

BALANCED MANIFOLDS AND *SKT* METRICS

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ABSTRACT. The equality between the balanced and the Gauduchon cones is discussed in several situations. In particular, it is shown that equality does not hold on many twistor spaces, and it holds on Moishezon manifolds. Moreover, it is proved that a *SKT* manifold of dimension three on which the balanced cone equals the Gauduchon cone is in fact Kähler.

1. INTRODUCTION

Let X be a closed complex manifold of dimension n . A class in the Bott-Chern cohomology group $H_{BC}^{1,1}(X, \mathbb{R})$ is called pseudoeffective if it contains a closed positive current. The set of such classes forms a closed convex cone in $H_{BC}^{1,1}(X, \mathbb{R})$ called the pseudoeffective cone and it is denoted it by \mathcal{E}_{BC}^1 . If furthermore X is Kähler manifold, let $\mathcal{M} \subseteq H_{BC}^{n-1, n-1}(X, \mathbb{R})$ the closure of the convex cone generated by classes of currents of the form $p_*(\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_{n-1})$, where $p : \tilde{X} \rightarrow X$ is some modification and $\tilde{\omega}_i$ are Kähler forms on \tilde{X} . The cone \mathcal{M} , called the movable cone, was introduced by Boucksom, Demailly, Păun and Peternell [5] who made the following conjecture:

Conjecture 1.1 (Conjecture 2.3, [5]). *For any Kähler manifold,*

$$(\mathcal{E}_{BC}^1)^* = \mathcal{M}.$$

This remarkable conjecture has recently been confirmed for projective manifolds by Witt Nyström [27], while the general case is still open. Extending the work of Toma [25] from projective to Kähler setting, it was observed by Fu and Xiao [16, Theorem A.2] (see also [8, Remark 2.8]) that Conjecture 1.1 implies that for Kähler manifolds the movable cone is in fact the balanced cone \mathcal{B} of all positive d -closed smooth $(n-1, n-1)$ -forms in $H_{BC}^{n-1, n-1}(X, \mathbb{R})$.

A Hermitian metric g on X with co-closed Kähler form ω is called balanced. The class of balanced manifolds, i.e., the class of closed complex manifolds carrying balanced metrics, was introduced by Michelsohn [19] who observed that prescribing a balanced metric (or equivalently its Kähler form) is the same as prescribing a positive d -closed smooth $(n-1, n-1)$ -form. This class of manifolds has attracted considerable interest in the recent years. Most notably, Alessandrini and Bassanelli proved in [2] that unlike the class of Kähler manifolds, the class of balanced manifolds is closed under bimeromorphisms. Furthermore, Fu, Li and Yau [14] stressed the importance of balanced manifolds from the perspective of heterotic string theory and constructed interesting non-Kähler examples in dimension three. Motivated by Conjecture 1.1, Fu and Xiao formulated the following:

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Conjecture 1.2 (Conjecture A.4., [16]). *For any compact balanced manifold*

$$(\mathcal{E}_{BC}^1)^* = \overline{\mathcal{B}}.$$

We give first many counter-examples to Conjecture 1.2. To formulate our result, recall that on a closed complex manifold X one can define the Gauduchon cone \mathcal{G} as the set of all classes in the Aeppli cohomology group $H_A^{n-1, n-1}(X, \mathbb{R})$ which can be represented by a Gauduchon metric, i.e, by a $\partial\bar{\partial}$ -closed positive $(n-1, n-1)$ -form. Lamari's positivity criterion [18, Lemme 3.3] can be stated as $(\mathcal{E}_{BC}^1)^* = \overline{\mathcal{G}}$. Furthermore, let

$$\iota_{n-1} : H_{BC}^{n-1, n-1}(X, \mathbb{R}) \rightarrow H_A^{n-1, n-1}(X, \mathbb{R})$$

be the map induced by the identity. Since a balanced metric is also a Gauduchon metric, we have $\iota_{n-1}(\mathcal{B}) \subseteq \mathcal{G}$. Therefore, the claim in Conjecture 1.2 is $\iota_{n-1}(\mathcal{B}) = \mathcal{G}$, provided the ambient manifold is balanced.

Theorem 1.3. *There exists twistor spaces such that $\iota_{n-1}(\mathcal{B}) \subsetneq \mathcal{G}$.*

Since the twistor spaces are known to carry balanced metrics [19], we obtain many counter-examples to the Fu-Xiao conjecture.

On the other hand, based on the main result of Witt Nyström [27], we confirm the validity of Conjecture 1.2 for Moishezon manifolds. For such manifolds, the $\partial\bar{\partial}$ -lemma holds, and so the map ι_{n-1} is an isomorphism. We prove:

Theorem 1.4. *For any Moishezon manifold $\mathcal{B} = \mathcal{G}$.*

In particular, Theorems 1.3 and 1.4 are pieces of evidence in favor of a conjecture of Popovici [22, Conjecture 6.1].

Conjecture 1.5. *If X is a compact complex manifold on which the $\partial\bar{\partial}$ -lemma holds, then $\mathcal{B} = \mathcal{G}$.*

Our last result is motivated by a conjecture of Fino and Vezzoni [12]. Recall that a Hermitian metric g with Kähler form ω on a compact complex manifold X of dimension n is called strongly Kähler with torsion (SKT for short) if ω is $\partial\bar{\partial}$ -closed. It is known that a metric which is both balanced and SKT is d -closed, hence Kähler [3]. Moreover, all the known examples of manifolds admitting a balanced metric and a SKT metric are Kähler. For instance, in [15] Fu, Li and Yau show that the examples of balanced non-Kähler manifolds they constructed do not carry SKT metrics. Verbitsky [26] showed that a twistor space with a SKT metric is Kähler. In [6], it is shown that a manifold in the Fujiki class \mathcal{C} (which is a balanced manifold by [1]) and which supports a SKT metric is Kähler. In [13], it is proved that a nilmanifold which is balanced and SKT is Kähler. It is therefore tempting to make the following conjecture

Conjecture 1.6 (Problem 3, [12]). *A balanced and SKT compact complex manifold is Kähler.*

We address here this conjecture for the class of complex manifolds of dimension three satisfying $\iota_2(\mathcal{B}) = \mathcal{G}$.

Theorem 1.7. *Let X be a compact complex manifold of dimension three such that $\iota_2(\mathcal{B}) = \mathcal{G}$. If X carries a SKT metric, then X is Kähler.*

2. PRELIMINARIES

Definition 2.1. Let (X, g) be a compact complex manifold of complex dimension n equipped with a Hermitian metric g , and let ω denote its Kähler form.

- i) If $d(\omega^{n-1}) = 0$, then g is called a balanced metric. A complex manifold which admits a balanced metric is called a balanced manifold.
- ii) If $\partial\bar{\partial}\omega = 0$, then g is called a strongly Kähler with torsion (SKT) metric. A complex manifold which admits a SKT metric is called a SKT manifold.
- iii) If $d\omega = 0$, then g is called a Kähler metric. A complex manifold which admits a Kähler metric is called a Kähler manifold.

Since the Kähler form of a Hermitian metric determines the metric, by an abuse of terminology we will not distinguish between the two notions. Moreover, according to Michelsohn [19, page 279], given a positive $(n-1, n-1)$ -form Φ on an n -dimensional manifold, there exists a positive $(1, 1)$ -form η such that $\Phi = \eta^{n-1}$. Therefore, prescribing a balanced or a Gauduchon metric is equivalent to prescribing a positive $(n-1, n-1)$ -form which is d or $\partial\bar{\partial}$ -closed, respectively.

2.1. Bott-Chern and Aeppli cohomologies and positive cones. Given a compact complex manifold X of dimension n , we define the Bott-Chern cohomology groups

$$H_{BC}^{p,q}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{C}_{p,q}^\infty(X) \mid d\alpha = 0\}}{\{i\partial\bar{\partial}\beta \mid \beta \in \mathcal{C}_{p-1,q-1}^\infty(X)\}},$$

and the Aeppli cohomology groups

$$H_A^{p,q}(X, \mathbb{C}) = \frac{\{\alpha \in \mathcal{C}_{p,q}^\infty(X) \mid i\partial\bar{\partial}\alpha = 0\}}{\{\partial\beta + \bar{\partial}\gamma \mid \beta \in \mathcal{C}_{p-1,q}^\infty(X), \gamma \in \mathcal{C}_{p,q-1}^\infty(X)\}}$$

As \mathbb{C} -vector spaces, $H_{BC}^{p,q}(X, \mathbb{C})$ and $H_A^{p,q}(X, \mathbb{C})$ are finite dimensional for every $p, q \geq 0$, as it follows from the Hodge theory developed by M. Schweitzer [23].

We use the notation $[s]$ for the class of a d -closed form or current s in $H_{BC}^{\bullet,\bullet}$ and $\{t\}$ for the class of a $i\partial\bar{\partial}$ -closed form or current t in $H_A^{\bullet,\bullet}$.

The groups $H_{BC}^{p,q}(X, \mathbb{C})$ and $H_A^{n-p,n-q}(X, \mathbb{C})$ are dual via the pairing

$$H_{BC}^{p,q}(X, \mathbb{C}) \times H_A^{n-p,n-q}(X, \mathbb{C}) \rightarrow \mathbb{C}, ([\alpha], \{\beta\}) \rightarrow \int_X \alpha \wedge \beta.$$

Let X be a compact complex manifold of dimension n . The Gauduchon cone of X is

$$\mathcal{G}_X = \{[\Omega] \in H_A^{n-1,n-1}(X, \mathbb{R}) \mid \Omega \text{ is a Gauduchon metric}\}$$

Similarly, we define the balanced cone:

$$\mathcal{B}_X = \{[\Omega] \in H_{BC}^{n-1,n-1}(X, \mathbb{R}) \mid \Omega \text{ is a balanced metric}\}.$$

The Gauduchon cone is an open convex cone. According to Gauduchon [17], it is never empty. The balanced cone is open and convex. It can be empty, as there are examples compact complex manifolds which do not admit balanced metrics (e.g., see [19]).

The natural morphisms induced by the identity

$$\iota_1 : H_{BC}^{1,1}(X, \mathbb{R}) \rightarrow H_A^{1,1}(X, \mathbb{R})$$

and

$$\iota_{n-1} : H_{BC}^{n-1,n-1}(X, \mathbb{R}) \rightarrow H_A^{n-1,n-1}(X, \mathbb{R})$$

are well-defined, but in general, they are neither injective, nor surjective. They are however isomorphisms if X is Kähler, or more generally on manifolds satisfying the $\partial\bar{\partial}$ -lemma. Nevertheless, we have

$$\iota_{n-1}(\mathcal{B}_X) \subseteq \mathcal{G}_X,$$

and so $\iota_{n-1}(\overline{\mathcal{B}_X}) \subseteq \overline{\mathcal{G}_X}$.

For $\# \in \{BC, A\}$ and $p \in \{1, n-1\}$ we define the following cones:

- (1) the $\#$ -pseudoeffective cone

$$\mathcal{E}_{X,\#}^p = \{\gamma \in H_{\#}^{p,p}(X, \mathbb{R}) \mid \exists T \geq 0, T \in \gamma\},$$

where by T we denote here a current.

- (2) the $\#$ -nef cone

$$\mathcal{N}_{X,\#}^p = \{\gamma \in H_{\#}^{p,p}(X, \mathbb{R}) \mid \forall \varepsilon > 0, \exists \alpha_\varepsilon \in \gamma, \alpha_\varepsilon \geq -\varepsilon \omega^p\},$$

where ω is the Kähler form of a fixed Hermitian metric on X and α_ε denotes a smooth (p, p) -form.

Remark 2.2. The pseudoeffective and nef cones $\mathcal{E}_{X,BC}^1$ and $\mathcal{N}_{X,BC}^1$ were first introduced by Demailly [9, Definition 1.3].

We recall next some of the properties and relations between the above cones.

Proposition 2.1. *Let X be a compact complex manifold of dimension n . Then*

- i) *The cone $\mathcal{E}_{X,BC}^1$ is closed and $\mathcal{N}_{X,BC}^1 \subseteq \mathcal{E}_{X,BC}^1$.*
- ii) *The cones $\mathcal{N}_{X,\#}^p$ are closed, where $p \in \{1, n-1\}$ and $\# \in \{BC, A\}$.*
- iii) *$\mathcal{N}_{X,A}^{n-1} = \overline{\mathcal{F}_X}$.*

Moreover, if X is balanced, then

- iv) *$\mathcal{N}_{X,BC}^{n-1} = \overline{\mathcal{F}_X}$.*
- v) *$\mathcal{E}_{X,A}^1$ is closed.*

Proof. For complete proofs we refer the interested reader to Lemmas 2.2, 2.3 and 2.5 in [8]. \square

We will often use the following result [8, Theorem 2.4] (see also [16, Remark 3.3]), which we state for the convenience of the reader:

Theorem 2.2. *Let X be a compact complex manifold of dimension n . Then*

- i) *$\mathcal{N}_{X,BC}^1 = (\mathcal{E}_{X,A}^{n-1})^*$,*
- ii) *$\mathcal{N}_{X,A}^{n-1} = (\mathcal{E}_{X,BC}^1)^*$.*

Moreover, if X is balanced, then

- iii) *$\mathcal{N}_{X,A}^1 = (\mathcal{E}_{X,BC}^{n-1})^*$,*
- iv) *$\mathcal{N}_{X,BC}^{n-1} = (\mathcal{E}_{X,A}^1)^*$.*

We conclude this section with the following result which indicates that the balanced cone is a natural generalization on balanced manifolds of the movable cone whose definition is confined to the Fujiki class \mathcal{C} manifolds.

Proposition 2.3. *Let $\pi : X \rightarrow Y$ be a blow-up with smooth center between two balanced compact complex manifolds of dimension n . Then $\pi_* \mathcal{B}_X = \mathcal{B}_Y$.*

Proof. The inclusion $\mathcal{B}_Y \subseteq \pi_*\mathcal{B}_X$ is just Corollary 4.9 in [2]. Conversely, let ω^{n-1} be a balanced metric on X . From Theorem 2.2 iv), in order to show that the class $[\pi_*\omega^{n-1}]$ is balanced, it is enough to check that $([\pi_*\omega^{n-1}], \{T\}) \geq 0$ where T is an arbitrary $(1,1)$ -current on Y which is positive and $\partial\bar{\partial}$ -closed. By [2], given such a current T , there exists \tilde{T} a positive $(1,1)$ -current on X which is $\partial\bar{\partial}$ -closed, such that $\pi_*\tilde{T} = T$ and $\{\tilde{T}\} = \pi^*\{T\}$. Then

$$([\pi_*\omega^{n-1}], \{T\}) = ([\omega^{n-1}], \{\tilde{T}\}) = \int_X \tilde{T} \wedge \omega^{n-1} \geq 0$$

The above inequality is strict when the class $\{T\} \neq 0$, and this shows that $[\pi_*\omega^{n-1}]$ is in the interior of the cone \mathcal{N}_{BC}^{n-1} , which is the balanced cone. \square

Remark 2.3. In [28, Proposition 2.3], Xiao observed that one always has $\pi_*\mathcal{B}_X \subseteq \mathcal{B}_Y$.

3. $\mathcal{B} = \mathcal{G}$ MANIFOLDS

Let X be a closed Hermitian manifold such that $\iota_{n-1}(\mathcal{B}_X) = \mathcal{G}_X$. Imposing such condition has several implications on the complex structure of X .

Lemma 3.1. *Let X be a complex manifold such that $\iota_{n-1}(\mathcal{B}_X) = \mathcal{G}_X$. Then ι_{n-1} is onto. If in addition X is a SKT manifold, then ι_1 and ι_{n-1} are isomorphisms.*

Proof. Since \mathcal{G}_X is open and non-empty, we see that X is balanced and that ι_{n-1} is surjective. In particular, since X is balanced, it follows that \mathcal{E}_A^1 is closed. Since the cones \mathcal{N}_{BC}^{n-1} and \mathcal{N}_A^{n-1} are the closures of the cones \mathcal{B}_X and \mathcal{G}_X respectively, we get that

$$\iota_{n-1}(\mathcal{N}_{BC}^{n-1}) = \mathcal{N}_A^{n-1}. \quad (3.1)$$

Dualizing (3.1), from Theorem 2.2 we obtain that $\iota_1(\mathcal{E}_{BC}^1) = \mathcal{E}_A^1$ and that ι_1 is injective. Since X is SKT, it follows that the interior of \mathcal{E}_A^1 is non-empty, therefore ι_1 is also onto, hence an isomorphism. Therefore, ι_{n-1} is also an isomorphism. \square

3.1. Twistor spaces and counter-examples to the Fu-Xiao conjecture. One can interpret Lemma 3.1 as an obstruction to the equality of the balanced and Gauduchon cones. We adopt this point of view and disprove next Conjecture A.4 in [16]. The counter-examples we propose are certain twistor spaces.

3.1.1. Twistor spaces. Let (M, g) be an oriented Riemannian 4-manifold. The rank-6 vector bundle bundle of 2-forms Λ^2 on M decomposes as the direct sum of two rank-3 vector bundles

$$\Lambda^2 T^* M = \Lambda_+ \oplus \Lambda_-$$

By definition, Λ^\pm are the eigenspaces of the Hodge \star -operator

$$\star : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M,$$

corresponding to the (± 1) -eigenvalues of \star . The sections of Λ^+ are called self-dual 2-forms, whereas the sections of Λ^- are the anti-self-dual 2-forms.

The Riemannian curvature tensor can be thought of as an operator

$$\mathcal{R} : \Lambda^2 T^* M \rightarrow \Lambda^2 T^* M,$$

called be the Riemannian curvature operator. The Riemannian curvature operator decomposes under the action of $SO(4)$ as

$$\mathcal{R} = \frac{s}{6}Id + W^- + W^+ + \overset{\circ}{r},$$

where W^\pm are trace-free endomorphisms of Λ^\pm , and they are called the self-dual and anti-self-dual components of the Weyl curvature operator. The scalar curvature s acts by scalar multiplication and $\overset{\circ}{r}$ is the trace-free Ricci curvature operator.

Definition 3.1. An oriented Riemannian 4-manifold (M, g) is said to be anti-self-dual (ASD) if $W^+ = 0$.

Remark 3.2. This definition is conformally invariant [4], i.e. if (M, g) is ASD, so is (M, ag) for any smooth positive function a .

A plethora of anti-self-dual 4-manifolds is rendered by a result of Taubes [24] asserting that for any smooth, compact, oriented, 4-dimensional manifold X , the connect sum $M = X \# k\overline{\mathbb{C}\mathbb{P}^2}$ of X with k copies of the complex projective 2-space equipped with the opposite of its complex orientation admits a metric with $W^+ = 0$ for k sufficiently large. In particular, one can find ASD manifolds with arbitrarily large first Betti number.

The twistor space of a conformal Riemannian manifold $(M, [g])$ is the total space of the sphere bundle of the rank three real vector bundle of self-dual 2-forms

$$\mathcal{Z} := S(\Lambda_+).$$

Atiyah, Hitchin, and Singer [4] show that that \mathcal{Z} comes naturally equipped with an almost complex structure, which is integrable if and only if $W^+ = 0$.

In [19], Michelsohn states that the twistor space of a closed ASD manifold always carries a balanced metric, a result proved in [20] (see also [8, Sect. 4]).

3.1.2. *Proof of Theorem 1.3.* Let \mathcal{Z} be the twistor space of a closed anti-self-dual manifold M of real dimension four. If $\mathcal{B}_{\mathcal{Z}} = \mathcal{G}_{\mathcal{Z}}$ by Lemma 3.1

$$j_2 : H_{BC}^{2,2}(\mathcal{Z}, \mathbb{C}) \rightarrow H_A^{2,2}(\mathcal{Z}, \mathbb{C})$$

is surjective. Hence the natural morphism

$$\bar{\partial} : H_A^{2,2}(\mathcal{Z}, \mathbb{C}) \rightarrow H_{BC}^{2,3}(\mathcal{Z}, \mathbb{C})$$

is zero. By duality, we obtain that

$$\bar{\partial} : H_A^{1,0}(\mathcal{Z}, \mathbb{C}) \rightarrow H_{BC}^{1,1}(\mathcal{Z}, \mathbb{C})$$

is zero, which in turn implies that the natural morphism

$$H_{\bar{\partial}}^{1,0}(\mathcal{Z}, \mathbb{C}) \rightarrow H_A^{1,0}(\mathcal{Z}, \mathbb{C})$$

is surjective. Here $H_{\bar{\partial}}^{\bullet,\bullet}$ denotes the usual Dolbeault cohomology. From [11] we know that $H_{\bar{\partial}}^{1,0}(\mathcal{Z}, \mathbb{C}) = 0$, hence $H_A^{1,0}(\mathcal{Z}, \mathbb{C}) = H_A^{0,1}(\mathcal{Z}, \mathbb{C}) = 0$. On the other hand, the morphism

$$H_{\bar{\partial}}^{0,1}(\mathcal{Z}, \mathbb{C}) \rightarrow H_A^{0,1}(\mathcal{Z}, \mathbb{C})$$

is always injective, therefore $H_{\bar{\partial}}^{0,1}(\mathcal{Z}, \mathbb{C}) = 0$. But, from Corollary 3.2 in [11], it follows that

$$\dim H_{\bar{\partial}}^{0,1}(\mathcal{Z}, \mathbb{C}) = \dim H_{dR}^1(M, \mathbb{C})$$

where H_{dR}^{\bullet} denotes the de Rham cohomology.

Summing up, on a twistor space on which $\mathcal{B}_X = \mathcal{G}_X$ one has $H_{dR}^1(M, \mathbb{C}) = 0$. Therefore on the twistor spaces X over the anti-self-dual 4-folds M with $H_{dR}^1(M, \mathbb{C}) \neq 0$ the balanced cone cannot be equal to the Gauduchon cone. The existence of such anti-self-dual manifolds is ensured by the aforementioned theorem of Taubes [24]. \square

3.2. The balanced and Gauduchon cones on Moishezon manifolds. The result of the previous section indicates that a generalization of Conjecture 2.3 in [5] to balanced manifolds fails. For projective manifolds, the recent work of Witt Nyström [27] implies that $\mathcal{B} = \mathcal{G}$. We extend next Witt Nyström's result to Moishezon manifolds.

Proposition 3.2. *If $\pi : X \rightarrow Y$ is a blow-up with smooth center in Y and if $\iota_{n-1}(\mathcal{B}_X) = \mathcal{G}_X$, then $\iota_{n-1}(\mathcal{B}_Y) = \mathcal{G}_Y$.*

Proof. Since the Gauduchon cone on X is never empty, it follows that the balanced cone on X is non-empty, hence X is balanced. Therefore Y is balanced [1]. Consequently, $\mathcal{E}_{A,X}^1$ and $\mathcal{E}_{A,Y}^1$ are closed and the equality $\iota_{n-1}(\mathcal{B}_X) = \mathcal{G}_X$ is equivalent to $\mathcal{E}_{A,X}^1 = \mathcal{E}_{BC,X}^1$ ([8] Theorem 2.4). So let T be a positive current, $i\partial\bar{\partial}T = 0$ on Y . Let π^*T be its total transform on X as defined in [2]. Since $\mathcal{E}_{A,X}^1 = \mathcal{E}_{BC,X}^1$, the class $\{\pi^*T\} \in H_A^{1,1}(X, \mathbb{R})$ contains a d -closed positive $(1,1)$ -current R . Therefore the class of $T = \pi_*\pi^*T$ contains π_*R , a d -closed positive current. \square

As a consequence of Proposition 3.2, we have:

Proof of Theorem 1.4. If Y is projective, from [27], as in the proof of Proposition 2.10 in [8] we have $\mathcal{B}_Y = \mathcal{G}_Y$. In general, a Moishezon manifold can be made projective by a sequence of blow-ups with smooth centers. For each blow-up in the sequence we can apply Proposition 3.2 and the conclusion follows. \square

Remark 3.3. An interesting question is whether the condition $\iota_{n-1}(\mathcal{B}) = \mathcal{G}$ is a bimeromorphic invariant, i.e., given X and Y two bimeromorphic compact complex manifolds, is it true that $\iota_{n-1}(\mathcal{B}_X) = \mathcal{G}_X$ if and only if $\iota_{n-1}(\mathcal{B}_Y) = \mathcal{G}_Y$?

We conclude with the following observation which serves as an introduction to the next section.

Proposition 3.3. *Let X be a compact complex surface. Then $\iota_1(\mathcal{B}_X) = \mathcal{G}_X$ if and only if X is Kähler.*

Proof. If $\iota_1(\mathcal{B}_X) = \mathcal{G}_X$, since $\mathcal{G}_X \neq \emptyset$ then $\mathcal{B}_X \neq \emptyset$, therefore X is balanced. Therefore X is Kähler since a balanced metric on a surface is Kähler. Conversely, if X is Kähler, from Proposition 2.7 in [8] it follows that $\mathcal{B}_X = \mathcal{G}_X$. \square

4. $\mathcal{B} = \mathcal{G}$ ON SKT THREEFOLDS

According to Popovici's Conjecture 1.5, every complex manifold on which the $\partial\bar{\partial}$ -lemma holds, has the property $\mathcal{B} = \mathcal{G}$. In particular, since the Gauduchon cone \mathcal{G} is open and non-empty, the manifold is balanced. While Conjecture 1.5 is still open, there are known examples on which $\mathcal{B} = \mathcal{G}$, and it's natural to consider the class of manifolds which satisfies that condition $\iota_{n-1}(\mathcal{B}) = \mathcal{G}$. We address here Conjecture 1.6 of Fino and Vezzoni on this class of balanced manifolds.

Proof of Theorem 1.7. Let η be a SKT metric on X .

Step 1. By Lemma 3.1, ι_1 is an isomorphism. Hence, there exists a $(1, 0)$ form α on X such that $\gamma = \bar{\partial}\alpha + \eta + \partial\bar{\alpha}$ is a d -closed $(1, 1)$ form. From $\iota_1(\mathcal{E}_{BC}^1) = \mathcal{E}_A^1$ it follows that the class of γ in $H_{BC}^{1,1}(X, \mathbb{R})$ is in \mathcal{E}_{BC}^1 . This means that there exists a d -closed positive $(1, 1)$ -current T such that $[\gamma] = [T]$ in $H_{BC}^{1,1}(X, \mathbb{R})$.

Step 2. We will show that $[\gamma]$ is also in \mathcal{N}_{BC}^1 , i.e., that it is nef. If the irreducible components of $\cup_{c>0} E_c(T)$ are all smooth, then we can use Théorème 2 in [21]. We have already checked that $[\gamma]$ is pseudoeffective, and let Z be an irreducible analytic subset of $\cup_{c>0} E_c(T)$. If Z is a curve, then $\int_Z \gamma = \int_Z \eta \geq 0$, hence $[\gamma]$ is nef on Z . If Z is a surface, then (see Lemma 2.1 in [6])

$$\int_Z \gamma \wedge \gamma = \int_Z \eta \wedge \eta + 2 \int_Z \partial\alpha \wedge \bar{\partial}\alpha > 0 \quad (4.1)$$

and it is well known that in this case Z is a Kähler surface. Let ω be a d -closed positive $(1, 1)$ form on Z . Then clearly from Stokes' theorem we have $\int_Z \omega \wedge \gamma = \int_Z \omega \wedge \eta > 0$ and this implies (see Theorem 4.5 (iii) in [10]) that $[\gamma]$ is nef on Z . This implies that $[\gamma]$ is nef on X .

In general, fix g a Hermitian metric on X and let $\varepsilon > 0$ be arbitrary. Then, from Theorem 3.2 in [10] it follows that there exists a closed current in the same class as γ , denoted $T_\varepsilon = \gamma + i\partial\bar{\partial}\varphi_\varepsilon \geq -\varepsilon g$ and $\pi_\varepsilon : X^\varepsilon \rightarrow X$ a sequence of blow-ups with smooth centers such that

$$\pi_\varepsilon^* T_\varepsilon = \sum_i \lambda_i [D_i] + \omega_\varepsilon \quad (4.2)$$

where D_i are smooth surfaces in X^ε , $\lambda_i > 0$ and ω_ε is a smooth d -closed $(1, 1)$ -form on X^ε .

Suppose

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{N-1}} X_{N-1} \xleftarrow{\pi_N} X_N = X^\varepsilon$$

is the sequence of blow-ups $\pi_\varepsilon : X^\varepsilon \rightarrow X$ and denote by C_j the center of the blow-up $\pi_{j+1} : X_{j+1} \rightarrow X_j$ and by E_j the exceptional divisor of the blow-up $\pi_j : X_j \rightarrow X_{j-1}$.

Now we construct by induction Hermitian metrics g_j on X_j as follows: set $g_0 = g$ on $X_0 = X$ and suppose that g_j has been constructed on X_j . It is well-known that one can put a metric on the line bundle $\mathcal{O}(-[E_{j+1}])$ on X_{j+1} such that its curvature, denoted β_{j+1} , is supported in a small neighborhood of E_{j+1} , that β_{j+1} is positive in a smaller neighborhood of E_{j+1} , and that, for c_{j+1} a small enough non-negative constant, $g_{j+1} = \pi_{j+1}^* g_j + c_{j+1} \beta_{j+1}$ is a Hermitian metric on X_{j+1} . We choose c_{j+1} such that

$$g_{j+1} \geq e^{-\frac{1}{(j+1)^2}} \pi_{j+1}^* g_j \quad (4.3)$$

From (4.3) and from $T_\varepsilon \geq -\varepsilon g$ it follows that

$$T_N \geq -\varepsilon e^{\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{(N-1)^2}} g_N \geq -e^{\frac{\pi^2}{6}} \varepsilon g_N$$

On $X^\varepsilon = X_N$ we consider the current $T_N = \pi_\varepsilon^* T_\varepsilon$ which is in the same class as $\gamma_\varepsilon = \pi_\varepsilon^* \gamma$. Each of the divisors D_i are either proper transforms of divisors on X or else components of the exceptional divisor of $\pi_\varepsilon : X^\varepsilon \rightarrow X$. In the first case, as above, we have

$$\int_{D_i} \gamma_\varepsilon \wedge \gamma_\varepsilon = \int_Z \eta_\varepsilon \wedge \eta_\varepsilon + 2 \int_Z \partial\alpha_\varepsilon \wedge \bar{\partial}\alpha_\varepsilon > 0$$

where $\eta_\varepsilon = \pi_\varepsilon^* \eta$ and $\alpha_\varepsilon = \pi_\varepsilon^* \alpha$, hence D_i is a Kähler surface with a Kähler form ω_i , and from

$$\int_C \gamma_\varepsilon \geq 0, \int_{D_i} \gamma_\varepsilon \wedge \omega_i \geq 0$$

where C is some curve in D_i , it follows that $[\gamma_\varepsilon]$ is nef on D_i . In the second case, D_i is the projectivization of a rank 2 vector bundle on a curve, hence it is Kähler, and again, from the inequalities

$$\int_C \gamma_\varepsilon = \int_C \eta_\varepsilon \geq 0, \int_{D_i} \gamma_\varepsilon \wedge \omega_i = \int_{D_i} \eta_\varepsilon \wedge \omega_i \geq 0$$

it follows that $[\gamma_\varepsilon]$ is nef on D_i .

Therefore, the current $T_N = \gamma_\varepsilon + i\partial\bar{\partial}\pi_\varepsilon^*\varphi_\varepsilon$ satisfies $T_N \geq -e^{\frac{\pi^2}{6}}\varepsilon g_N$ and its Bott-Chern class $[\gamma_\varepsilon]$ is nef on each D_i . By using the techniques in [21], we can find a \mathcal{C}^∞ function ψ_N on X_ε such that

$$\gamma_\varepsilon + i\partial\bar{\partial}\psi_N \geq -2e^{\frac{\pi^2}{6}}\varepsilon g_N$$

Namely, from Lemme 1 in [21], we first find a smooth function f_ε in a neighborhood of $\cup_i D_i$ such that $\gamma_\varepsilon + i\partial\bar{\partial}f_\varepsilon \geq -e^{\frac{\pi^2}{6}}\varepsilon g_N$ and then take for ψ_N a regularization of the maximum between $f_\varepsilon - C$ and $\pi_\varepsilon^*\varphi_\varepsilon$, where C is some large constant. Note that $\pi_\varepsilon^*\varphi_\varepsilon$ takes the value $-\infty$ on $\cup_i D_i$.

Now we construct by induction \mathcal{C}^∞ functions $\psi_{\varepsilon,j}$ on X_j such that

$$\gamma_j + i\partial\bar{\partial}\psi_{\varepsilon,j} \geq -2e^{\frac{\pi^2}{6}} \left(\frac{1}{2^0} + \frac{1}{2^1} + \dots + \frac{1}{2^{N-j}} \right) \varepsilon g_j \quad (4.4)$$

where γ_j is the pull-back of γ to X_j .

Set $\psi_{\varepsilon,N} = \psi_N$ and suppose that $\psi_{\varepsilon,j+1}$ has been constructed on X_{j+1} such that

$$\gamma_{j+1} + i\partial\bar{\partial}\psi_{\varepsilon,j+1} \geq -2e^{\frac{\pi^2}{6}} \left(\frac{1}{2^0} + \frac{1}{2^1} + \dots + \frac{1}{2^{N-j-1}} \right) \varepsilon g_{j+1} \quad (4.5)$$

If C_j is the center of the blow-up $\pi_{j+1} : X_{j+1} \rightarrow X_j$, then $\gamma_j|_{C_j}$ is nef since C_j is a curve or a point and if C_j is a curve, $\int_{C_j} \gamma_j \geq 0$, so from Lemme 1 in [21], there exists U_j a neighborhood of C_j and λ_j a \mathcal{C}^∞ function on U_j such that

$$\gamma_j + i\partial\bar{\partial}\lambda_j \geq -e^{\frac{\pi^2}{6}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{N-j-1}} + \frac{1}{2^{N-j+1}} \right) \varepsilon g_j$$

Pushing forward (4.5) to X_j we obtain

$$\gamma_j + i\partial\bar{\partial}\pi_{j+1,*}\psi_{\varepsilon,j+1} \geq -2e^{\frac{\pi^2}{6}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{N-j-1}} \right) \varepsilon (g_j + c_{j+1}\pi_{j+1,*}\beta_{j+1})$$

It is well-known that $\pi_{j+1,*}\beta_{j+1}$ is $i\partial\bar{\partial}$ exact (since β_{j+1} is in the same class as $-[E_{j+1}]$ and the push forward of $[E_{j+1}]$ is 0), so let $\pi_{j+1,*}\beta_{j+1} = i\partial\bar{\partial}\mu_j$. Therefore

$$\begin{aligned} \gamma_j + i\partial\bar{\partial} \left[\pi_{j+1,*}\psi_{\varepsilon,j+1} + 2e^{\frac{\pi^2}{6}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{N-j-1}} \right) \varepsilon c_{j+1}\mu_j \right] \geq \\ -2e^{\frac{\pi^2}{6}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{N-j-1}} \right) \varepsilon g_j \end{aligned}$$

and, as in [21], using the function λ_j constructed above, and a function ζ as in Lemme 2 in [21], we obtain a \mathcal{C}^∞ function $\psi_{\varepsilon,j}$ on X_j which satisfies (4.4).

For $j = 0$ we obtain a \mathcal{C}^∞ function $\psi_{\varepsilon,0}$ such that

$$\gamma + i\partial\bar{\partial}\psi_{\varepsilon,0} \geq -4e^{\frac{\pi^2}{6}}\varepsilon g$$

This means that $[\gamma]$ is nef.

Step 3. From Lemma 2.1 in [6] with $k = 3$ we obtain

$$\int_X \gamma^3 = \int_X \eta^3 + 6 \int_X \eta \wedge \partial\alpha \wedge \bar{\partial}\bar{\alpha} > 0$$

and we can use Theorem 4.1 in [7]. Indeed, $[\gamma]$ is a nef class, of positive self-intersection, and X is a 3-fold that supports a SKT metric (see Remark 4.3 in [7]). This implies that X is Kähler. \square

Remark 4.1. Given Z a singular component of $\cup_{c>0} E_c T$, it is clear that, if $p : \tilde{X} \rightarrow X$ is a resolution of singularities of Z , then \tilde{Z} (the proper transform of Z) is Kähler and that $p^*\gamma$ is nef on \tilde{Z} . This implies that, for every $\varepsilon > 0$, there exists $\tilde{\varphi}_\varepsilon$ a \mathcal{C}^∞ function on \tilde{Z} such that $p^*\gamma + i\partial\bar{\partial}\tilde{\varphi}_\varepsilon \geq -\varepsilon\tilde{g}$ on \tilde{Z} . However, it is not obvious to us that from this data one can construct a \mathcal{C}^∞ function φ_ε on Z which satisfies $\gamma + i\partial\bar{\partial}\varphi_\varepsilon \geq -\varepsilon g$ (so that it is as in Définition 3 in [21]) since such a function has to be locally the restriction of a \mathcal{C}^∞ function on a neighborhood of Z . This is the reason for looking at the approximation current T_ε in the proof of Theorem 1.7 above instead of working directly with the closed positive current T .

Remark 4.2. In [26] Verbitsky showed that a twistor space which supports a SKT metric is Kähler. Our result does not imply Verbitsky's result. On a twistor space, it is not always true that the balanced cone equals the Gauduchon cone, as noticed in the proof of Theorem 1.3.

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