UNEXPECTED LOSS OF MAXIMALITY: THE CASE OF HILBERT SQUARES OF REAL SURFACES

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Dedicated to the memory of Mark Sapir

ABSTRACT. We explore the maximality of the Hilbert square of maximal real surfaces, and find that in many cases the Hilbert square is maximal if and only if the surface has connected real locus. In particular, the Hilbert square of no maximal $K3$-surface is maximal. Nevertheless, we exhibit maximal surfaces with disconnected real locus whose Hilbert square is maximal.

> On dédaigne volontiers un but qu'on n'a pas réussi à atteindre, ou qu'on a atteint définitivement.
>
> M. Proust, A la recherche du temps perdu.

1. INTRODUCTION

A result of fundamental importance in understanding the topology of real algebraic varieties is the Smith inequality [26] that bounds from above the sum of the $\mathbb{F}_2$-Betti numbers of the fixed point set of an involution by the sum of $\mathbb{F}_2$-Betti numbers of the ambient space itself. Applied to the complex conjugation on a n-dimensional algebraic variety X defined over the field $\mathbb{R}$ it states that

$$\sum_{i=0}^{n} \beta_i(X(\mathbb{R})) \leq \sum_{i=0}^{2n} \beta_i(X(\mathbb{C})). \tag{1.1}$$

A real algebraic variety realizing equality in the Smith inequality is said to be *maximal* or an *M-variety*. The study of M-varieties is one of central themes in the study of the topology of real algebraic varieties [8].

In dimension one, examples of planar M-curves are given by the Harnack curves [15], and one can easily construct abstract maximal real algebraic curves of arbitrary genus. In higher dimensions, the question of sharpness is far from being completely understood despite the existence of a powerful patchworking method due to O. Viro. The latter one is presumed to provide maximal projective hypersurfaces of any dimension and degree, but yet only asymptotic (by degree) maximality is achieved [16]. Otherwise, apart from

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abelian varieties, flag manifolds equipped with the standard real structure, and smooth toric varieties [2], only sporadic examples are known.

An interesting phenomenon was detected in the work of G. Weichold [30] and F. Klein [20]. They established a correspondence between the topology of the real locus of a curve and that of its Jacobian which, in modern language, shows that the Jacobian of a real algebraic curve with non-empty real part is maximal if and only if the curve is maximal (see [14] for a contemporary presentation). Recently, relying on the work of M. Liu and F. Schaffhauser [24], E. Brugallé and F. Schaffhauser [5] provided a new insight into Weichold and Klein's work by proving that the moduli spaces of vector bundles of coprime rank and degree over a real algebraic curve are maximal if and only if the base curve itself is maximal.

The question of extending the maximality phenomenon to the symmetric products of curves was addressed by I. Biswas and S. D'Mello in [3], where they obtained partial results. A complete answer was found by M. Franz [11] who disclosed a much more general result: a symmetric product of a space with an involution having fixed points is maximal if and only if the space itself is maximal.

Closely related to the symmetric product is the Hilbert scheme of points. In this paper we begin a study of its maximality and observe a rather different behavior. As the following two theorems show, already in the case of Hilbert squares of surfaces, while the statement in "only if" direction is preserved, it drastically fails in the opposite direction.

Theorem 1.1. *Let X be a real nonsingular projective surface. If the Hilbert square $X^{[2]}$ is maximal, then X is maximal.*

Theorem 1.2. *Let X be a maximal real nonsingular projective surface with $H_1(X(\mathbb{C}), \mathbb{F}_2) = 0$. Then its Hilbert square $X^{[2]}$ is maximal if and only if the real locus $X(\mathbb{R})$ of X is connected.*

As a consequence of Theorem 1.2, we get the following results:

Corollary 1.3. *Let X be a real nonsingular projective surface satisfying $H_1(X(\mathbb{C}), \mathbb{F}_2) = 0$ and $h^{2,0}(X(\mathbb{C})) > 0$, then $X^{[2]}$ is not maximal. In particular, this is the case if X is a K3-surface, or a surface birational to it.*

Corollary 1.4. *If X is a maximal real nonsingular rational surface with $X(\mathbb{R})$ connected, then $X^{[2]}$ is maximal. In particular, this is the case of the projective plane, and of ruled surfaces $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ where $\mathcal{E}$ is a rank 2 vector bundle equipped with a real structure that lifts the standard real structure on $\mathbb{P}^1$.*

When $X(\mathbb{C})$ has a positive first Betti number, we have only a partial answer.

Theorem 1.5. *Let X be a maximal real nonsingular projective surface with $\operatorname{Tors}_2 H_1(X(\mathbb{C}), \mathbb{Z}) = 0$. Then:*

1) *If $X(\mathbb{R})$ is connected, then the Hilbert square $X^{[2]}$ is maximal.*

2) *If* $\beta_0(X(\mathbb{R})) > 1 + \beta_1(X(\mathbb{C}))$, *then the Hilbert square* $X^{[2]}$ *is not maximal.*

Examples of maximal surfaces X with torsion free homology, and such that $1 < \beta_0(X(\mathbb{R})) < \beta_1(X)$ are given by the ruled surfaces $X = \mathbb{P}_C(E)$, where E is a rank 2 complex vector bundle over a maximal curve C of positive genus. For such surfaces, the real locus $X(\mathbb{R})$ is disconnected, and we show that their Hilbert square is maximal, in contrast with situations described in Corollary 1.3. On the other hand, if X is a maximal torus of complex dimension 2, we notice that $X^{[2]}$ is not maximal while $\beta_0(X(\mathbb{R})) = \beta_1(X(\mathbb{C}))$.

For surfaces X with 2-torsion in the homology with integer coefficients, one can still compute the Betti numbers of the Hilbert square $X^{[2]}$ [28]. For real maximal Enriques surfaces, we notice that $X^{[2]}$ is never maximal. We should point out that the real locus of maximal Enriques surfaces is always disconnected (see [7] for a full classification of topological types).

In all the enumerated results, as well as throughout the whole paper, when we speak about the maximality of a Hilbert scheme of points $X^{[n]}$, we mean the maximality of $X^{[n]}$ with respect to the real structure which is *canonically inherited* from a real structure on X. Thus, non maximality with respect to this kind of real structures does not exclude that $X^{[n]}$ may admit another real structure with respect to which it becomes maximal. In particular, at the current stage we do not know if there exist (real or complex) $K3$, or Enriques, surfaces X for which $X^{[n]}$ admits however a maximal real structure.

Remark 1.6. All the above definitions and the results, including their proofs, literally extend from real algebraic setting to compact complex analytic manifolds equipped with an anti-holomorphic involution.

Remark 1.7. Theorem 1.1, under additional assumption $X(\mathbb{R}) \neq \emptyset$, was also obtained by L. Fu in a recent preprint [12], where the maximality of various moduli spaces of sheaves/bundles/subschemes on maximal varieties is disclosed.

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Notations and conventions:

1) By a variety equipped with a real structure we mean a pair (Y, c) consisting of a complex variety Y and an anti-holomorphic involution $c : Y \to Y$.

2) Let Y be an algebraic variety defined over $\mathbb{R}$, and G denote the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The group G is a cyclic group of order 2 and acts on the locus of complex points $Y(\mathbb{C})$. The non-trivial element of G acts as an anti-holomorphic involution, which we will denote by c, and the fixed point set of the action coincides with the set of real points of Y. The pair (Y, c) is a variety equipped with a real structure. To mediate between the notations traditionally used for varieties equipped with real structures and for algebraic varieties defined over $\mathbb{R}$, we will use from now on Y to denote the set of complex points, and $Y(\mathbb{R})$ the set of real points.
3) Unless explicitly stated, all the homology and cohomology groups have coefficients in the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. We use $\beta_i(\,\cdot\,)$ and $b_i(\,\cdot\,)$ to denote the Betti numbers when the coefficients are in $\mathbb{F}_2$ or in $\mathbb{Q}$, respectively. We will use the notation $\beta_*(\,\cdot\,)$ and $b_*(\,\cdot\,)$ for the corresponding total Betti numbers.

2. Preliminaries

2.1. **Smith theory.** Most results cited in this section are due to P.A. Smith; proofs can be found, e.g., in [4, Chapter 3] and [6, Chapter 1].

Throughout the section we consider a topological space X with a cellular involution $c : X \to X$, i.e., c transforms cells into cells and acts identically on each invariant cell. Denote by $F = \operatorname{Fix} c$, $\bar{X} = X/c$, and let $\mathrm{in} : F \hookrightarrow X$ and $\mathrm{pr} : X \to \bar{X}$ be the natural inclusion and projection, respectively.

Introduce the *Smith chain complexes*

$$\begin{aligned}\mathrm{Sm}_*(X) &= \ker[(1 + c_*)\ : S_*(X) \to S_*(X)],\\ \mathrm{Sm}_*(X, F) &= \ker[(1 + c_*)\ : S_*(X, F) \to S_*(X, F)].\end{aligned}$$

and *Smith homology* $H_r(\mathrm{Sm}_*(X))$ and $H_r(\mathrm{Sm}_*(X, F))$. There is a canonical isomorphism $\mathrm{Sm}_*(X, F) = \mathrm{Im}[(1 + c_*)\ : S_*(X) \to S_*(X)]$. The *Smith sequences* are the long homology and cohomology exact sequences associated with the short exact sequence of complexes

$$0 \to \mathrm{Sm}_*(X) \xrightarrow{\text{inclusion}} S_*(X) \xrightarrow{1+c_*} \mathrm{Sm}_*(X, F) \to 0. \tag{2.1}$$

Analyzing this sequence notice first that we have a canonical canonical splitting $\mathrm{Sm}_*(X) = S_*(F) \oplus \mathrm{Im}(1 + c_*)$. The transfer homomorphism $\mathrm{tr}^* : S_*(\bar{X}, F) \to \mathrm{Sm}_*(X, F)$ is an isomorphism [4, Chapter 3] (see also *op. cit.* for the cohomology version). In view of the above identifications the long exact sequences associated to (2.1) yields:

Theorem 2.1. *There are two natural, in respect to equivariant maps, exact sequences, called (homology and cohomology)* Smith sequences *of* (X, c)*:*

$$\cdots \to H_{p+1}(\bar{X}, F) \xrightarrow{\Delta} H_p(\bar{X}, F) \oplus H_p(F) \xrightarrow{\mathrm{tr}^* + \mathrm{in}_*} H_p(X) \xrightarrow{\mathrm{pr}_*} H_p(\bar{X}, F) \to ,$$

$$\to H^p(\bar{X}, F) \xrightarrow{\mathrm{pr}^*} H^p(X) \xrightarrow{\mathrm{tr}_* \oplus \mathrm{in}^*} H^p(\bar{X}, F) \oplus H^p(F) \xrightarrow{\Delta} H^{p+1}(\bar{X}, F) \to \cdots .$$

The homology and cohomology connecting homomorphisms Δ *are given by*

$$x \mapsto x \cap \omega \oplus \partial x \quad \text{and} \quad x \oplus f \mapsto x \cup \omega + \delta f,$$

respectively, where $\omega \in H^1(\bar{X} \setminus F)$ *is the characteristic class of the double covering* $X \setminus F \to \bar{X} \setminus F$*. The images of* $\mathrm{tr}^* + \mathrm{in}_*$ *and* pr^* *consist of invariant classes:* $\operatorname{Im} \mathrm{tr}^* \subset \ker(1 + c_*)$ *and* $\operatorname{Im} \mathrm{pr}^* \subset \ker(1 + c^*)$.

The following immediate consequences of Theorem 2.1 which we state in the homology setting, have an obvious counterpart for cohomology:

Corollary 2.2. *Let* (X, c) *be a topological space equipped with a cellular involution. Then:*

1) $\dim H_*(F) + 2\sum_p \dim \operatorname{Coker}(\mathrm{tr}^p + \mathrm{in}_p) = \dim H_*(X)$ *(Smith identity);*
2) $\dim H_*(F) \leq \dim H_*(X)$ *(Smith inequality);*
3) $\dim H_*(F) \leq \dim H^1(\mathbb{F}_2; H_*(X))$ *(Borel-Swan inequality);*
4) $\dim H_*(F) = \dim H_*(X) \bmod 2$*;*
5) $\chi(X) = \chi(F) + 2\chi(\bar{X}, F)$*;*
6) $\chi(X) = 2\chi(\bar{X}) - \chi(F)$

Recall that by definition $H^1(\mathbb{F}_2; H_*(X)) = \ker(1 + c_*)/\operatorname{Im}(1 + c_*)$.

Definition 2.3. *Let* (X, c) *be a topological space equipped with a cellular involution. If* $\dim H_*(F) = \dim H_*(X)$*, one says that* c *is an* M-involution*, and* X *is called* maximal*, or an* M-variety*.*

Corollary 2.4. *Let* (X, c) *be a topological space equipped with a cellular involution. Then* X *is an an* M*-variety if and only if for every* $k \geq 0$*, the sequence*

$$0 \to H_{k+1}(\bar{X}, F) \xrightarrow{\Delta} H_k(\bar{X}, F) \oplus H_k(F) \to H_k(X) \to 0$$

is exact.

Corollary 2.5 (Smith Lemma)**.** *A space with involution is maximal if and only if each* $\mathbb{F}_2$*-homology class of the space can be realized by an equivariant cycle.*

2.2. **Equivariant (co)homology and the Borel spectral sequence.** Let $BG = \mathbb{RP}^\infty$ be the classifying space of a cyclic group of order 2 denoted by G, and $EG = S^\infty$ its universal cover. Let $X_G := (X \times EG)/G$ be the Borel construction, where G acts freely, as the diagonal action, and denote by $H^*_G(X, \mathbb{F}_2) := H^*(X_G, \mathbb{F}_2)$ the equivariant cohomology ring with $\mathbb{F}_2$-coefficients. To the fiber sequence

$$X \hookrightarrow X_G \to BG;$$

we associate the Leray-Serre spectral sequence:

$$E_2^{pq} = H^p(G, H^q(X, \mathbb{F}_2)) \longrightarrow H_G^{p+q}(X, \mathbb{F}_2). \tag{2.2}$$

The maximality of (X, c) can be reformulated into the surjectivity of the restriction map from the equivariant cohomology to the usual cohomology (see, for example, [27, Chapter III, Proposition 4.16]):

Proposition 2.6. *Let (X, c) be a topological space equipped with a cellular involution. The following conditions are equivalent:*

1) *X is an M-variety.*
2) *The action of G on $H^*(X, \mathbb{F}_2)$ is trivial and the Leray-Serre spectral sequence (2.2) degenerates at the second page.*
3) *The restriction homomorphism $R : H^*_G(X, \mathbb{F}_2) \to H^*(X, \mathbb{F}_2)$ is surjective.*

3. Cut-and-Paste construction of Hilbert squares over the reals

For smooth varieties a simple, well known, construction of the Hilbert square consists in the following. Given a smooth variety X, one lifts the involution $\tau : X \times X \to X \times X$ permuting the factors to an involution $Bl(\tau)$ on the blowup $Bl_\Delta(X \times X)$ of $X \times X$ along the diagonal $\Delta \subset X \times X$. The quotient of $Bl_\Delta(X \times X)$ by $Bl(\tau)$ is then naturally isomorphic to the Hilbert square $X^{[2]}$.

The quotient map $Bl_\Delta(X \times X) \to X^{[2]}$ is a double ramified covering. By construction, the branch locus $E \subset X^{[2]}$ is naturally isomorphic to $\mathbb{P}(T^*X)$ (in the Grothendieck notation) and coincides with the exceptional divisor of the canonical projection $X^{[2]} \to X^{(2)}$ to the symmetric square $X^{(2)}$ of X.

This construction works over any field. Applying it to a smooth variety X defined over the reals, we observe that $X^{[2]}(\mathbb{R})$ is a disjoint union of connected components

$$X^{[2]}(\mathbb{R}) = X^{[2]}_{\text{main}}(\mathbb{R}) \coprod X^{[2]}_{\text{extra}}(\mathbb{R}), \quad X^{[2]}_{\text{extra}}(\mathbb{R}) = \coprod_{i<j} (F_i \times F_j)$$

where $F_1, \dots, F_r$ are the connected components of $X(\mathbb{R})$ and $X^{[2]}_{\text{main}}(\mathbb{R})$ is the component of $X^{[2]}(\mathbb{R})$ that contains $E(\mathbb{R})$.

Furthermore, $E(\mathbb{R}) \subset X^{[2]}_{\text{main}}(\mathbb{R})$ is naturally diffeomorphic to $\mathbb{P}_\mathbb{R}(T^*X(\mathbb{R}))$. The normal line bundle of $E(\mathbb{R})$ in $X^{[2]}_{\text{main}}(\mathbb{R})$ is trivial, and $E(\mathbb{R})$ divides $X^{[2]}_{\text{main}}(\mathbb{R})$ in $r + 1$ submanifolds with boundary:

$$X^{[2]}_{\text{main}}(\mathbb{R}) = \bigcup_{i=0}^{r} \mathbb{H}_i, \quad \partial\mathbb{H}_0 = \bigsqcup_{i=1}^{r} \partial\mathbb{H}_i$$

where

$$\partial\mathbb{H}_0 = E(\mathbb{R}), \quad \operatorname{Int}\mathbb{H}_0 \cong (X/\mathrm{c}) \setminus X(\mathbb{R})$$

$$\partial\mathbb{H}_i = \mathbb{P}_\mathbb{R}(T^*F_i), \quad \operatorname{Int}\mathbb{H}_i \cong F_i^{(2)} \setminus \Delta F_i, i = 1, \dots, r.$$

Here $\mathrm{c} : X \to X$ denotes the anti-holomorphic involution on X induced the non-trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, while ΔF_i is the diagonal in $F_i^{(i)}$, for every $i = 1, \dots, r$. The manifolds $\mathbb{H}_i$ with $i = 1, \dots, r$ are glued to $\mathbb{H}_0$ along common with $\mathbb{H}_0$ boundaries $\mathbb{P}_{\mathbb{R}}(T^*F_i) \subset \mathbb{P}_{\mathbb{R}}(T^*X(\mathbb{R}))$.

Proposition 3.1. *Let X be a real smooth projective surface. Then:*

1) *For each $k = 1, \dots, r$ the inclusion homomorphisms*

$$\mathfrak{i}_*^{\mathbb{H}_k} : H_1(\mathbb{P}_{\mathbb{R}}T^*F_k) \to H_1(\mathbb{H}_k)$$

vanish at $\xi \in H_1(\mathbb{P}_{\mathbb{R}}T^*F_k)$ if and only if $\xi \cap \omega = 0$ where ω is the first Stiefel-Whitney class of the tautological line bundle over $\mathbb{P}_{\mathbb{R}}T^*F_k$.

2) *The inclusion homomorphism*

$$\mathfrak{i}_*^{\mathbb{H}_0} : H_1(\coprod_{k=1}^{r} \mathbb{P}_{\mathbb{R}}T^*F_k) \to H_1(\mathbb{H}_0)$$

is non-vanishing on each of the fiber-classes $w_k \in H_1(\mathbb{P}_{\mathbb{R}}T^*F_k)$.

3) *If X is maximal, then the homomorphism $\mathfrak{i}_*^{\mathbb{H}_0}$ is surjective.*

Proof. The given definition of ω is equivalent to saying that it is the characteristic class of the tautological double covering $UT^*F_k \to \mathbb{P}_{\mathbb{R}}T^*F_k$ (where UT^* stands for the unit cotangent bundle). By the latter reason we may also interpret ω as (a restriction of) the characteristic class of the double coverings $X \setminus X(\mathbb{R}) \to (X/\mathrm{c}) \setminus X(\mathbb{R})$ and $(F_k \times F_k) \setminus \Delta F_k \to (F_k)^{(2)} \setminus \Delta F_k$, $k = 1, \dots, r$. For that, it is sufficient, for each $k = 1, \dots, r$, to identify UT^*F_k and $\mathbb{P}_{\mathbb{R}}T^*F_k$ with the boundary of a tubular neighborhood of ΔF_k in $F_k \times F_k$ and $(F_k)^{(2)}$, respectively. Similarly, when it is a question of $X \setminus X(\mathbb{R}) \to (X/\mathrm{c}) \setminus X(\mathbb{R})$, we identify $\cup_k UT^*F_k$ and $\cup_k \mathbb{P}_{\mathbb{R}}T^*F_k$ with the boundary of a tubular neighborhood of $\cup F_k$ in X and X/c, respectively.

Due to ω being restricted from $H^1(\mathbb{H}_k)$ for each $k = 0, \dots, r$, the non-vanishing of $\mathfrak{i}_*^{\mathbb{H}_k}(\xi)$ when $\xi \cap \omega = 1$ is evident. This proves the second statement of the lemma and the "only if part" of the first one.

To prove the "if part" of the first statement, let $k \geq 1$ and consider a loop $\Gamma \subset \mathbb{P}_{\mathbb{R}}T^*F_k$ representing a class ξ with $\xi \cap \omega = 0$. Such a loop lifts to a loop $\tilde{\Gamma}$ on the boundary of a tubular neighborhood of ΔF_k in $F_k \times F_k$. Therefore, the homology class $[\tilde{\Gamma}] \in H_1(F_k \times F_k)$ belongs to the image of $H_1(\Delta F_k)$. But, according to the Künneth formula and Poincaré duality, the class $[\tilde{\Gamma}]$, as every class in the image of $H_1(\Delta F_k)$, is of the form $\sum(x_j \otimes 1 + 1 \otimes x_j)$, for some $x_j \in H_1(F_k)$. Since, in addition, the inclusion map

$$\mathrm{in}_*^{\complement \Delta F_k} : H_1(F_k \times F_k \setminus \Delta F_k) \to H_1(F_k \times F_k)$$

is an epimorphism and its kernel is generated by tubular circles w around ΔF_k, we may write the homology class $\tilde{\xi}$ realized in $H_1(F_k \times F_k \setminus \Delta F_k)$ by $\tilde{\Gamma}$ as $x + \tau_* x + \epsilon w$, $\epsilon = 0, 1$, for some $x \in H_1(F_k \times F_k \setminus \Delta F_k)$ with τ being the deck transformation of our covering, which in the case under consideration is just transposition of the factors. Now, we are done:

$$\begin{aligned}
\mathfrak{i}_*^{\mathbb{H}_k}(\xi) &= (\mathrm{pr}_* \circ \mathrm{in}_*^{\complement \Delta F_k})(\tilde{\Gamma}) \\
&= \mathrm{pr}_*(x + \tau_* x + \epsilon w) \\
&= \mathrm{pr}_*(x) + \mathrm{pr}_*(\tau_* x) + \epsilon\, \mathrm{pr}_*(w) \\
&= 0,
\end{aligned}$$

where $\mathrm{pr} : F_k \times F_k \to F_k^{(2)}$ is the projection.

To prove the third statement, note, first, that $X(\mathbb{R})$ is a codimension 2 smooth submanifold of X/c, while $\coprod_{k=1}^r \mathbb{P}_\mathbb{R} T^* F_k$ is the boundary of a tubular neighborhood of $X(\mathbb{R})$ in X/c. Therefore, the surjectivity of $\mathfrak{i}_*^{\mathbb{H}_0}$ follows from the surjectivity of the inclusion homomorphism $H_1(X(\mathbb{R})) \to H_1(X/\mathrm{c})$, which it in its turn follows from the exactness of the sequence $0 \to H_1(X/\mathrm{c}, X(\mathbb{R})) \to H_0(X(\mathbb{R})) \to H_0(X) \to 0$ that holds due to maximality of X (see Corollary 2.4). □

We collect next a few results which will be used several times in the next sections. For the convenience of the reader, we indicate the main ideas of the proof.

Proposition 3.2. *Let X be a smooth compact complex surface. In an abbreviated notation $\beta_* = \beta_*(X)$ and $\beta_1 = \beta_1(X)$, we have:*

1) *The relation*

$$\chi(X^{[2]}(\mathbb{R})) = \frac{1}{2}\beta_* - 2\beta_1 + \frac{1}{2}\chi(X(\mathbb{R}))^2 - \chi(X(\mathbb{R})). \tag{3.1}$$

2) *If* $\mathrm{Tors}_2\, H_*(X;\mathbb{Z}) = 0$, *the relation*

$$\beta_*(X^{[2]}) = \frac{1}{2}\beta_*(\beta_* + 1) + \beta_* - 2\beta_1. \tag{3.2}$$

3) *If* $\mathrm{Tors}_2\, H_*(X;\mathbb{Z}) \neq 0$, *the relation*

$$\beta_*(X^{[2]}) \geq \frac{1}{2}\beta_*(\beta_* + 1) + \beta_* - 2\beta_1. \tag{3.3}$$

Proof. The first item follows, for example, from the more general formula computing the Euler characteristic of the Hilbert scheme of points of surfaces defined over the reals [19, page 5452], and the observation that $\chi(X) = \beta_*(X) - 4\beta_1(X)$.

The other two items are implicitly contained in the proof of Theorem 2.2 in [28]. According to [28, Theorem 2.2], $\mathrm{Tors}_2\, H_*(X^{[2]};\mathbb{Z}) = 0$ as soon as $\mathrm{Tors}_2\, H_*(X;\mathbb{Z}) = 0$. Therefore, both parts of the statement follow from the following relation for the ordinary Betti numbers

$$\dim H_*(X^{[2]};\mathbb{Q}) = \frac{1}{2}b_*(b_* + 1) + b_* - 2b_1 \tag{3.4}$$

where we notate

$$b_i = \dim H_i(X;\mathbb{Q}), \quad b_* = \sum b_i.$$

This relation follows from the presentation of $X^{[2]}$ as the quotient of the blowup $Bl_\Delta(X\times X)$ of $X\times X$ due to the following arguments: [1]

(1) $H_*(X^{[2]};\mathbb{Q})$ is canonically (by pull-back) isomorphic to the $Bl(\tau)$ invariant subspace of $H_*(Bl_\Delta(X\times X);\mathbb{Q})$.
(2) $H_*(Bl_\Delta(X\times X);\mathbb{Q})$ splits canonically as
$$H_*(Bl_\Delta(X\times X);\mathbb{Q}) = H_*(X\times X;\mathbb{Q})\oplus H_*(\Delta(X\times X);\mathbb{Q}).$$
(3) $Bl(\tau)$ acts identically on the second summand, while on the first summand it maps the basic elements $v_i\otimes v_j$, $v_i\in H_i(X\times X;\mathbb{Q}), v_j\in H_j(X\times X;\mathbb{Q})$ to $(-1)^{ij}v_i\otimes v_j$.

□

4. Proof of Theorem 1.1

As a first step, we show the following:

Proposition 4.1. *Let X be a real smooth projective surface. If $X^{[2]}$ is maximal, then $X(\mathbb{R})\neq\emptyset$.*

Proof. By contradiction, let assume that $X^{[2]}$ is maximal and $X(\mathbb{R})=\emptyset$. Then $X^{[2]}(\mathbb{R})$ is the quotient smooth 4-manifold $X/\,\mathrm{c}$, and so
$$\beta_*(X^{[2]}(\mathbb{R})) = \beta_*(X/\,\mathrm{c}).$$
Since X and $X/\,\mathrm{c}$ are connected, by applying the homology Smith sequence we find
$$\cdots\to H_1(X)\to H_1(X/\,\mathrm{c})\to H_0(X/\,\mathrm{c}) = \mathbb{F}_2,$$
and so $\beta_1(X/\,\mathrm{c})\leq\beta_1+1$ where β_i states for $\beta_i(X)$. From Poincaré duality and the Riemann-Hurwitz formula we find
$$\begin{aligned}\beta_*(X/\,\mathrm{c}) &= \chi(X/\,\mathrm{c}) + 4\beta_1(X/\,\mathrm{c})\\ &\leq \frac{1}{2}\,\chi(X)+4\beta_1+4\\ &= \frac{1}{2}\,\beta_*+2\beta_1+4,\end{aligned}$$
where β_* is the total Betti number of X with $\mathbb{F}_2$-coefficients. Thus,
$$\beta_*(X^{[2]}(\mathbb{R})) = \beta_*(X/\,\mathrm{c}) \leq \frac{1}{2}\beta_*+2\beta_1+4 = 5+3\beta_1+\frac{1}{2}\beta_2.$$

[1] Alternatively, a different proof of the relation (3.4) follows from a simple inspection of Götsche's formula [13] for the Betti numbers of the Hilbert scheme of points on complex surfaces.

Using the estimate (3.3) in Proposition 3.2, we notice now that

$$\begin{aligned}\beta_*(X^{[2]}) \geq& \frac{1}{2}\beta_*(\beta_*+1)+\beta_*-2\beta_1\\ &=\frac{1}{2}(2+2\beta_1+\beta_2)^2+3+\beta_1+\frac{3}{2}\beta_2\\ &> 5+3\beta_1+\frac{1}{2}\beta_2\\ &\geq \beta_*(X^{[2]}(\mathbb{R})),\end{aligned}$$

contradicting the maximality of $X^{[2]}$. □

Proof of Theorem 1.1. Pick a point $p \in X(\mathbb{R})$, whose existence is ensured by Proposition 4.1, and consider the map $f : X \to X^{(2)}$ given by

$$f(x) = \{p, x\}.$$

Since the Hilbert-Chow map $\pi : X^{[2]} \to X^{(2)}$ is an isomorphism when restricted to $X^{[2]} \setminus E$ and $f(X \setminus \{p\}) \cap \pi(E) = \emptyset$, the restriction of f to $X \setminus \{p\}$ induces a map

$$\phi : X \setminus \{p\} \to X^{[2]}.$$

The map ϕ extends to the blowup $Bl_p(X)$ of X at the point p, and so we have a commutative diagram

$$\begin{tikzcd} Bl_p(X) \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}"'] & X^{[2]} \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & X^{(2)}. \end{tikzcd}$$

Lemma 4.2. *The map $\phi^* : H^*(X^{[2]}) \to H^*(Bl_p(X))$ is surjective.*

Proof. Note, first, that $H^i(Bl_p(X)) = H^i(X) \oplus H^i(\mathbb{P}^1)$ for any $i > 0$. Here, the first summand coincide with $\operatorname{Im}\mathrm{pr}^*$. Since $f^* : H^*(X^{(2)}) \to H^*(X)$ is surjective (see, for example, [11, Lemma 2.5] for a much more general statement), this implies that the first summand is contained in $\operatorname{Im}\phi^*$.

The second summand is nontrivial only for $i = 2$. It is generated by $\mathbb{F}_2$-reduction of the integer class $\mathrm{in}^* c_1(\theta)$, where θ is the tautological line bundle over $\mathbb{P}(T^*X) = E \subset X^{[2]}$ and $\mathrm{in} : \mathbb{P}^1 \hookrightarrow E$ is the inclusion. Since $c_1(\theta)$ is the restriction of an appropriate element $e \in H^2(X^{[2]};\mathbb{Z})$ (see, for example, [28, page 4]), the second summand is also contained $\operatorname{Im}\phi^*$. □

To finish the proof of Theorem 1.1, consider now the commutative diagram

$$\begin{tikzcd} H^*_G(X^{[2]}) \arrow[r] \arrow[d, "R^{[2]}"'] & H^*_G(Bl_p(X)) \arrow[d, "R"] \\ H^*(X^{[2]}) \arrow[r, "\phi^*"] & H^*(Bl_p(X)). \end{tikzcd}$$

Since $X^{[2]}$ is maximal the induced restriction map $R^{[2]}$ is onto, which together with the surjectivity of ϕ^* implies that the restriction map R is onto, as well. Hence, by Proposition 2.6, X is maximal. $\square$

5. Proof of Theorem 1.2 and Corollaries 1.3, 1.4

5.1. **Reduction of the proof to computation of $\beta_1(X^{[2]}(\mathbb{R}))$.** As before, let β_i, $i = 0, \dots, 4$ denote the $\mathbb{F}_2$ - Betti numbers of X. Under assumption of Theorem 1.2, we have $\beta_1 = \beta_3 = 0$, while β_2 coincides with the ordinary Betti number $b_2 = \dim H_2(X; \mathbb{Q})$.

According to Proposition 3.2, we find

$$\beta_*(X^{[2]}) = 5 + 3\beta_2 + \frac{1}{2}\beta_2(\beta_2 + 1),$$
$$\chi(X^{[2]}(\mathbb{R})) = 1 + \frac{1}{2}\beta_2 + \frac{1}{2}\chi(X(\mathbb{R}))^2 - \chi(X(\mathbb{R})).$$

Since $\beta_1(X^{[2]}(\mathbb{R})) = \dfrac{1}{4}\left[\beta_*(X^{[2]}(\mathbb{R})) - \chi(X^{[2]}(\mathbb{R}))\right]$, the above two relations imply that $X^{[2]}$ is maximal if and only if

$$\begin{aligned}\beta_1(X^{[2]}(\mathbb{R})) =& \frac{1}{4}\left[\beta_*(X^{[2]}) - \chi(X^{[2]}(\mathbb{R}))\right] \\ =& \frac{1}{4}\left[4 + 3\beta_2 + \frac{1}{2}\beta_2^2 + \chi(X(\mathbb{R})) - \frac{1}{2}\chi^2(X(\mathbb{R}))\right].\end{aligned} \tag{5.1}$$

It is convenient to rewrite this expression in terms of the number of the connected components of $X(\mathbb{R})$, which will be denoted by r. For that, we notice, first, that the maximality of $X(\mathbb{R})$ is equivalent to

$$2r + \beta_1(X(\mathbb{R})) = 2 + \beta_2,$$

which implies $\chi(X(\mathbb{R})) = 2r - \beta_1(X(\mathbb{R})) = 4r - 2 - \beta_2$. Then, an immediate computation transforms the criterium (5.1) into

$$\beta_1(X^{[2]}(\mathbb{R})) = 3r - 2r^2 + r\beta_2. \tag{5.2}$$

5.2. **Direct computation of $\beta_1(X^{[2]}(\mathbb{R}))$.** For this computation, we use the cut-and-paste construction described in Section 3, and follow the notations introduced therein.

Lemma 5.1. $\beta_1(X^{[2]}(\mathbb{R})_{\text{extra}}) = r\beta_2 - 2r^2 + 4r - \beta_2 - 2.$

Proof. Using the Künneth formula, we get

$$\begin{aligned}\beta_1(X^{[2]}_{\text{extra}}(\mathbb{R})) &= \sum_{i<j}(\beta_1(F_i)+\beta_1(F_j)) \\ &= \sum_{i<j}(\beta_*(F_i)+\beta_*(F_j)-4) \\ &= \frac{1}{2}\sum_{i,j}(\beta_*(F_i)+\beta_*(F_j)-4)-\frac{1}{2}\sum_i(2\beta_*(F_i)-4) \\ &= r\beta_*(X(\mathbb{R}))-2r^2-\beta_*(X(\mathbb{R}))+2r.\end{aligned} \tag{5.3}$$

Due to the maximality of $X(\mathbb{R})$, we have

$$\beta_*(X(\mathbb{R}))=\beta_*(X)=2+\beta_2,$$

and the proof of the lemma follows from (5.3). □

Lemma 5.2. $\beta_1(X^{[2]}_{\text{main}}(\mathbb{R}))=3+\beta_2-2r.$

Proof. We start by computing $\beta_1(\mathbb{H}_0)$. By the Poincaré-Lefschetz duality we have $\beta_1(\mathbb{H}_0)=\beta_3(X/\,\mathrm{c},X(\mathbb{R}))$. On the other hand, using the Smith exact sequence

$$\begin{aligned}0\to &H_4(X/\,\mathrm{c},X(\mathbb{R}))\xrightarrow{\mathrm{tr}^*}H_4(X)=\mathbb{F}_2\xrightarrow{\mathrm{pr}_*}H_4(X/\,\mathrm{c},X(\mathbb{R}))\xrightarrow{\Delta} \\ &\xrightarrow{\Delta}H_3(X/\,\mathrm{c},X(\mathbb{R}))\xrightarrow{\mathrm{tr}^*}H_3(X)=0.\end{aligned}$$

we conclude that $H_4(X/\,\mathrm{c},X(\mathbb{R}))=\mathbb{F}_2$ and the map Δ is an isomorphism. Therefore, $\beta_1(\mathbb{H}_0)=1$.

For $\beta_1(\mathbb{H}_i)$ with $i=1,\ldots,r$, once more by Poincaré-Lefschetz duality, we have $\beta_1(\mathbb{H}_i)=\beta_1(F_i^{(2)}\setminus\Delta F_i)=\beta_3(F_i^{(2)},\Delta F_i)$. Since the diagonal embedding $\Delta F_i\hookrightarrow F_i^{(2)}$ represents $0\in H_2(F_i^{(2)})$, the following short exact sequence holds

$$0\to H_3(F_i^{(2)})\to H_3(F_i^{(2)},\Delta F_i)\to H_2(\Delta F_i)\to 0,$$

and implies $\beta_3(F_i^{(2)},\Delta F_i)=\beta_3(F_i^{(2)})+1$. Furthermore, applying the Macdonald formula when F_i is oriented, and the Sadok-Salvatore formula [25] (see also [1]) in the non-orientable case, we find $\beta_3(F_i^{(2)})=\beta_1(F_i)$. Summing-up, we find $\beta_1(\mathbb{H}_i)=1+\beta_1(F_i)=\beta_*(F_i)-1$.

Thus, we obtain

$$\begin{aligned}\sum_{i=0}^{r}\beta_1(\mathbb{H}_i) &= 1+\sum_{i=1}^{r}(\beta_*(F_i)-1) \\ &= 1+\beta_*(X(\mathbb{R}))-r \\ &= 3+\beta_2-r.\end{aligned} \tag{5.4}$$

To finish the proof we apply the Mayer-Vietoris sequence

$$\bigoplus_{i=1}^{r}H_1(\mathbb{P}_{\mathbb{R}}T^*F_i)\xrightarrow{\mu}\bigoplus_{i=0}^{r}H_1(\mathbb{H}_i)\to H_1(X^{[2]}_{\text{main}}(\mathbb{R}))\to 0$$

and calculate the rank of μ as follows.

Lemma 5.3. $\operatorname{rank} \mu = r$.

Proof. According to Proposition 3.1, for each $i = 1, \dots, r$ the inclusion map

$$\mathfrak{i}_*^{\mathbb{H}_i} : H_1(\mathbb{P}_{\mathbb{R}} T^* F_i) \to H_1(\mathbb{H}_i)$$

vanishes on $\xi \in H_1(\mathbb{P}_{\mathbb{R}} T^* F_i)$ if and only if $\xi \cap \omega = 0$, where ω is the first Stiefel-Whitney class of the tautological line bundle over $\mathbb{P}_{\mathbb{R}} T^* F_i$. Under the assumption that $\beta_1(X) = 0$, we can see that the same conclusion holds for $i = 0$, as well. Namely, since $H_1(X) = 0$, we find that $H_1(\mathbb{H}_0) = H_1((X/\operatorname{c}) \setminus X(\mathbb{R})) = \mathbb{F}_2$ and the only its nontrivial element is represented by any of the tubular circles w_i around $F_i \subset X(\mathbb{R})$. Since for each of them, $w_i \cap \omega = 1$, by applying the second statement of Proposition 3.1 we conclude that the inclusion map

$$\mathfrak{i}_*^{\mathbb{H}_0} : H_1(\coprod_{i=1}^{r} \mathbb{P}_{\mathbb{R}} T^* F_i) = \bigoplus_{i=1}^{r} H_1(\mathbb{P}_{\mathbb{R}} T^* F_i) \to H_1(\mathbb{H}_0)$$

vanishes on $\xi \in H_1(\mathbb{P}_{\mathbb{R}} T^* F_i)$ if and only if $\xi \cap \omega = 0$.

To finish the proof, we represent μ by a block matrix $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ where $\begin{bmatrix} A & 0 \end{bmatrix}$ is a row matrix of $i_*^{\mathbb{H}_0} : \bigoplus_{i=1}^{r} H_1(\mathbb{P}_{\mathbb{R}} T^* F_i) \to H_1(\mathbb{H}_0) = \mathbb{F}_2$ and $\begin{bmatrix} C & 0 \end{bmatrix}$ is a matrix of

$$\bigoplus_{i=1}^{r} \mathfrak{i}_*^{\mathbb{H}_i} : \bigoplus_{i=1}^{r} H_1(\mathbb{P}_{\mathbb{R}} T^* F_i) \to \bigoplus_{i=1}^{r} H_1(\mathbb{H}_i).$$

Since, as we have established at the beginning of the proof, the rank of the second one is r, it remains to apply a simple matrix rule: the rank of a block matrix $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}$ where A, C are matrices of size $m_a \times r$, $m_c \times r$ is equal to $\operatorname{rank} C$ as soon as $\operatorname{rank} C = r$. □

From (5.4) and Lemma 5.3 we deduce

$$\beta_1(X^{[2]}_{\text{main}}(\mathbb{R})) = \sum_{i=0}^{r} \beta_1(\mathbb{H}_i) - r = 3 + \beta_2 - 2r,$$

which concludes the proof of Lemma . □

5.3. **End of the proof of Theorem 1.2.** As we proved in Section 5.1, $X^{[2]}$ is maximal if and only if $\beta_1(X^{[2]}(\mathbb{R})) = 3r - 2r^2 + r\beta_2$. On the other hand, combining the results of Lemmas 5.1 and 5.2 we get

$$\beta_1(X^{[2]}(\mathbb{R})) = 1 + r\beta_2 + 2r - 2r^2.$$

Obviously, the two results agree if and only if $r = 1$. □

5.4. **Proof of Corollaries 1.3, 1.4.** Both corollaries are immediate consequences of Theorem 1.2 plus, in the case of Corollary 1.3, the following statement.

Lemma 5.4. *If $H_1(X;\mathbb{Z})$ has no 2-torsion and $h^{2,0}(X)+h^{1,0}(X)>0$, then X does not admit a maximal real structure with connected $X(\mathbb{R})$.*

Proof. Due to absence of 2-torsion, the maximality relation can be rewritten in terms of ordinary Betti numbers as follows:

$$2b_0(X(\mathbb{R}))+b_1(X(\mathbb{R}))=b_*(X)=\sum h^{p,q}(X).$$

On the other hand, by Comessatti inequality (see, f.e., [8]),

$$2-(2b_0(X(\mathbb{R}))-b_1(X(\mathbb{R})))\leq h^{1,1}(X).$$

Combining them together we get $b_0(X(\mathbb{R}))\geq 1+\frac{1}{2}h^{2,0}+h^{1,0}\geq 2$. □

6. Proof of Theorem 1.5

We follow the same strategy as in the proof of Theorem 1.2, and the same notations.

Notice first that the maximality of $X(\mathbb{R})$ is equivalent to $\chi(X(\mathbb{R}))=4r-\beta_*$. By applying Proposition 3.2 to the maximal surface X we conclude that $X^{[2]}$ is maximal if and only if we have

$$\begin{aligned}\beta_1(X^{[2]}(\mathbb{R}))&=\frac{1}{4}\left[\frac{1}{2}\beta_*^2+\beta_*+\chi(X(\mathbb{R}))-\frac{1}{2}\chi(X(\mathbb{R}))^2\right]\\&=r\beta_*-2r^2+r.\end{aligned}\tag{6.1}$$

Furthermore, using (5.3) we find that (6.1) is equivalent with

$$\beta_1(X^{[2]}_{\mathrm{main}}(\mathbb{R}))=\beta_*-r.$$

On the other hand, as in the proof of Lemma 5.2, from Corollary 2.4 and (5.4), we deduce that

$$\beta_1(\mathbb{H}_0)=1+\beta_1,\quad \sum_1^r\beta_1(\mathbb{H}_i)=\beta_*-r.\tag{6.2}$$

Hence $X^{[2]}$ is maximal if and only if in the Mayer-Vietoris sequence

$$\bigoplus_{i=1}^r H_1(\mathbb{P}_\mathbb{R}T^*F_i)\xrightarrow{\mu}\bigoplus_{i=0}^r H_1(\mathbb{H}_i)\to H_1(X^{[2]}_{\mathrm{main}}(\mathbb{R}))\to 0$$

we have $\operatorname{rank}\mu=1+\beta_1$. Therefore, to finish the proof of Theorem 1.5 it remains to prove the following lemma.

Lemma 6.1. *Under assumptions of Theorem 1.5 we have:*

- $\operatorname{rank}\mu=1+\beta_1$ *if $X(\mathbb{R})$ is connected.*
- $\operatorname{rank}\mu\geq 2+\beta_1$ *if $r>1+\beta_1$.*

Proof. Assume that $r > 1 + \beta_1$ and show, first, that under this assumption the homomorphism $\mathfrak{i}_*^{\mathbb{H}_0} : H_1(\coprod_{i=1}^r \mathbb{P}_\mathbb{R} T^* F_i) \to H_1(\mathbb{H}_0)$ vanishes on at least one nontrivial combination of the fiber-classes $w_i, i = 1, \dots, r$.

Pick a point p_i on each $F_i \subset X$ with $i = 1, \dots, r$. For each $i = 2 \dots, r$, join p_1 with p_i by a generic path γ_i in X and consider the circles C_i formed by $\gamma_i \cup \mathrm{c}(\gamma_i)$. Since their number is strictly bigger than $\beta_1(X)$, there exists a 2-dimensional submanifold $\Upsilon \subset X$ bounding some nontrivial combination of circles C_i. The projection $\mathrm{pr}(\Upsilon)$ of Υ into $X/\,\mathrm{c}$ gives a 2-cycle which has a nontrivial $\mathbb{F}_2$-intersection with $F_i \subset X/\,\mathrm{c}$ for at least one of $i = 2 \dots, r$. This implies that at least one of F_i represents a nonzero element of $H_2(X/\,\mathrm{c})$.

To get a nontrivial combination of fiber-classes mapped to zero by

$$\mathfrak{i}_*^{\mathbb{H}_0} : H_1(\coprod_{i=1}^r \mathbb{P}_\mathbb{R} T^* F_i) \to H_1(\mathbb{H}_0),$$

it is sufficient now to pick an F_k representing a nonzero element and to take a generic smooth 2-dimensional submanifold $\Sigma \subset X/\,\mathrm{c}$ which has nontrivial $\mathbb{F}_2$-intersection with this F_k. Then, the wanted combination of the fiber-classes w_i is provided by those F_i which have a non-zero $\mathbb{F}_2$-intersection with Σ.

To finish the proof of Lemma 6.1, we use Proposition 3.1 and apply the following matrix rules:

- (case $r = 1$) the rank of a block matrix $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ where A, C are matrices of size $m_a \times r$, $m_c \times r$ is equal to rank $\begin{bmatrix} A & B \end{bmatrix}$ as soon as $\operatorname{rank} A = \operatorname{rank} C = r$.
- (case $r > 1 + \beta_1(X)$) the rank of a block matrix $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ where A, C are matrices of size $m_a \times r$, $m_c \times r$ is strictly bigger than rank $\begin{bmatrix} A & B \end{bmatrix}$ as soon as $\operatorname{rank} A < \operatorname{rank} C = r$.

Here we choose as $\begin{bmatrix} A & B \end{bmatrix}$ and $\begin{bmatrix} C & 0 \end{bmatrix}$ a matrix of $\mathfrak{i}_*^{\mathbb{H}_0}$ and a matrix of

$$\bigoplus_{i=1}^r \mathfrak{i}_*^{\mathbb{H}_i} : \bigoplus_{i=1}^r H_1(\mathbb{P}_\mathbb{R} T^* F_i) \to \bigoplus_{i=1}^r H_1(\mathbb{H}_i),$$

respectively. The rank of the first one is equal to $\beta_1(\mathbb{H}_0) = 1 + \beta_1$ by (6.2) and Proposition 3.1.3), while the rank of the second one is equal to r due to Proposition 3.1.1). In the case $r > 1 + \beta_1(X)$, we have $\operatorname{rank} A < \operatorname{rank} C = r$, since, as we have established at the beginning of the proof, the homomorphism $\mathfrak{i}_*^{\mathbb{H}_0}$ has, in such a case, a nontrivial kernel. □

7. Examples and applications

We start by presenting several examples, emphasizing the results obtained in Theorems 1.1, 1.2 and 1.5.

7.1. **Examples.** Theorem 1.1 provides a quick way to produce examples of Hilbert squares which are not maximal, and examples are abundant. Here we would like to draw the reader's attention to a special phenomenon. There exist complex deformation classes of surfaces that contain surfaces with real structure, but none of these real structures is maximal. This is the case, for example, of Campedelli surfaces [18] and of Miayoka-Yau surfaces with $h^{2,0} \leq 3$ [17]. Such surfaces cannot have maximal Hilbert square.

We will address next the Hilbert square of maximal surfaces only.

7.1.1. *Elliptic surfaces.* For simplicity, we restrict ourselves to minimal elliptic surfaces with rational base and without multiple fibers. Over the complex field, such elliptic surfaces form a countable number of deformation classes determined just by the Euler characteristic $\chi(X) = 12k$, $k \in \mathbb{Z}_{\geq 1}$. Such elliptic surfaces satisfy $H_1(X;\mathbb{Z}) = 0$. A straightforward application of Theorem 1.2 and Corollary 1.3 yields the following answer:

Proposition 7.1. *For $k = 1$, there exist maximal real elliptic surfaces X with $\chi(X(\mathbb{R})) = 12$ and connected $X(\mathbb{R})$; for these surfaces $X^{[2]}$ is maximal. For each $k \geq 2$, all maximal real elliptic surfaces X have disconnected $X(\mathbb{R})$; for these surfaces $X^{[2]}$ is never maximal.*

Proof. For $k = 1$, it is sufficient to pick two real plane cubics intersecting each other at 9 pairwise distinct real points and to take as X the blowup of $\mathbb{P}^2$ at these 9 points. Then the maximality of $X^{[2]}$ follows from Theorem 1.2.

If $k \geq 2$, it is sufficient to observe that $h^{2,0}(X) > 0$. After that there remains to apply Corollary 1.3. □

7.1.2. *Ruled surfaces.* As a byproduct of the proof of Theorem 1.5, we obtain the following result:

Proposition 7.2. *Let C be a maximal curve, and $\mathcal{E}$ a rank 2 complex vector bundle with a real structure which lifts the real structure of C. Then, the Hilbert square $X^{[2]}$ of the ruled surface $X = \mathbb{P}_C(\mathcal{E})$ is maximal.*

Proof. As it is established in the proof of Theorem 1.5, it is sufficient to check that $\operatorname{rank} \mu = 1 + \beta_1$. But $\beta_1 = \dim H_1(X) = 2g$, while $\dim H_1(X/\operatorname{c}) = g$, and thus $\operatorname{rank} \mu = 2g + 1 = 1 + \beta_1$ follows from $[F_i] = 0 \in H_1(X/\operatorname{c})$ for every connected component $F_i \subset X(\mathbb{R})$, $i = 1, \dots, g+1$. □

When C is of genus $g \geq 1$, the ruled surfaces $X = \mathbb{P}_C(\mathcal{E})$ provide examples of maximal surfaces with torsion free homology, whose real locus is disconnected (namely, $1 < \beta_0(X(\mathbb{R})) = g + 1 \leq \beta_1(X) = 2g$), but whose Hilbert square is maximal.

7.1.3. *Abelian surfaces.* Contrary to the case of ruled surfaces, for real abelian surfaces we obtain the following result.

Theorem 7.3. *No real abelian surfaces has a maximal Hilbert square.*

Proof. It is sufficient to consider the case of maximal surfaces and to check that $\operatorname{rank}\mu > 1+\beta_1$. But, in such a case, the action of the conjugation c on X is diffeomorphic to the action of $(-1)\times id$ on $E\times E$ where E is an elliptic curve. Thus, $\mathbb{H}_0$ is diffeomorphic to $(S^2 \setminus 4\ points)\times E$. Therefore, μ is represented by a matrix of the shape $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ where A, C are matrices of size $m_a\times 4, m_c\times 4$ with $\operatorname{rank} A = 3 < \operatorname{rank} C = 4$ and $\operatorname{rank}(A,B) = \beta_1(\mathbb{H}_0) = 5$, wherefrom $\operatorname{rank}\mu > \operatorname{rank}(A,B) = 5$. □

For comparison with the case of ruled surfaces, let us note that if X is a maximal abelian surface, we have $1 < \beta_0(X(\mathbb{R})) = \beta_1(X) = 4$.

7.1.4. *Rational surfaces.* As it follows from Comessatti's classification of real rational surfaces, every maximal real rational surface is either the real projective plane $\mathbb{P}^2$, or a real ruled surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ as precised in Corollary 1.4, or a real del Pezzo surface X with $K_X^2 = 1$ and $X(\mathbb{R})$ homeomorphic to $\mathbb{P}^2(\mathbb{R})\sqcup 4S^2$, or any of them blown-up at some number of real points. Since $\beta_1(X)$ and the number of real connected components are not changing under blow-ups, to each of these surfaces we may apply Theorem 1.2 and find the following complement to Corollary 1.4.

Proposition 7.4. *The Hilbert square $X^{[2]}$ is not maximal if X is a real del Pezzo surface with $K_X^2 = 1$ and $X(\mathbb{R})$ homeomorphic to $\mathbb{P}^2(\mathbb{R})\sqcup 4S^2$, or its blowup at some number of real points. For all other maximal real rational surfaces X, their Hilbert square is maximal.*

7.1.5. *Product of curves.* According to Theorem 1.5, if X is a maximal real surface such that $b_0(X(\mathbb{R})) > 1+\beta_1$, its Hilbert square is not maximal. Examples of such surfaces with $\beta_1 > 0$ exist in abundance. The simplest ones are given by products $C_1\times C_2$, where C_1 and C_2 are maximal curves of genus g_1 and g_2, respectively, where $g_1, g_2 \geq 2$ and $g_1+g_2 > 4$.

7.1.6. *Enriques surfaces.* For surfaces X with $\operatorname{Tors}_2 H^2(X;\mathbb{Z}) \neq 0$, one can still compute the $\mathbb{F}_2$-Betti numbers of the Hilbert square $X^{[2]}$ for a given example, albeit a general formula is lacking [28, Example 2.5]. We address here the maximality of the Hilbert square for real Enriques surfaces, and we find:

Theorem 7.5. *No real Enriques surfaces has a maximal Hilbert square.*

Proof. The proof follows the same lines as that of Theorem 1.5. There are only 2 essential changes. First, the formula (3.2) is to be replaced by

$$\beta_*(X^{[2]}) = \frac{1}{2}\beta_*(\beta_*+1) + \beta_* + 2\beta_1$$

which follows, for example, from the computation of $\mathbb{F}_2$-Betti numbers performed by B. Totaro [28]. Second, at the final step, Lemma 6.1 is to be replaced by a much simpler observation: $\operatorname{rank}\mu \geq \operatorname{rank} \oplus_1^r \mu_i = r$. □

Remark 7.6. As mentioned in Introduction, the real locus of a maximal real Enriques surface is always disconnected. This can be proved directly, i.e., without appealing to a full classification [7], in the following manner. Namely, let us assume that the real locus $X(\mathbb{R})$ of some maximal real Enriques surface X is connected. Then, $X(\mathbb{R})$ is homeomorphic to a real projective plane blown-up at $16 - 3 = 13$ real points. Its universal covering is a double covering $Y \to X$, where Y is a K3-surface. Y can be equipped with two possible lifts of the real structure. We choose the one for which $Y(\mathbb{R}) \neq \emptyset$. Then, $Y(\mathbb{R})$ is the orientation double cover of $X(\mathbb{R})$, and so $\chi(Y(\mathbb{R})) = 2\chi(X(\mathbb{R})) = -24$. Since $Y(\mathbb{R})$ is orientable, it follows that $Y(\mathbb{R})$ is homeomorphic to a sphere with 13 handles, which contradicts the Smith inequality $\beta_*(Y(\mathbb{R})) \leq \beta_*(Y) = 24$.

7.2. **Application to cubic 4-folds.** The deformation classification of real nonsingular cubic 4-folds, established by S. Finashin and V. Kharlamov [9], associates with all but one class (called by them *irregular*) a deformation class of a real nonsingular K3 surface. On the other hand, due to V. Krasnov [21], for each regular class of cubic 4-folds (in Finashin-Kharlamov's sense) except the one which corresponds to a K3 with 10 spheres as the real locus, the Fano variety is equivariantly diffeomorphic to the Hilbert square of the corresponding K3. In a separate paper [22], Krasnov proved that for the cubics in the irregular class, the real locus of the Fano variety is a disjoint union of six disjoint copies of $S^2 \times S^2$ and one more component which is homeomorphic to $X^{[2]}_{\text{main}}(\mathbb{R})$, where X is a K3-surface with $X(\mathbb{R}) = 3S^2$. Therefore, combining these results with Corollary 1.3 and the computations from the proof of Theorem 1.2, we get the following statement.

Theorem 7.7. *If a real nonsingular cubic 4-fold belongs to a regular class and the real locus of the associated real K3 is not the union of 10 spheres, as well as if the cubic belongs to the irregular class, then the Fano variety of the cubic is not maximal.*

Remark 7.8. Notice that maximal real nonsingular cubic 4-folds do exist. They form 3 real deformation classes, see [9], distinguished by the topology of their real loci: $\mathbb{P}^4(\mathbb{R})\#10(S^2\times S^2)\#(S^1\times S^3)$, or $\mathbb{P}^4(\mathbb{R})\#6(S^2\times S^2)\#5(S^1\times S^3)$, or $\mathbb{P}^4(\mathbb{R})\#2(S^2\times S^2)\#9(S^1\times S^3)$, see [10]. As a consequence of Theorem 7.7, the Fano variety of a maximal real nonsingular cubic 4-fold is not maximal. This result contrasts with Krasnov's theorem [23] stating that the Fano surface of a real nonsingular cubic 3-fold is maximal if and only if the cubic is maximal.

7.3. **On third Hilbert power.** As was noticed by L. Fu [12], maximality of $X^{[2]}$ implies that of $X^{[3]}$. Combining this with Theorem 1.2 and Propositions 7.1, 7.2, each time when the corresponding statement insures maximality of $X^{[2]}$ we deduce maximality of $X^{[3]}$. But it leaves open the non-maximality results: for example, is $X^{[3]}$ non-maximal for K3-surfaces, and, more generally, for surfaces with $H_1(X) = 0$ and disconnected real part?

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