2014).
- [CFGU] L. A. Cordero, M. Fernández, A. Gray, L. Ugarte, *Compact nilmanifolds with nilpotent complex structure: Dolbeault cohomology*. Trans. Amer. Math. Soc. **352** (2000) 5405–5433.
- [DGMS] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*. Invent. Math. **29** (1975), no. 3, 245–274.

- [FPS] A. Fino, M. Parton, S. Salamon, *Families of strong KT structures in six dimensions*. Comment. Math. Helv. **79** (2), (2004), 317–340.
- [FGV] A. Fino, G. Grantcharov, L. Vezzoni, *Asthen-Kähler and balanced structures on fibrations*. Int. Math. Res. Not. **IMRN** **2019**, no. 22, 7093–7117.
- [FT1] A. Fino, A. Tomassini, *Blow-ups and resolutions of strong Kähler with torsion metrics*, Adv. Math. **221** (2009) 914–935.
- [FT2] A. Fino, A. Tomassini, *On asthen-Kähler metrics*. J. Lond. Math. Soc. (2) **83** (2) (2011) 290–308.
- [FKV] A. Fino, H. Kasuya, L. Vezzoni, *SKT and tamed symplectic structures on solvmanifolds*. Tohoku Math. J. (2) **67** (2015), no. 1, 19–37.
- [FWW] J. Fu, Z. Wang, D. Wu, *Semilinear equations, the γ_k -function, and generalized Gauduchon metrics*, J.Eur. Math. Soc., **15**, 2013, pp. 659–680.
- [Ga] P. Gauduchon, *Le théorème de l'excentricité nulle*. C.R Acad. Sci. Paris Sér. A-B **2285** (1977), no.5, A387–A390.
- [GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*. Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.
- [HL] R. Harvey, H.B. Lawson, *An intrinsic characterization of Kähler manifolds*. Invent. Math. **74** (1983), no. 2, 169–198.
- [Hi] H. Hironaka, *An example of a non-Kählerian complex-analytic deformation of Kählerian complex structures*. Ann. of Math. (2) **75** (1962), 190–208.
- [In] M. Inoue, *On surfaces of class VII₀*. Invent. Math., **24** (1974), 269–310.
- [IO] N. Istrati, A. Otman, *Bott-Chern cohomology of compact Vaisman manifolds*. [arXiv:2206.07312v2 \[math.DG\]](https://arxiv.org/abs/2206.07312v2), to appear in Trans. Amer. Math. Soc.
- [IP] S. Ivanov, G. Papadopoulos, *Vanishing theorems on $(\ell|k)$ -strong Kähler manifolds with torsion*. Adv. Math. **237** (2013), 147–164.
- [JY] J. Jost, S.-T. Yau. *A non-linear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry*. Acta Math. **170** (1993), 221–54; Corrigendum Acta Math. **173** (1994), 307.
- [KO] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformally Kähler manifolds*. Tohoku Math. J. (2) **57** (2005), no. 2, 201–221.
- [Ka1] H. Kasuya, *Vaisman metrics on solvmanifolds and Oeljeklaus-Toma manifolds*. Bull. Lond. Math. Soc. **45** (2013), no. 1, p. 15–26.
- [Ka2] H. Kasuya, *Hodge symmetry and decomposition on non-Kähler solvmanifolds*. J. Geom. Phys. **76** (2014), 61–65.
- [Ko] K. Kodaira, *On the Structure of Compact Complex Analytic Surfaces, II, III* Amer. J. Math. **88** (1966), 682–721, Amer. J. Math. **90** (1968), 55–83.
- [LU] A. Latorre, L. Ugarte, *On non-Kähler compact complex manifolds with balanced and asthen-Kähler metrics*. C. R. Math. Acad. Sci. Paris **355** (2017), no. 1, 90–93.
- [LUV] A. Latorre, L. Ugarte, R. Villacampa, *On the Bott-Chern cohomology and balanced Hermitian nilmanifolds*. Internat. J. Math. **25** (2014), no. 6, 1450057, 24 pp.
- [LYZ] J. Li, S.T. Yau, F. Zheng, *On projectively flat Hermitian manifolds*, Comm. Anal. Geom. **2** (1994), no. 1, 103–109.
- [Mal] A. I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Translation Ser. 1, **9** (1962), 276–307.
- [Mat] K. Matsuo, *Asthen-Kähler structures on Calabi-Eckmann manifolds*. Colloq. Math. **115**, no. 4 (2009), 33–99.
- [Na] I. Nakamura, *Complex parallelisable manifolds and their small deformations*. J. Differ. Geom. **10** (1) (1975), 85–112.
- [No] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. (2) **59** (1954), 531–538.
- [OT] K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*. Ann. Inst. Fourier **55** (2005), no. 1, 161–171.
- [OV] L. Ornea, M. Verbitsky, *Locally conformally Kähler metrics obtained from pseudoconvex shells*. Proc. Amer. Math. Soc. **144** (2016), no. 1, 325–335.
- [RYY] S. Rao, S. Yang, Song, X. Yang, *Dolbeault cohomologies of blowing up complex manifolds*. J. Math. Pures Appl. (9) **130** (2019), 68–92.
- [RT] F. A. Rossi, A. Tomassini, *On strong Kähler and asthen-Kähler metrics on nilmanifolds*. Adv. Geom. **12** (2012), no. 3, 431–446.

- [Sa] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001) 311–333.
- [Sc] M. Schweitzer, *Autour de la cohomologie de Bott-Chern*. [arXiv:0709.3528](https://arxiv.org/abs/0709.3528) [math.AG].
- [ST] T. Sferruzza, A. Tomassini, *On cohomological and formal properties of Strong Kähler with torsion and astheno-Kähler metrics*. [arXiv:2206.06904](https://arxiv.org/abs/2206.06904) [math.DG].
- [St1] J. Stelzig, *The double complex of a blow-up*. Int. Math. Res. Not. IMRN **2021**, no. 14, 10731–10744.
- [St2] J. Stelzig, *On the structure of double complexes*. J. Lond. Math. Soc. (2) **104** (2021), no. 2, 956–988.
- [St3] J. Stelzig, *Unpublished notes*.
- [Sw] A. Swann, *Twisting Hermitian and hypercomplex geometries*. Duke Math. J. **155** (2010), 403–431.
- [Ug] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*. Transform. Groups **12** (2007), no. 1, 175–202.
- [Va] I. Vaisman, *Generalized Hopf manifolds*. Geom. Dedicata **13** (1982), no. 3, 231–255.
- [Ve1] M. Verbitsky *Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds*. Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 64–91; translation in Proc. Steklov Inst. Math. 2004, no. 3 (246), 54–78, [arXiv:math/0302219v4](https://arxiv.org/abs/math/0302219v4) [math.DG].
- [Ve2] M. Verbitsky, *Rational curves and special metrics on twistor spaces*. Geom. Topol. **18** (2014), no. 2, 897–909.
- [Vo] C. Voisin, *Hodge Theory and Complex Algebraic Geometry. I*. Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, Cambridge, 2002.
- [YY1] S. Yang, X. Yang, *Bott-Chern blow-up formulae and the bimeromorphic invariance of the $\partial\bar{\partial}$ -lemma for threefolds*. Trans. Amer. Math. Soc. **373** (2020), no. 12, 8885–8909.
- [YY2] S. Yang, X. Yang, *Bott-Chern hypercohomology and bimeromorphic invariants*, <https://arxiv.org/pdf/2204.07938.pdf>

INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST 014700, ROMANIA

E-mail address: `Ionut.Chiose@imar.ro`

DEPARTMENT OF MATHEMATICS, 1326 STEVENSON CENTER, VANDERBILT UNIVERSITY, NASHVILLE, TN, 37240, USA

E-mail address: `rares.rasdeaconu@vanderbilt.edu`