BALANCED METRICS ON UNIRULED MANIFOLDS

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ABSTRACT. We show that an $n$–dimensional Moishezon manifold is uniruled if and only if it supports a balanced metric $\omega$ of positive total scalar Chern curvature. A similar statement also holds true for class $\mathcal{C}$ manifolds of dimension three.

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INTRODUCTION

A compact complex manifold $M$ is called uniruled if there exists a rational curve passing through every point of $M$. A differential geometric characterization of uniruledness in complex dimension two was given by Yau [Ya]. He proved that a Kähler surface $S$ has Kodaira dimension $-\infty$ (equivalently, uniruled) if and only if it admits a Kähler metric $\omega$ of positive total scalar curvature. This is equivalent to

$$\int_S c_1(K_S) \wedge \omega < 0,$$

where $K_S$ denotes the canonical line bundle of $S$. The aim of this article is to extend Yau's differential geometric characterization in higher dimensions.

Yau’s approach to find Kähler metrics of positive total scalar curvature on uniruled surfaces relies on the minimal model theory. His proof follows in two steps:

A) Bimeromorphic invariance: Yau shows that the existence of such metrics is an invariant property under bimeromorphic maps. In the case of surfaces, the invariance under blow-ups suffices.

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B) Existence of a Kähler metric of positive total scalar Chern curvature on an exhaustive list of bimeromorphism classes of uniruled surfaces: Yau proved the existence of Kähler metrics satisfying (0.1) on all geometrically ruled surfaces. Our approach follows closely Yau’s ideas.

To extend Step A in higher dimensions, recall that any bimeromorphic map decomposes by the weak factorization theorem [AKMW, Theorem 0.3.1] into a sequence of blow-ups and blow-downs with smooth centers. A well-known fact is that the class of Kähler manifolds of dimension greater than three is not closed under bimeromorphisms. We are led to consider a larger class of manifolds which is invariant under bimeromorphisms. From the work of Alessandrini and Bassanelli [AB1, AB2], it is known that the class of manifolds carrying balanced metrics, i.e., Hermitian metrics with co-closed Kähler form (see [Mi] and Section 1), satisfies such a property. In dimension two, a balanced metric is in fact Kähler, but in higher dimensions there exist non-Kähler manifolds which admit balanced metrics or Kähler manifolds which admit non-Kähler balanced metrics. We prove:

**Theorem A.** Let $X$ and $Y$ be two bimeromorphic compact complex manifolds of dimension $n$. If there exists a balanced metric $\omega_{X}^{n-1}$ on $X$ such that
\[ \int_{X} c_1(K_X) \wedge \omega_{X}^{n-1} < 0, \]
then there exists $\omega_{Y}^{n-1}$ a balanced metric on $Y$ such that
\[ \int_{Y} c_1(K_Y) \wedge \omega_{Y}^{n-1} < 0. \]

An extension of Step B to uniruled manifolds of higher dimensions relies on the state of the art of the minimal model program. For projective uniruled manifolds one can find a bimeromorphic simpler model in any dimension [BCHM]. These bimeromorphic models are higher dimensional analogs of the geometrically ruled surfaces, called Mori fiber spaces (see Section 3.2). We obtain:

**Theorem B.** Every $n-$dimensional, Moishezon, uniruled manifold $X$ admits a balanced metric $\omega_{X}^{n-1}$ such that
\[ \int_{X} c_1(K_X) \wedge \omega_{X}^{n-1} < 0. \]

Recall that a compact complex manifold is Moishezon if it is bimeromorphic to a projective manifold.

We provide two proofs for this result. One proof uses the minimal models program. A second proof of this result is based on ideas of Toma [To], and it relies on the results of Boucksom, Demailly, Păun and Peternell [BDPP], avoiding the minimal models program.

In a recent work, Heier and Wong discuss the existence of a Kähler metric of positive total scalar curvature on projective uniruled manifolds [HW, Section 5], but a definite conclusion is elusive. Such metrics are known to exist on some uniruled manifolds. Most notably, they exist on projective Mori fiber spaces of dimension three, as established by Demailly, Peternell and Schneider [DPS]. An approach to this existence question is proposed by the second author in the case of rationally connected threefolds [Ră].
A generalization of the minimal model program to the class of Kähler manifolds is known only in complex dimension three [HP1, HP2]. We prove the following extension of Theorem B in dimension three:

**Theorem C.** Every uniruled threefold $X$ of class $\mathcal{C}$ admits a balanced metric $\omega^2$ such that

$$\int_X c_1(K_X) \wedge \omega^2 < 0.$$ 

Recall that a complex manifold is called of class $\mathcal{C}$ if it is bimeromorphic to a Kähler manifold. This class of manifolds is strictly larger than the class of Kähler manifolds in dimension three or more, and it contains the class of Moishezon manifolds. Every class $\mathcal{C}$ manifold carries balanced metrics by [AB1, AB2].

Conversely, Heier and Wong [HW, Theorem 1.1] show that every projective manifold which admits a Kähler metric of positive total scalar curvature is uniruled. We strengthen their result and extend it to Moishezon manifolds:

**Theorem D.** Every smooth, Moishezon manifold carrying a balanced metric of positive total scalar Chern curvature is uniruled.

The proof relies on the work of Boucksom, Demailly, Păun and Peternell [BDPP].

In dimension three, we extend Theorem D to manifolds of class $\mathcal{C}$. We rely on a remarkable result of Brunella [Br], which is restricted to dimension three.

**Theorem E.** Every three dimensional smooth manifold of class $\mathcal{C}$ carrying a balanced metric of positive total scalar Chern curvature is uniruled.

We explore next the possibility of extending the above characterization of uniruledness in terms of the positivity of the total scalar Chern curvature of a balanced metric beyond class $\mathcal{C}$. In general, the existence of a balanced metric fails. However, in dimension three, a large class of uniruled manifolds admitting such metrics is given by complex manifolds bimeromorphic to twistor spaces [AHS]. We prove:

**Theorem F.** Every complex manifold $X$ bimeromorphic to a twistor space admits a balanced metric $\omega^2$ such that

$$\int_X c_1(K_X) \wedge \omega^2 < 0.$$ 

1. **Total scalar curvatures**

In this section we briefly recall some well-known background material in complex differential geometry to introduce the terminology.

Let $(M, g)$ be a Hermitian manifold and $\omega$ its Kähler form. On $(M, g)$ one can consider two canonical connections: the Levi-Civita connection, and the Chern connection.

Let $s$ denote the scalar curvature of the Levi-Civita connection. The *total scalar Riemannian curvature* is defined as

$$\int_M s_{\mu_g} = \int_M \frac{s\omega^n}{n!},$$

where $\mu_g = \frac{\omega^n}{n!}$ is the volume form.
Let $s_C$ denote the scalar curvature of the Chern connection associated to the Hermitian metric $g$. The total scalar Chern curvature is defined by

$$\int_M s_C \mu_g.$$  

The Ricci curvature form of the Chern connection represents the first Chern class of $M$ rescaled by a factor of $2\pi$, and $c_1(M) = -c_1(K_M)$, where $K_M$ is the canonical line bundle of $M$. Since the scalar curvature is the trace of the Ricci curvature form, we can write

$$\int_M s_C \mu_g = \int_M \frac{s_C \omega^n}{n!} = -2\pi n \int_M c_1(K_M) \wedge \omega^{n-1}. \quad (1.1)$$

A result due to Gauduchon [Ga2, page 506] (see also [LY, Corollary 1.11]) compares the total scalar Riemannian curvature and the total scalar Chern curvature:

**Proposition 1.1.** Let $(M, g)$ be a compact, complex manifold equipped with a Hermitian metric. Then

$$\int_M s_C \mu_g \geq \frac{1}{2} \int_M s \mu_g,$$

with equality if and only if the metric is Kähler.

**Corollary 1.2.** Let $(M, g)$ be a compact, complex manifold of dimension $n$ equipped with a Hermitian metric. If the scalar Riemannian curvature of $M$ is positive, then

$$\int_M c_1(K_M) \wedge \omega^{n-1} < 0.$$  

□

**Definition 1.1.** Let $(M, g)$ be a compact complex manifold of complex dimension $n$ equipped with a Hermitian metric $g$, and let $\omega$ denote its Kähler form. If $d\omega = 0$, then $g$ is called a Kähler metric. A complex manifold which admits a Kähler metric is called a Kähler manifold.

If $g$ is a Kähler metric, then its Kähler form $\omega$ is a real, $d$-closed, strictly positive $(1,1)$-form. Conversely, given a strictly positive, $d$-closed $(1,1)$-form $\omega$, there exists a Hermitian metric $g$ whose Kähler form is $\omega$. We will use the notation $(M, \omega)$ to denote a Kähler manifold with prescribed Kähler form.

**Definition 1.2.** Let $(M, g)$ be a compact complex manifold of complex dimension $n$ equipped with a Hermitian metric $g$, and let $\omega$ denote its Kähler form. If $d(\omega^{n-1}) = 0$, then $g$ is called a balanced metric. A complex manifold which admits a balanced metric is called a balanced manifold. We will use the notation $(M, \omega^{n-1})$ to denote a balanced manifold.

Given a balanced metric of Kähler form $\omega$, the $(n-1, n-1)$-form $\omega^{n-1}$ is real, strictly positive and $d$-closed. Conversely, it is an easy exercise in linear algebra to see that given a real, strictly positive, $d$-closed $(n-1, n-1)$-form $\Omega$, there exists a unique Hermitian metric of Kähler form $\omega$ such that $\Omega = \omega^{n-1}$ ([Mi, page 279]). Throughout the paper, by a balanced metric we mean a real, $d$-closed, strictly positive $(n-1, n-1)$-form, denoted by $\omega^{n-1}$.

A Kähler manifold is balanced, and if $n = 2$ the converse is also true. In higher dimensions the converse is not necessarily true. A large class of examples is provided the twistor spaces of closed anti-self-dual four-manifolds. These are
complex manifolds of dimension three \cite{AHS}, equipped with a one-parameter family of balanced metrics \cite{Mi, Mu, dBN}. However, the only Kählerian twistor space are those associated to $S^4$ and $\mathbb{CP}^2$, the complex projective plane, with the opposite orientation, \cite{Hi}. These balanced manifolds will be discussed in Section 4. We refer the interested reader to \cite{Mi, Section 6} for more examples. More recently, another interesting class of non-Kähler balanced has been found by Fu, Li and Yau. In \cite{FLY}, the authors constructed a complex structure on $\# k S^3 \times S^3, k \geq 2$ with trivial canonical bundle and carrying a balanced metric.

2. Positive cones in Bott-Chern and Aeppli cohomology groups

In this section we introduce the Bott-Chern and Aeppli cohomology groups and the pseudoeffective and the nef cones. In the Kähler case, these cohomology groups are isomorphic to the usual Dolbeault cohomology groups due to the $i \partial \bar{\partial}$-lemma. However, we prefer to work with the Bott-Chern and Aeppli cohomology groups since the class of a $d$- or $i \partial \bar{\partial}$-closed positive current lies naturally in these cohomology groups, and, moreover, the duality statements between the nef and pseudoeffective cones (Theorem 2.3) can be naturally stated in this setting. For more details, see \cite{Sc}.

Let $X$ be a compact complex manifold of dimension $n$. The Bott-Chern cohomology groups are defined as

$$H^{p,q}_{BC}(X, \mathbb{C}) = \{ \alpha \in \mathcal{C}^\infty_{p,q}(X) | d\alpha = 0 \} \cup \{ i \partial \bar{\partial} \beta | \beta \in \mathcal{C}^\infty_{p-1,q-1}(X) \},$$

and the Aeppli cohomology groups are

$$H^{p,q}_A(X, \mathbb{C}) = \{ \alpha \in \mathcal{C}^\infty_{p,q}(X) | i \partial \bar{\partial} \alpha = 0 \} \cup \{ \partial \beta + \bar{\partial} \gamma | \beta \in \mathcal{C}^\infty_{p-1,q}(X), \gamma \in \mathcal{C}^\infty_{p,q-1}(X) \}.$$

Since all the operators involved in the definitions of the above cohomology groups are real in bidegrees $(p,p)$, the real cohomology groups $H^{p,p}_{BC}(X, \mathbb{R})$ and $H^{n-p,n-q}_A(X, \mathbb{R})$ are well-defined. The above groups can be defined by using smooth forms or currents. We use the notation $[s]$ for the class of a $d$-closed form or current $s$ in $H^{p,p}_{BC}$ and $\{t\}$ for the class of a $i \partial \bar{\partial}$-closed form or current $t$ in $H^{n-p,n-q}_A$. The groups $H^{p,q}_{BC}(X, \mathbb{C})$ and $H^{n-p,n-q}_A(X, \mathbb{C})$ are dual via the pairing

$$H^{p,q}_{BC}(X, \mathbb{C}) \times H^{n-p,n-q}_A(X, \mathbb{C}) \to \mathbb{C}, ([\alpha], \{\beta\}) \to \int_X \alpha \wedge \beta \quad (2.1)$$

**Definition 2.1.** Let $T$ be a current of bi-dimension $(p,p)$. We say that $T$ is a positive current if $T \wedge i \alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i \alpha_p \wedge \bar{\alpha}_p$ is a positive measure, for all smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_p$.

For $\# = BC$ or $A$ and $p \in \{1, n-1\}$ we define the following cones:

1. the $\#-$pseudoeffective cone

$$\mathcal{E}^p_X,\# = \{ \gamma \in H^{p,p}_\#(X, \mathbb{R}) | \exists T \geq 0, T \in \gamma \}. \quad (2.2)$$

which is a convex cone. Here $T$ denotes a current.
(2) the $\# - \text{nef}$ cone
\[ \mathcal{A}_{X,\#}^p = \{ \gamma \in H^p_\#(X, \mathbb{R}) | \forall \varepsilon > 0, \exists \alpha_\varepsilon \in \gamma, \alpha_\varepsilon \geq -\varepsilon \omega^p \} \]  \hspace{1cm} (2.3)
where $\omega$ is the Kähler form of a fixed Hermitian metric on $X$ and $\alpha_\varepsilon$ denotes a smooth $(p, p)$-form. It is a convex cone.

Lemma 2.1. The cone $\mathcal{E}_{X,BC}^1$ is closed and $\mathcal{A}_{X,BC}^1 \subset \mathcal{E}_{X,BC}^1$.

Proof. This follows from the existence of Gauduchon metrics on any compact complex manifold [Ga1]. That means $X$ admits a Hermitian metric $g$ with Kähler form $\omega$ satisfying $\partial \overline{\partial} \omega^{n-1} = 0$. Indeed, suppose $([T_j])_j$ is a sequence of pseudo-effective classes represented by the closed positive currents $T_j$ such that $[T_j] \to \gamma \in H^{1,1}(X, \mathbb{R})$. Fix $g$ a Gauduchon metric with Kähler form $\omega$, and notice that $\int_X T_j \wedge \omega^{n-1}$ depends only on $g$ and the cohomology class $[T_j]$. Since the sequence $(\int_X T_j \wedge \omega^{n-1})_j$ is bounded, we can assume that the sequence $(T_j)_j$ is weakly convergent to a closed positive current $T$. Then $\gamma = [T] \in \mathcal{E}_{X,BC}^1$.

For the inclusion $\mathcal{A}_{X,BC}^1 \subset \mathcal{E}_{X,BC}^1$, let $[\alpha] \in \mathcal{A}_{X,BC}^1$ where $\alpha$ is a $d$ closed smooth $(1, 1)$ form. Then, by definition, for every $\varepsilon > 0$, there exists $\varphi_\varepsilon \in \mathcal{E}^\infty(X, \mathbb{R})$ such that $\alpha_\varepsilon := \varepsilon \omega + \alpha + i\partial \overline{\partial} \varphi_\varepsilon \geq 0$. Since $\int_X \alpha_\varepsilon \wedge \omega^{n-1}$ is bounded for $0 < \varepsilon \leq 1$, we extract a weakly convergent subsequence $\alpha_\varepsilon \to T$, where $T$ is a closed, positive current in class $[\alpha]$. Hence $[\alpha] \in \mathcal{E}_{X,BC}^1$. \hfill \Box

Lemma 2.2. The cone $\mathcal{A}_{X,\#}^p$ is closed.

Proof. Suppose we have a sequence $\gamma_j \in \mathcal{A}_{X,\#}^p$ such that $\gamma_j \to \gamma$. In each cohomology class $\gamma_j$ choose the unique harmonic representative $\beta_j$ and $\beta$ the unique harmonic representative in $\gamma$ (see [Sc] for more on the harmonic forms in Bott-Chern and Aeppli cohomology groups). Then, from the standard theory of elliptic operators, it follows that $\beta_j \to \beta$ in the uniform convergence. Now it is clear that $\gamma$ is nef. \hfill \Box

Given $V$ a real vector space, denote by $V^*$ its dual. If $C$ a convex cone in $V$, we denote by $C^* \subset V^*$ its dual:
\[ C^* = \{ v^* \in V^* | v^*(c) \geq 0, \forall c \in C \} \]
By the Hahn-Banach Theorem, we have $C^{**} = \overline{C}$.

Theorem 2.3. Let $X$ be a compact complex manifold of dimension $n$. Then

i) $\mathcal{A}_{X,BC}^1 = (\mathcal{E}_{X,BC}^{n-1})^*$

ii) $\mathcal{A}_{X,A}^n = (\mathcal{E}_{X,BC}^1)^*$

Moreover, if $X$ is balanced, then

iii) $\mathcal{A}_{X,A}^1 = (\mathcal{E}_{X,BC}^{n-1})^*$

iv) $\mathcal{A}_{X,BC}^{n-1} = (\mathcal{E}_{X,BC}^1)^*$

Proof. i) This is Théorème 1.2 (1) in [La1].

ii) Clearly $\mathcal{A}_{X,A}^n \subset (\mathcal{E}_{X,BC}^1)^*$. Conversely, $(\mathcal{E}_{X,BC}^1)^* \subset \mathcal{A}_{X,A}^{n-1}$ is equivalent to $(\mathcal{A}_{X,A}^{n-1})^* \subset \mathcal{E}_{X,BC}^1$ since $\mathcal{E}_{X,BC}^1$ is closed. But this is Lemme 1.4 in [La1].

iii) If $X$ is balanced, then the equality follows as in the proof of Lemme 1.3 in [La1] with only minor changes. Instead of a Gauduchon metric, we use a balanced metric.
iv) If $X$ is balanced, as in the proof of Lemma 2.1 we can see that $\mathcal{E}^{n-1}_{X,A}$ is closed. The assertion is equivalent to $(\mathcal{N}^{n-1}_{X,BC})^* \subset \mathcal{E}^{n-1}_{X,A}$ and this follows as in the proof of Lemme 1.4 in [La1]. □

Behind the above duality statements is the Hahn-Banach separation theorem.

Remark 2.2. For Kähler manifolds, Boucksom, Demailly, Păun and Peternell define the pseudoeffective cone [BDPP, Definition 1.1] as a subset of $H^{1,1}_{\mathbb{R}}(X) := H^{1,1}(X) \cap H^{2}_{dR}(X, \mathbb{R})$:

$$\mathcal{E}^{1}_{X,dR} = \{ \gamma \in H^{1,1}_{dR}(X, \mathbb{R}) | \exists T \geq 0, T \in \gamma \}.$$

From the $\partial \bar{\partial}$–lemma we can see that the canonical map

$$J : H^{1,1}_{BC}(X, \mathbb{R}) \to H^{1,1}_{dR}(X)$$

is an isomorphism, and that the $BC$–pseudoeffective cone defined in (2.2) coincides with $\mathcal{E}^{1}_{X,dR}$, via the map $J$. The similar observation can be made for the $BC$–nef cone.

Let

$$\mathcal{K}X = \{ [\omega] \in H^{1,1}_{BC}(X, \mathbb{R}) | \omega \text{ is a Kähler metric} \}$$

denote the Kähler cone of $X$. Similarly, we define the balanced cone:

$$\mathcal{B}X = \{ [\omega] \in H^{n-1,n-1}_{BC}(X, \mathbb{R}) | \omega^{n-1} \text{ is a balanced metric} \}.$$

Remark 2.3. If $X$ is a Kähler manifold, we have a natural map $\varpi : \mathcal{K} \to \mathcal{B}$ given by $\varpi([\omega]) = [\omega^{n-1}]$. Fu and Xiao [FX] showed that the map $p$ is injective [FX, Proposition 1.1]. Moreover, $p$ is not always surjective. More precisely, they provided examples of manifolds [FX, pages 11 and 12] where $\mathcal{B}X \setminus \varpi(\mathcal{K}X) \neq \emptyset$.

If $X$ is Kähler, then $\mathcal{N}^{1}_{X,BC} = \mathcal{K}X$. Also, as in the proof of Lemma 2.1, $\mathcal{E}^{n-1}_{X,A}$ is closed and we have, $\mathcal{N}^{n-1}_{X,A} \subset \mathcal{E}^{n-1}_{X,A}$.

If $X$ is balanced, then $\mathcal{N}^{n-1}_{X,BC} = \mathcal{B}X$. Similarly, $\mathcal{E}^{1}_{X,A}$ is closed and we have $\mathcal{N}^{1}_{X,BC} \subset \mathcal{E}^{1}_{X,A}$.

We have natural morphisms

$$j_1 : H^{1,1}_{BC}(X, \mathbb{R}) \to H^{1,1}_{A}(X, \mathbb{R})$$

and

$$j_{n-1} : H^{n-1,n-1}_{BC}(X, \mathbb{R}) \to H^{n-1,n-1}_{A}(X, \mathbb{R})$$

which are isomorphisms if $X$ is Kähler, due to the $\partial \bar{\partial}$–lemma.

Proposition 2.4. Let $X$ be a compact Kähler manifold of dimension $n$. Then

$$j_{n-1}(\mathcal{E}^{n-1}_{X,BC}) = \mathcal{E}^{n-1}_{X,A} \quad (2.4)$$

and

$$j_1(\mathcal{N}^{1}_{X,BC}) = \mathcal{N}^{1}_{X,A} \quad (2.5)$$
Proof. The second statement follows from the first one by duality. From Theorem 2.3, we have that \((\mathcal{N}_X^{n-1})^* = \mathcal{E}_{X,A}^{n-1}\) since \(\mathcal{E}_{X,A}^{n-1}\) is closed. Corollary 0.3 in [DP] implies that the currents of the form \(j_{n-1}(\int_Y \omega^{p-1} \wedge \bullet)\), where \(Y\) is a \(p\)-dimensional analytic subset of \(X\) and \(\omega\) is a Kähler metric on \(X\), generate the cone \(\mathcal{E}_{X,A}^{n-1}\). Since the currents \((\int_Y \omega^{p-1} \wedge \bullet)\) are \(d\)-closed and positive, we see that \(j_{n-1}(\mathcal{E}_{X,B}^{n-1}) = \mathcal{E}_{X,A}^{n-1}\). □

Remark 2.4. Given a compact complex Kähler manifold of dimension \(n\), it is conjectured that \(j_{n-1}(\mathcal{N}_X^{n-1}) = \mathcal{N}_X^{n-1}\), i.e., that the dual of the pseudoeffective cone \(\mathcal{E}_{X,B}^{1}\) is the closure of the cone of the balanced metrics. This conjecture follows from Conjecture 2.3 in [BDPP], as it can be see from the proof of (2.8) below.

2.1. Néron-Severi groups. For a compact complex manifold \(X\) of dimension \(n\) we have natural maps

\[
\alpha_p : H^{p,p}_{BC}(X, \mathbb{R}) \to H^{2p}_{dR}(X, \mathbb{R})
\]

\[
\beta_p : H^{2p}_{dR}(X, \mathbb{R}) \to H^{p,p}_A(X, \mathbb{R})
\]

\[
\gamma_p : H^{2p}(X, \mathbb{Z}) \to H^{2p}_{dR}(X, \mathbb{R})
\]

Define the Néron-Severi groups

\[
H^{p,p}_{BC,NS}(X, \mathbb{R}) = \alpha_p^{-1}(\gamma_p(H^{2p}(X, \mathbb{Z}))) \otimes_{\mathbb{Z}} \mathbb{R} \subset H^{p,p}_{BC}(X, \mathbb{R})
\]

and

\[
H^{p,p}_{A,NS}(X, \mathbb{R}) = \beta_p(\gamma_p(H^{2p}(X, \mathbb{Z}))) \otimes_{\mathbb{Z}} \mathbb{R} \subset H^{p,p}_A(X, \mathbb{R})
\]

If \(X\) is projective, then the canonical morphisms

\[
H^{1,1}_{BC,NS}(X, \mathbb{R}) \to H^{1,1}_{A,NS}(X, \mathbb{R})
\]

and

\[
H^{n-1,n-1}_{BC,NS}(X, \mathbb{R}) \to H^{n-1,n-1}_{A,NS}(X, \mathbb{R})
\]

are isomorphisms, and the standard notations for these groups are \(N^1\) or \(NS^1_X\), and \(N_1\), respectively. The group \(NS^1_X\) is generated by classes of divisors on \(X\), and by the Hard Lefschetz Theorem, it follows that \(N_1\) is generated by classes of curves on \(X\).

Let the subscript \(NS\) denote the intersection of a cone (nef or pseudoeffective) with the Néron-Severi groups.

Proposition 2.5. If \(X\) is compact Kähler of dimension \(n\), then the pairing

\[
H^{p,p}_{BC,NS}(X, \mathbb{R}) \times H^{n-p,n-p}_{A,NS}(X, \mathbb{R}) \to \mathbb{R}, ([\alpha], [\beta]) \to \int_X \alpha \wedge \beta \quad (2.6)
\]

is nondegenerate and all the equalities of Theorem 2.3 hold at the Néron-Severi level. Moreover,

\[
j_{n-1}(\mathcal{E}_{BC,NS}^{n-1}) = \mathcal{E}_{A,NS}^{n-1}
\]

and

\[
j_1(\mathcal{N}_X^{1}) = \mathcal{N}_A^{1}
\]

If \(X\) is projective, then

\[
j_{n-1}(\mathcal{N}_X^{n-1}) = \mathcal{N}_A^{n-1}
\]

(2.7)
and 
\[ j_1(\mathcal{E}_{BC,NS}^1) = \mathcal{E}_A^{1,NS} \] (2.8)

Proof. The only non-trivial statement is (2.8), as (2.7) follows by duality.

Let \( \{T\} \in \mathcal{E}_A^{1,NS} \) where \( T \) is a positive, \( i\partial\bar{\partial} \)-closed current, and let \( j_1([S]) = \{T\} \), \( [S] \in H_{BC,NS}^{1,1}(X, \mathbb{R}) \). We want to show that \( [S] \in \mathcal{E}_{BC,NS}^1 \). For the proof, we follow [To].

From [BDPP, Theorem 2.2] we see that it is enough to check that 
\[ ([S], \{ p_*(A_1 \cap \ldots \cap A_{n-1}) \}) \geq 0 \]
where \( p : Y \to X \) is a proper modification of \( X \) and \( A_1, \ldots, A_{n-1} \) are very ample line bundles on \( Y \). From Theorem 3 in [AB2], there exists \( T' \), a positive pluriharmonic current on \( Y \) which is the total transform of \( T \). Then 
\[ ([S], \{ p_*(A_1 \cap \ldots \cap A_{n-1}) \}) = ([T], [p_*(A_1 \cap \ldots \cap A_{n-1})]) = (T', A_1 \cap \ldots \cap A_{n-1}) \geq 0, \]
and we are done. \( \square \)

3. Uniruled manifolds and balanced metrics

3.1. Bimeromorphism invariance. We prove here that the existence of a balanced metric of positive total scalar Chern curvature is an invariant property under bimeromorphisms.

Proof of Theorem A. By [AKMW], we can assume that \( p : Y \to X \) is a blow-up with smooth center \( C \) and let \( E \) be the exceptional divisor of \( p \). Then \( K_Y = p^*K_X + aE \), where \( a = \text{codim}_X C - 1 > 0 \).

Suppose first that \( X \) admits a balanced metric \( \omega_X^{n-1} \) which is negative on the canonical line bundle of \( X \). Then 
\[ \int_Y c_1(K_Y) \wedge p^*\omega_X^{n-1} = \int_Y c_1(p^*K_X) \wedge p^*\omega_X^{n-1} = \int_X c_1(K_X) \wedge \omega_X^{n-1} < 0. \]
It is known that \( Y \) is also balanced [AB3], and if \( \omega_Y^{n-1} \) is a balanced metric on \( Y \), then \( p^*\omega_X^{n-1} + \varepsilon\omega_Y^{n-1} \) is a balanced metric and 
\[ \int_Y c_1(K_Y) \wedge (p^*\omega_X^{n-1} + \varepsilon\omega_Y^{n-1}) < 0, \]
for a small \( \varepsilon > 0 \).

Conversely, suppose that \( Y \) supports a balanced metric \( \omega_Y^{n-1} \) such that 
\[ \int_Y c_1(K_Y) \wedge \omega_Y^{n-1} < 0. \] (3.1)

and suppose that 
\[ \int_X c_1(K_X) \wedge \omega_X^{n-1} \geq 0 \]
for any balanced metric \( \omega_X^{n-1} \) on \( X \). Then 
\[ \int_X c_1(K_X) \wedge \eta \geq 0 \]
for any class \([\eta] \in \mathcal{M}^{-1}_{BC,X}\). Therefore, by Theorem 2.3 iv), \(c_1(K_X) \in \mathcal{E}^1_{X,A}\), i.e., there exists \(T\) a positive \(i\partial\bar{\partial}\)-closed current in the Aeppli cohomology class \(\{c_1(K_X)\}\). From [AB2], it follows that there exists a positive \(i\partial\bar{\partial}\)-closed current on \(Y\) denoted by \(T'\), which is the total transform of \(T\). This means that

\[
T' \in \{c_1(p^*K_X)\} = p^*\{c_1(K_X)\}.
\]

In particular, \(c_1(p^*K_X) \in \mathcal{E}^1_{Y,A}\) and therefore

\[
\{c_1(K_Y)\} = \{c_1(p^*K_X)\} + a\{[E]\} \in \mathcal{E}^1_{Y,A}
\]

which contradicts (3.1). \(\square\)

3.2. Metrics on Mori fiber spaces. We start by recalling background definitions from the minimal model program.

**Definition 3.1.** A compact complex variety \(Y\) is called \(\mathbb{Q}\)-factorial if every Weil divisor of \(Y\) is \(\mathbb{Q}\)-Cartier.

Let \(Y\) be normal variety such that \(mK_Y\) is Cartier for some \(m > 0\), and let \(f: Z \to Y\) be a resolution of singularities. Up to numerical equivalence, we can write

\[
K_Z \equiv f^*(K_Y) + \sum a_i E_i,
\]

where the \(E_i\)'s are the \(f\)-exceptional divisors, and \(a_i \in \mathbb{Q}\).

**Definition 3.2.** We say that \(Y\) has log-terminal singularities if \(a_i > -1\), for all \(i\).

It is well-known that this definition is independent of the choice of the resolution [KM].

**Definition 3.3.** A normal compact complex variety \(Y\) with only \(\mathbb{Q}\)-factorial log-terminal singularities equipped with a map \(\phi: Y \to B\) is called a Mori fiber space if the following conditions are satisfied:

i) The map \(\phi\) is a morphism with connected fibers onto a normal variety \(B\) with \(\dim B < \dim Y\).

ii) All the curves \(C\) in the fibers of \(\phi\) are numerically proportional and \(K_X \cdot C < 0\).

3.2.1. *The projective case.* We give here a first proof of Theorem B based on the minimal models program. A second proof, avoiding the minimal model program follows.

**Proposition 3.1.** Let \(\phi: Y \to B\) be a Mori fiber space, with \(Y\) and \(B\) projective. Then, there exists an ample line bundle \(H\) on \(Y\) such that

\[
K_Y \cdot H^{n-1} < 0.
\]

**Proof.** If \(\dim B = 0\), by Kleiman’s ampleness criterion \(-K_Y\) is ample, and so \(K_Y \cdot H^{n-1} < 0\) for all ample line bundles on \(Y\).

Assume now that \(\dim B > 0\) and fix an ample line bundle \(L\) on \(B\), and \(H_0\) an ample line bundle on \(Y\). Let

\[
H_m = m\phi^*L + H_0.
\]
Then $H_m$ is an ample line bundle on $Y$ for all $m > 0$, and
\[
K_Y \cdot H_m^{-1} = K_Y \cdot (m\phi^*L + H_0)^{-1} \\
= c(n, b)m^bK_Y \cdot (\phi^*L)^b \cdot H_0^{n-1-b} + O(m^{b-1}) \\
= c(n, b)m^b(L^b)(K_F \cdot H_0^{n-1-b}) + O(m^{b-1}),
\]
where $c(n, b)$ is a positive integer depending only on $n$ and $b$, and $F$ denotes the fiber of $\phi$. Since $-K_F$ is ample, it follows that $K_Y \cdot H_m^{n-1} < 0$ for $m \gg 0$. Take now $H_Z = H_m$ for some fixed $m \gg 0$. 

\[\square\]

The first proof of Theorem B. Let $X$ be a smooth, Moishezon, uniruled manifold of dimension $n > 0$. Then there exists a smooth projective manifold $Y$ of dimension $n$ bimeromorphic to $X$. Since uniruledness is preserved under bimeromorphic transformations, $Y$ is uniruled, and so its canonical divisor $K_Y$ is not pseudoeffective. According to [BCHM, Corollary 1.3.3], $Y$ is birational to a Mori fiber space $\phi : Z \to B$ with $Z$ and $B$ projective. In general, $Z$ is not smooth, and let $f : \hat{Z} \to Z$ be a desingularization. Then, there exists an ample line bundle $H_Z$ on $\hat{Z}$ such that
\[K_{\hat{Z}} \cdot H_Z^{n-1} < 0.\]
Indeed, from Proposition 3.1, we know that there exists an ample line bundle $H_Z$ on $Z$ such that $K_Z \cdot H_Z^{n-1} < 0$. Fix $H_0$ be an ample line bundle on $\hat{Z}$. For every $m > 0$, let
\[H_m = mf^*H_Z + H_0\]
Then $H_m$ is an ample line bundle on $\hat{Z}$, and
\[K_{\hat{Z}} \cdot H_m^{n-1} = (f^*K_Z + \sum a_iE_i) \cdot (mf^*H_Z + H_0)^{n-1}
= m^{n-1}K_{\hat{Z}} \cdot H_Z^{n-1} + O(m^{n-2}) < 0,
\]
for $m$ sufficiently large. Take now $H_{\hat{Z}} = H_m$ for fixed $m \gg 0$. Since $H_{\hat{Z}}$ is ample, the first Chern class of $kH_m$ is represented by the Kähler form of a Hodge metric $\omega_{\hat{Z}}$ for sufficiently large $k$. In particular, we found on $\hat{Z}$ a Kähler metric $\omega$ such that
\[\int_{\hat{Z}} c_1(K_{\hat{Z}}) \wedge \omega^{n-1} < 0.\]
Since $\hat{Z}$ and $X$ are smooth and bimeromorphic manifolds, we can apply now Theorem A to conclude that $X$ admits a balanced metric $\omega^{n-1}$ such that
\[\int_X c_1(K_X) \wedge \omega^{n-1} < 0.
\]
\[\square\]

We can give a short second proof of Theorem B using the results presented in Section 2. In fact, for projective manifolds we can prove a slightly more precise result:

**Proposition 3.2.** Let $X$ be a uniruled projective manifold of dimension $n$. Then there exists $\omega^{n-1}$ a balanced metric, $[\omega^{n-1}] \in H^{n-1,n-1}_{BC,NS}(X, \mathbb{R})$ such that
\[\int_X c_1(K_X) \wedge \omega^{n-1} < 0 \quad (3.2)\]
Proof. By [BDPP], on a uniruled manifold the canonical bundle \( K_X \) is not pseudoeffective, hence \( c_1(K_X) \not\in \mathcal{E}_{BC,NS} \). Therefore, from Proposition 2.5 we see \( c_1(K_X) \not\in \mathcal{E}_{A,NS} \). Theorem 2.3 now implies the existence of a balanced metric \( \omega^{n-1} \) which is negative when paired with \( c_1(K_X) \). \( \square \)

The proof of Theorem C now follows from Proposition 3.2 and Theorem A.

3.2.2. The Kähler case. The first proof of Theorem B can be adapted in Kähler setting.

Proposition 3.3. Let \( \phi : Z \to S \) be a Mori fiber space where \( Z \) and \( S \) are Kähler spaces. Then there exists a Kähler form \( \eta \) on \( Z \) such that

\[
K_Z \cdot [\eta^{n-1}] < 0.
\]

Proof. Fix \( \omega_S \) and \( \omega_Z \) Kähler forms on \( S \) and \( Z \), respectively and consider the family of Kähler forms

\[
\eta_t = t \phi^* \omega_S + \omega_Y, \quad t > 0.
\]

As in the proof of Proposition 3.1 we see that \( K_Z \cdot [\eta_t^{n-1}] < 0 \) for \( t \gg 0 \). We omit the details. \( \square \)

Proof of Theorem C. Let \( X \) be a smooth, uniruled, 3−dimensional manifold of class \( \mathcal{E} \). That means there exists a uniruled, 3−dimensional, Kähler manifold \( Y \) bimeromorphic to \( X \).

According to Höring and Peternell [HP2, Theorem 1.1], \( Y \) is bimeromorphic to a Kähler Mori fiber space \( Z \) as in Proposition 3.3. In general, \( Z \) is not smooth. Let \( f : \tilde{Z} \to Z \) be a desingularization. By [Va, 1.3.1], \( \tilde{Z} \) is a smooth Kähler manifold. Then the argument in the proof of Theorem B yields a Kähler metric \( \omega \) on \( \tilde{Z} \) such that

\[
\int_Y c_1(K_{\tilde{Z}}) \wedge \omega^2 < 0.
\]

Applying now Theorem A, we can conclude that \( X \) admits a balanced metric with the property claimed in Theorem C. \( \square \)

3.3. Characterization of uniruledness. In this short section, we prove Theorems D and E.

Proof of Theorem D. Let \( X \) be a Moishezon manifold of dimension \( n \) admitting a balanced metric \( \omega^{n-1} \) such that

\[
\int_X c_1(K_X) \wedge \omega^{n-1} < 0.
\]

Let \( Y \) be a projective manifold bimeromorphic to \( X \). By Theorem A, \( Y \) admits a balanced metric \( \eta^{n-1} \) such that

\[
\int_Y c_1(K_Y) \wedge \eta^{n-1} < 0.
\]

Then \( [c_1(K_Y)] \not\in (\mathcal{A}^{n-1}_{BC,Y})^* \), and so, by Theorem 2.3, \( \{c_1(K_Y)\} \not\in \mathcal{E}_{A,Y} \). That is \( K_Y \) is not pseudoeffective. Hence \( Y \) is uniruled by [BDPP, Corollary 0.3]. Since uniruledness is preserved by bimeromorphic maps, it follows that \( X \) is uniruled. \( \square \)

Here we have to work with singular Kähler spaces. For the basic notions in the theory of a Kähler space we refer the interested reader to the sections 2 and 3 in [HP2].
Proof of Theorem E. Let \( X \) be a class \( \mathcal{C} \) manifold of dimension three which admits a balanced metric \( \omega^2 \) such that
\[
\int_X c_1(K_X) \wedge \omega^2 < 0.
\]
Let \( Z \) be a Kähler manifold bimeromorphic to \( X \). By Theorem A, \( Z \) admits a balanced metric \( \eta^2 \) such that
\[
\int_Z c_1(K_Z) \wedge \eta^2 < 0.
\]
Hence \( K_Z \) is not pseudoeffective.

If \( Z \) is projective, from Theorem E or [BDPP, Corollary 0.3], we conclude that \( Z \) is uniruled. As uniruledness is invariant under bimeromorphisms, it follows that \( X \) is uniruled, as well. If \( Z \) is Kähler and non-projective, a remarkable result of Brunella [Br, Corollary 1.2] implies that \( X \) is covered by rational curves, hence it is uniruled. Hence, \( X \) is uniruled. \( \square \)

4. Balanced metrics on twistor spaces

A large class of examples of uniruled complex manifolds which are seldom of class \( \mathcal{C} \) (see Remark 4.1) is provided by the manifolds bimeromorphic to the twistor spaces of closed anti-self-dual four-manifolds. Moreover, the twistor spaces come naturally equipped with a one-parameter family of balanced metrics. We show that among these metrics there exists a balanced metric of positive total Chern scalar curvature.

Let \((M,g)\) be an oriented Riemannian 4−manifold. Under the action of the Hodge \( \ast \)−operator
\[
\ast : \Lambda^2 M \to \Lambda^2 M,
\]
one has a decomposition \( \Lambda^2 M = \Lambda_+ \oplus \Lambda_- \) into self-dual and anti-self-dual forms, corresponding to the \((\pm 1)\)−eigenvalues of \( \ast \).

Let \( \mathcal{R} : \Lambda^2 \to \Lambda^2 \) be the Riemannian the curvature operator. Under the action of \( SO(4) \), the Riemannian curvature operator decomposes as
\[
\mathcal{R} = \frac{s}{6} Id + W^- + W^+ + \mathring{r},
\]
where \( s \) denotes the scalar curvature, \( W^\pm \) are the self-dual and anti-self-dual components of the Weyl curvature operator, and \( \mathring{r} \) is the trace-free Ricci curvature operator. Let \( \mathcal{R}_- = \frac{s}{12} Id + W^- + \mathring{r} \) be its restriction to \( \Lambda_- \). The oriented Riemannian 4−manifold \((M,g)\) is said to be anti-self-dual (ASD) if \( W^+ = 0 \). This definition is conformally invariant, i.e. if \( g \) is ASD, so is \( ag \) for any smooth positive function \( a \).

The twistor space of a conformal Riemannian manifold \((M,[g])\) is the total space of the sphere bundle of the rank three real vector bundle of self-dual \(2−\)forms \( \mathcal{Z} := S(\Lambda_+) \). Let \( \varpi : \mathcal{Z} \to M \) be the projection onto \( M \). For every \( x \in M \), the fiber \( \varpi^{-1}(x) \) corresponds to the set of \( g \)−orthogonal complex structures compatible with the given orientation. More precisely, any such \( j \) defines the unit length self-dual form
\[
\omega(v, w) = \frac{1}{\sqrt{2}} g(v, jw).
\]
The real six-dimensional manifold $\mathcal{Z}$, comes equipped with an almost complex structure, that is an endomorphism $\mathcal{J} : T\mathcal{Z} \rightarrow T\mathcal{Z}$ satisfying $\mathcal{J}^2 = -1$. The Levi-Civita connection $\nabla$ of $M$ gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. At a point $(\sigma, x) \in \mathcal{Z}$, the vertical distribution $\mathcal{V}$ consists of the vectors tangent to the fiber of $\varpi$, which is an oriented metric 2–sphere, and hence equipped with a compatible complex structure $I$. On the other hand, the almost complex structure $\mathcal{J}$ associated to $\sigma$ discussed above naturally lifts to the horizontal distribution $\mathcal{H}$. Then, $\mathcal{J}$ is defined as $\mathcal{J} = (j, I)$. A remarkable result of Atiyah, Hitchin and Singer [AHS] asserts that $\mathcal{J}$ is integrable if and only if the metric $g$ is ASD. In such a case, the fibers $\varpi^{-1}(x), x \in M$ are smooth rational curves, and so $\mathcal{Z}$ is uniruled.

Let $h_t$ be the family of Riemannian metrics on $\mathcal{Z}$ defined by

$$h_t = \varpi^* g + t g^{\text{vert}},$$

where $t > 0$, $g$ is the metric of $M$ and $g^{\text{vert}}$ is the restriction of the metric induced on $\Lambda_+$ to the vertical distribution $\mathcal{V}$. Then $\varpi : (\mathcal{Z}, h_t) \rightarrow (M, g)$ is a Riemannian submersion with totally geodesic fibers. Moreover, the metrics $h_t$ are compatible with $\mathcal{J}$, and according to [Mi, Mu, dBN] they are balanced.

The Riemannian scalar curvature of the metrics $h_t$ is computed by Davidov and Muskarov [DM]. More precisely, in [DM, Corollary 4.2] it is proved that for every $(\sigma, x) \in \mathcal{Z}$,

$$s_\mathcal{Z}(\sigma, x) = s_M(x) + \frac{1}{4} \left( \| \mathcal{R}(\sigma) \|^2 - \| \mathcal{R}_\sigma \|^2 \right) + \frac{2}{t},$$

where $s_\mathcal{Z}$ and $s_M$ denote the scalar curvatures of $\mathcal{Z}$ and $M$, respectively. In particular, for $0 < t \ll 1$, we see that the metric $h_t$ satisfies $s_\mathcal{Z} > 0$.

**Proof of Theorem F.** Let $X$ be a complex manifold bimeromorphic to a twistor space $\mathcal{Z}$. Let $\omega_t$ be the Kähler 2-form of the metrics balanced metrics $h_t$ on $\mathcal{Z}$ discussed above. By Corollary 1.2, we have

$$\int_{\mathcal{Z}} c_1(\mathcal{Z}) \wedge \omega_t^n \geq \frac{1}{12\pi} \int_{\mathcal{Z}} s_\mathcal{Z} \omega_t^n > 0,$$

for $0 < t \ll 1$. The conclusion of Theorem F now follows from Theorem A. □

**Remark 4.1.** Twistor spaces of class $\mathcal{C}$ are rather scarce. LeBrun and Poon [LP], and Campana [Ca2], independently proved that a twistor space $\mathcal{Z}$ of an ASD four-manifold $M$ is of Fujiki class $\mathcal{C}$, then $\mathcal{Z}$ is Moishezon and $M$ is homeomorphic to either $S^4$ or the connected sum of $n \geq 1$ copies of $\mathbb{CP}^2$, the complex projective plane endowed with the opposite orientation. However, a result of Taubes [Ta] asserts that every Riemannian manifold $M$ can be equipped with an ASD metric after taking the connected sum with sufficiently many copies of $\mathbb{CP}^2$.

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