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Constructive logic with strong negation as a substructural logic I

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Joint research with **Manuela Busaniche**

Recently, M. Spinks and R. Veroff showed that Nelson constructive logic with strong negation can be considered as a substructural logic. The aim of this talk, based on joint work with M. Busaniche, is to show how this approach can be exploited to obtain information on the algebraic semantics of Nelson logic and to relate it to some non-classical logics existing in the literature.

Constructive logic with strong negation as a substructural logic II

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Joint research with **Roberto Cignoli**

In his talk "Constructive logic with strong negation as a substructural logic" Roberto Cignoli showed how Nelson constructive logic with strong negation can be considered as a substructural logic. Under this point of view it can be shown that Nilpotent Minimum logic is an axiomatic extension of Nelson constructive logic with strong negation. In the present talk we shall extend some results concerning Nilpotent Minimum algebras to subvarieties of Nelson algebras. We will also use a well known representation of Nelson algebras to study subvarieties of involutive residuated lattices.

Structural Completeness for Fuzzy Logics

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Joint research with **George Metcalfe**

A consequence relation is structurally complete (SC) iff all of its proper extensions have new theorems. That is, if a schematic rule is admissible (preserves the set of theorems), then it is derivable in any formal system axiomatizing the consequence relation. Classical logic has this property, but for many families of non-classical logics such as intermediate, modal, and substructural logics, it is relatively rare, often existing only for certain fragments of the systems in question. We investigate structural completeness for the class of (t-norm based) fuzzy logics including Gödel-Dummett logic, known to be structurally complete, and Łukasiewicz logic, where the positive fragment is known to have the property, but not the full logic. By combining approaches described below we are able to answer the question of structural completeness for most (if not quite all) prominent fuzzy logics.

We observe that many existing proofs of SC construct embeddings of generators of the algebraic semantics of the logic in question into the free algebra in this class of algebras. We show that this observation is characteristic of SC in a wide class of logics. Using this method we first reformulate the existing results for Łukasiewicz logic and show that all fragments of the third fundamental fuzzy logic, Product logic, are structurally complete, a result that extends to related logics such as Cancellative Hoop Logic and Abelian Logic.

Then we study the relation of hereditary SC (SC in all extensions of the logic) and the notion of a 'hereditary' (local) deduction theorem. We obtain a general method to show that a wide range of logics (fuzzy or otherwise) are hereditary structurally complete (this method in fact generalizes the well-known trick of Prucnal). Finally, we study the weakened notion of SC — so-called passive SC, where we restrict to a relatively small subclass of rules. This notion is not so interesting on its own, but can easily be used to disprove SC for a wide class of logics.

On Projective MV-algebras

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Joint research with **Revaz Grigolia**

We describe the class of finitely generated projective subalgebras of the finitely generated free MV-algebras. As main results we characterize such algebras by means a functional description of their generators, obtaining in such a way, up to isomorphism, a characterization of all finitely generated projective MV-algebras. Also we get a characterization of finitely generated projective MV-algebras via special formulas, the *projective formulas*, of Łukasiewicz logic.

Symmetric Gaggles

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Joint research with **Katalin Bimbó**

In Dunn and Hardegree (2001) there are several ideas relevant to *algebra and proof theory.* One of these is the notion of a *gaggle.* I introduced gaggle theory in a series of papers in the early 1990's as a general framework for studying algebras closely related to various logics. A second idea is *symmetric consequence* (sometimes called *multiple-conclusion logic*, derived from Gentzen's calculus **LK** and previously studied by Kneale, Scott, Shoesmith and Smiley, etc.). A third idea, which Hardegree and I related to the second idea, is that of a *hemi-distributoid* which arises naturally from Gentzen's rule of *Cut.* On this occasion I shall be relating the second idea to the first by way of the third.

Gaggles may be quickly described as a generalization of the Jónsson-Tarski Boolean algebras with operators, except Jónsson and Tarski required that each operation distribute over join in each of its places. In gaggles operators can do this in some places, but in other places they can also distribute over meet, or they can "co-distribute" over meet (changing it to join), or vice versa. Also operators can form a family related by an "abstract law of residuation" that generalizes the notion of a Galois connection (hence the name *gaggle*, a pronunciation of *ggl*, which is the acronym for *generalized Galois logics*). To make things even more complicated gaggles live on a distributive lattice that is not necessarily a Boolean algebra (and there are weaker structures yet that live on lattices, semi-lattices, and partial orders).

The proto-typical case of a binary gaggle is a (distributive lattice-ordered) residuated groupoid with its characteristic law:

$$a \leq c \leftarrow b \quad \text{iff} \quad a \circ b \leq c \quad \text{iff} \quad b \leq a \rightarrow c.$$

It is by now well-know (due to work by K. Došen, J.M. Dunn etc.) that a Gentzen sequent in his proof system for intuitionistic logic **LJ**, say $A, B \Rightarrow C$, can be interpreted as $a \circ b \leq c$, (adding associativity, commutativity, and idempotence for $\circ$) and that sequent calculi for various substructural logics can be interpreted by dropping one or more of the parenthesized properties.

But how does one interpret a sequent in Gentzen's system for classical logic, **LK**, say $A, B \Rightarrow C, D$? There is of course the dual residuated groupoid satisfying:

$$a \geq c \leftharpoondown b \quad \text{iff} \quad a + b \geq c \quad \text{iff} \quad b \geq a \rightharpoondown c$$

We explore "gluing" these together into a single "symmetric gaggle" so that one can now write $a \circ b \leq c + d$. But how should $\circ$ and $+$ interact? It turns out (because of a symmetric form of Cut) that the hemi-distribution law

$$a \circ (b + c) \leq (a \circ b) + c$$

and its three symmetric variants are the answer.

A key idea of gaggle theory is to give a general representation of gaggles where an n-ary operator is interpreted using an $(n+1)$-ary relation, similar to the Kripke semantics for modal logic and the Routley-Meyer semantics for relevance logic (and related substructural logics). We shall give such a relational representation of hemi-distribution and talk about how hemi-distribution and this representation might be generalized to n-ary operators. If time permits this work will be related to other work having certain commonalities but different motivations from ours (the Cut rule). These are by V. N. Grishin (1983) (a common generalization of ordered groups and Heyting-Brouwer algebras) and more recently by M. Moortgat and his collaborators (symmetrical categorical grammar).

References

[1] J. M. Dunn and G. Hardegree (2001), *Algebraic Methods in Philosophical Logic*, Oxford University Press.

[2] K. Bimbó and J. M. Dunn (2008), *Generalized Galois Logics: Relational Semantics of Non-Classical Logical Calculi*, CSLI Lecture Notes, **187**, scheduled for March publication.

[3] B. Jónsson and A. Tarski (1951), *Boolean Algebras with Operators*, American Journal of Mathematics, **73**, 891–939.

Axiomatizations for intersections of substructural logics

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Joint research with **J.G. Raftery** and **J.S. Olson**

Substructural logics include classical, intuitionistic, many-valued, relevance and linear logics. These logics have been studied extensively individually, and a uniform theory encompassing them has been developed recently.

Substructural logics, ordered by inclusion, form a lattice. An axiomatization for the join of two logics (for the least logic containing both) is obtained simply by the union of the two axiomatizations. On the other hand an axiomatization for the meet of the two logics (for the greatest logic contained in both) is more elusive, as for example the intersection of the two axiomatizations may be empty. In [1] such an axiomatization is recursively constructed from the initial axiomatizations, by studying the join of the corresponding varieties of residuated lattices (in the dual subvariety lattice).

In the study of substructural logics and residuated lattices one easily finds that the meet operation is more important than the join (disjunction connective), for example in the structure theory of the congruences and the deductive rules of logics without weakening. Other reasons also motivate the study of the case where there is no disjunction connective in the language.

The axiomatizations provided in [1] make use of the join operation. In this talk we extend these results to the smaller language, by giving axiomatizations without using disjunction. Moreover, we describe cases where the new axiomatization is finite, provided the initial ones are also finite. We further investigate the finitely subdirectly irreducible algebras of residuated (semi)lattices (if join is not included in the language) and show that if they form an elementary class, the finitely axiomatized extensions of the corresponding substructural logic are closed under intersections.

References

[1] N. Galatos, *Equational bases for joins of residuated-lattice varieties*, Studia Logica 76(2) (2004), 227–240.

[2] N. Galatos, J.S. Olson and J.G. Raftery, *Irreducible residuated semilattices and finitely based varieties*, to appear in Reports on Mathematical Logic.

Varieties of unital lattice-ordered groups and pseudo-MV-algebras

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MV-algebras are the algebras associated with multi-valued logic, and pseudo-MV-algebras are the not necessarily commutative versions of these. They are categorically equivalent to unit intervals in unital lattice-ordered groups, which are those lattice-ordered groups with a prescribed unit u as part of the language. Varieties of lattice-ordered groups have been intensely studied for many decades. But the study of varieties (more properly, equationally defined classes) of unital lattice-ordered groups has only recently begun. It is of even greater interest because the language is much richer, due to the addition of the constant u. We will discuss some interesting new varieties, and raise many unsolved questions.

A Weaker Form of Wajsberg's Axiom

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It is well known that integral commutative representable residuated lattices (ICRRLs) satisfying Wajsberg's axiom $(a \to b) \to b \approx (b \to a) \to a$ are already Wajsberg hoops, i.e. each such algebra is either an MV-algebra or a negative cone of a totally ordered Abelian group. In our talk we are going to discuss a weaker form of this axiom, namely we replace the identity by a quasi-identity $(a \to b) \to b \approx 1 \;\Rightarrow\; (b \to a) \to a \approx 1$. We will show that the corresponding quasi-variety is in fact a variety axiomatized by $((a \to b) \to b)^2 \leq (b \to a) \to a$. The members of this variety are ICRRLs which are "almost" Wajsberg hoops. More precisely, each totally ordered ICRRL $\mathbf{L}$ is either a Wajsberg hoop or $\mathbf{L}$ is subdirectly irreducible and $\mathbf{L}/\theta$ is a Wajsberg hoop where θ is the monolith.

Thus the above-mentioned variety consists of chains which contain at most one component which is not a Wajsberg hoop (by a component we mean a quotient of a principal convex subalgebra by its predecessor in the chain of convex subalgebras). We will also consider for each n the varieties whose chains contain at most n components which are not Wajsberg hoops and give their axiomatizations. These varieties form a strictly increasing sequence of subvarieties whose limit is the variety of ICRRLs.

Translations in Constructive Set Theory

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Constructive set theory has been introduced as a foundational theory for constructive mathematics. We will discuss the system and explain why it is considered to be constructive. Then we will define certain translations on the theory from which several of its constructive properties can be derived and discuss the Kripke models of the system.

Using Gentzen system techniques with existing automated theorem provers

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The aim of this talk is to show that Gentzen sequent calculi can be used in standard resolution theorem provers to improve their search space characteristics. This is mostly of use with lattice ordered algebras, and does not require cut-free Gentzen systems. For example it is currently not known if there is a cut-free Gentzen system for residuated Kleene algebras or residuated Kleene lattices. However if we present axiomatizations for these equational theories in the style of Gentzen sequent rules, then a theorem prover such as Prover9 becomes quite an effective reasoning tool in these otherwise rather untractable theories.

The assembly of an algebraic frame

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Joint research with **Brian Boucher, William M. McGovern** and **Eric Zenk**

The assembly of a frame A is the frame NA of all nuclei defined on A, under pointwise ordering. A may be embedded in NA by assigning x to the closed nucleus $z \mapsto z \vee x$. By restricting to compact algebraic frames, on the one hand, and to the inductive nuclei, on the other, one obtains a monoreflection to compact regular algebraic frames; this reflection is the epicompletion.

Todd Wilson, in his 1994 dissertation, introduces free meets; that is to say, meets which are preserved by all nuclei. He proves that all meets of A are free iff A is dually a frame AND its assembly is boolean. We show that NA is, in fact, atomic, for all algebraic frames with disjointification.

Interpolation and amalgamation for abelian ℓ-groups and MV-algebras

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In this talk I will explain how the logical property of interpolation can be used as a stepping stone to the algebraic property of amalgamation. In particular, I will give a simple proof of the deductive interpolation property for abelian ℓ-groups. This provides as nice by-products, not only amalgamation but also decidability and generation of the variety by the integers $\mathbb{Z}$. Moreover, the approach can be modified to establish the amalgamation property for the variety of MV-algebras, the algebraic semantics for Łukasiewicz logic.

Classification of lattice-ordered abelian groups with unit

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Finitely presented unital ℓ-groups are completely classified by weighted abstract simplicial complexes. Two such complexes classify the same unital ℓ-group iff they are connected by a path of Alexander stellar operations and their inverses. The proof uses the Wlodarczyk-Morelli solution of the weak Oda conjecture for toric varieties. We describe various kinds of computable invariants for finitely presented unital ℓ-groups, and their MV-algebraic equivalents.

An algebraic approach to the comparison of propositional logics

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The interpretability of one logic into another is a very common issue in Mathematical Logic but, as far as we know, a precise and general definition of the concept of "interpretation" is still lacking, the meaning of this word usually relying on a logician's intuition.

We propose a general and purely syntactic definition of interpretability for the case of propositional deductive systems and show some algebraic characterizations in the wake of the order-theoretic approach proposed by Galatos and Tsinakis in [1] — where consequence relations are represented as structural closure operators on complete posets that are, also, subject to an action from a complete po-monoid — and revised in [2] in terms of quantales, quantale modules and $\mathcal{Q}$-module structural closure operators.

References

[1] N. Galatos, C. Tsinakis, *Equivalence of consequence relations: an order-theoretic and categorical perspective*, to appear.

[2] C. Russo, *Quantale Modules, with Applications to Logic and Image Processing*, Ph.D. thesis, University of Salerno – Italy, 2007.