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ch. 5 (~~32~~, ~~34~~, ~~37~~, ~~39~~, ~~40~~; ~~30~~)
ch. ~~6~~ (~~1~~, ~~10~~, ~~13~~, ~~23~~)

32. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$. What is
a) the probability that a repair time exceeds 2 hours?
b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

Solution:
Let X denote the amount of time for a repair

a) $P(X > 2) = 1 - F(2)$
$= 1 - (1 - e^{-\frac{1}{2}(2)})$
$= e^{-1} \approx .368$

b) $P(X \geq 10 \mid X > 9) = \dfrac{P(X \geq 10 \cap X > 9)}{P(X > 9)}$

$= \dfrac{1 - F(10)}{1 - F(9)} = \dfrac{e^{-5}}{e^{-9/2}} = e^{-.5} \approx .607$

34. Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has used a car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime of mileage is not exponentially distributed, but rather is (in 1000s of miles) uniformly distributed over (0, 40).

Solution: Let x denote # of miles (in 1000s)
using exponential r.v.:

$\lambda = \frac{1}{20}$

$$P(\text{remaining life} \geq 20) = 1 - F(20)$$
$$= e^{-(\frac{1}{20})(20)}$$
$$= e^{-1} \approx .368$$

However, if lifetime is distributed normally with $(0, 40)$, then:

$$P(\text{lifetime} > 30000 \mid \text{lifetime} > 10000)$$
$$= \frac{P(X > 30 \cap X > 10)}{P(X > 10)}$$
$$= \frac{1 - \int_0^{30} (\frac{1}{40})\, dx}{1 - \int_0^{10} (\frac{1}{40})\, dx}$$
$$= \frac{.25}{.75} = \boxed{\frac{1}{3}}$$

37. If X is uniformly distributed over $(-1, 1)$, find

a) $P\{|X| > \frac{1}{2}\}$

b) the density function of the random variable $|X|$.

Solution

a) If X is uniformly distributed over $(-1, 1)$, then

$$f(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2} & -1 < x < 1 \\ 0 & x \geq 1 \end{cases} \Rightarrow F(x) = \begin{cases} 0 & x \leq -1 \\ \frac{1}{2} & -1 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

If $Y = |X|$, then $F_Y(y) = P\{Y \leq y\} = P\{|X| \leq y\}$
$= P\{-y \leq X \leq y\} = P\{X \leq y\} - P\{Y \leq -y\}$
$= F(y) - F(-y)$ when $y \geq 0$

b) $\therefore F_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases}$

$\therefore P\{|X| > \frac{1}{2}\} = P\{Y > \frac{1}{2}\}$

$= 1 - F(\frac{1}{2})$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

b) $f_Y(y) = F_X(y) = \begin{cases} 0 & y < 0 \\ 1 & 0 \le y < 1 \\ 0 & y \ge 1 \end{cases}$

39. If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

Solution:

We know the probability density function of X is

$$f_X(a) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We want f_Y. Let $Y = \log X$

$$\begin{aligned} F_Y(a) &= P\{Y \leq a\} \\ &= P\{\log X \leq a\} \\ &= P\{X \leq e^a\} \\ &= F_X(e^a) \end{aligned}$$

$$f_Y(a) = e^a f_X(e^a)$$

40. If X is uniformly distributed over $(0,1)$ find the density function $Y = e^X$.

Solution:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases} \Rightarrow F_X(x) = P\{X \leq x\} = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\begin{aligned} F_Y(y) = P\{Y \leq y\} &= P\{e^X \leq y\} \\ &= P\{X \leq \ln y\} \\ &= F_X(\ln y) \end{aligned}$$

$$= \begin{cases} 0 & \ln y \leq 0 \\ \ln y & 0 < \ln y < 1 \\ 1 & \ln y \geq 1 \end{cases} \Rightarrow \begin{cases} 0 & 0 < y \leq 1 \\ \ln y & 1 < y < e \\ 1 & y \geq e \end{cases}$$

$$\therefore F_Y(y) = \begin{cases} 0 & y \leq 1 \\ \ln y & 1 < y < e \\ 1 & y \geq e \end{cases} \Rightarrow f(y) = \begin{cases} 0 & y \leq 1 \\ \frac{1}{y} & 1 < y < e \\ 0 & y \geq e \end{cases}$$

30. Let X have probability density function f_X. Find the probability density function of the random variable Y defined by $Y = aX + b$.

Solution:

X has the probability density function f_X with distribution function $F(x) = \int_{-\infty}^{x} f_X(t)\,dt = P(Y \le x)$

Suppose the distribution function for Y is $F_Y(y)$. Then $F_Y(y) = P\{Y \le y\}$
$= P\{aX + b \le y\}$
$= P\{aX \le y - b\}$

if $a > 0$, $F_Y(y) = P\{X \le \frac{y}{a} - \frac{b}{a}\} = F(\frac{y}{a} - \frac{b}{a})$
$= \int_{-\infty}^{\frac{y}{a}-\frac{b}{a}} f_X(t)\,dt$

$\therefore f_Y(y) \Rightarrow F_Y'(y) = \frac{1}{a} f_X(\frac{y}{a} - \frac{b}{a})$

if $a = 0$ then $Y = b$. $\therefore f_Y(y) = 1$ $(y = b)$.

if $a < 0$ then Y
$P(aX \le y - b\}$
$= P\{X \ge \frac{y}{a} - \frac{b}{a}\}$
$= 1 - F(\frac{y}{a} - \frac{b}{a}\}$

$\therefore f_Y(y) \Rightarrow F_Y'(y) = \{1 - F(\frac{1}{a}y - \frac{b}{a})\}$
$= -\frac{1}{a} f_X(\frac{1}{a}y - \frac{b}{a})$.

Chapter 6

1. Two fair dice are rolled. Find the joint probability mass function of X and Y when,
a) X is the largest value obtained on any die and Y is the sum of the values.
b) X is the value on the first die and Y is the larger of the two values.
c) X is the smallest and Y is the largest value obtained on the dice.

Solution:

a) X and Y are discrete random variables

$P(x,y) = P(X=x, Y=y)$

$\therefore P(1,2) = \frac{1}{36}$ $P(2,3) = P(X=2, Y=3) = \frac{2}{36} = \frac{1}{18}$

$P(2,4) = \frac{1}{36}$, $P(3,4) = \frac{2}{36}$, $P(3,5) = \frac{2}{36}$

$P(3,6) = \frac{1}{36}$ $P(4,5) = \frac{2}{36}$, $P(4,6) = \frac{2}{36}$,

$P(4,7) = \frac{2}{36}$ $P(4,8) = \frac{1}{36}$, $P(5,6) = \frac{2}{36}$

$P(5,7) = \frac{2}{36}$, $P(5,8) = \frac{2}{36}$ $P(5,9) = \frac{2}{36}$

$P(5,10) = \frac{1}{36}$ $P(6,7) = \frac{2}{36}$ $P(6,8) = \frac{2}{36}$

$P(6,9) = \frac{2}{36}$ $P(6,10) = \frac{2}{36}$ $P(6,11) = \frac{2}{36}$

$P(6,12) = \frac{1}{36}$.

b) $(X \leq Y)$ $P(1,1) = \frac{1}{36}$ $P(1,2) = \frac{1}{36}$ $P(1,3) = \frac{1}{36}$

$P(1,4) = \frac{1}{36}$ $P(1,5) = \frac{1}{36}$ $P(1,6) = \frac{1}{36}$

$P(2,2) = P($the 2^{nd} roll can be 1 or 2$) = \frac{2}{36}$

$P(2,3) = \frac{1}{36}$ $P(2,4) = \frac{1}{36}$ $P(2,5) = \frac{1}{36}$ $P(2,6) = \frac{1}{36}$

$P(3,3) = P($the 2^{nd} roll can be 1, 2 or 3$) = \frac{3}{36}$

$P(3,4) = \frac{1}{36}$ $P(3,5) = \frac{1}{36}$ $P(3,6) = \frac{1}{36}$

$P(4,4) = P($the 2^{nd} roll can be 1 or 2 or 3 or 4$) = \frac{4}{36}$.

$P(4,5) = \frac{1}{36}$, $P(4,6) = \frac{1}{36}$ $P(5,5) = P($the 2^{nd} roll can be 1, 2, 3, 4, 5 or 6$) = \frac{5}{36}$

$P(5,6) = \frac{1}{36}$.

$P(6,6) = P($2nd roll a 1, 2, 3, 4, 5 or 6$) = \frac{6}{36}$

c) ($X = Y$ or $X < Y$) If $X = Y$, then the faces of the two die have the same value.

$P(i,i) = P(1 \le i \le 6) = \frac{1}{36}$

for $P(i,j)$; $(1 < i < j < 6) = P(X = i, Y = j)$, which gives different values for each dice. Either the first gives the largest or the second. $\therefore P(i,j)\ (1 < i < j < 6) = \frac{2}{36}$

10. The joint probability density function of X and Y is given by $f(x,y) = e^{-(x+y)}$ $(0 \le x < \infty,\ 0 \le y < \infty)$ Find a) $P(X < Y)$
b) $P(X < a)$

a) $P(X < Y) = P\{(x,y) \mid 0 < x < y,\ 0 < y < \infty\}$

$= P\{(x,y) \mid [0, Y) \times [0, \infty)\}$

$= \int_0^\infty \int_0^y e^{-(x+y)}\,dx\,dy$

$= \int_0^\infty e^{-y}\,dy\,(-1)\int_0^y e^{-x}\,dX$

$= \int_0^\infty -e^{-y}\,dy\, e^{-x}\Big|_0^y$

$= \int_0^\infty -e^{-y}\,dy\,(e^{-y} - 1)$

$= \int_0^\infty (e^{-2y}\,dy + e^{-y}\,dy)$

$= \int_0^\infty -e^{-2y}\,dy + \int_0^\infty e^{-y}\,dy$

$= \frac{1}{2}e^{-2y}\Big|_0^\infty - e^{-y}\Big|_0^\infty$

$= \frac{1}{2}[0-1] - [0-1] = \boxed{\frac{1}{2}}$

$\therefore P\{X < y\} = \frac{1}{2}$.

b) $P\{X < a\} = P\{(x,y) \mid [0, a) \times [0, \infty)\}$

$= \int_0^\infty \int_0^a e^{-(x+y)}\,dx\,dy$

$= \int_0^\infty e^{-y}\,dy \int_0^a e^{-x}\,dx = \int_0^\infty e^{-y}\,dy\,(-1)\int_0^a e^{-x}\,d(x)$

$= (1 - \frac{1}{e^a})^a = 1 - (\frac{1}{e})^a$

13- A man and a woman agree to meet at a certain location about 12:30 p.m. If the man arrives at a time uniformly distributed b/w 12:15 and 12:45, and the woman independently arrives at a time uniformly distributed b/w 12:00 and 1:00 p.m. Find the probability that the 1st to arrive waits no longer than 5 minutes. What is the prob. the man arrives first?

Solution:

X = time man arrives

Y = time woman arrives

Time within 12:00–1:00 p.m. recorded per minute.

$$\therefore \quad f(X) = \begin{cases} 0 & X \le 15 \\ 1/30 & 15 < X < 45 \\ 0 & X \ge 45 \end{cases} \qquad f(Y) = \begin{cases} 0 & Y \le 0 \\ \frac{1}{60} & 0 < Y < 60 \\ 0 & Y \ge 60 \end{cases}$$

$$P\{|X-Y| \le 5\} = P\{-5 \le X - Y \le 5\}$$

$$= P\{(X,Y) \mid -5 \le X - Y \le 5\}$$

$$\therefore P\{(X,Y) \mid (X,Y) = (15, 45) \times [X-5, X+5]\}$$

$$\therefore \quad P(a,b) = P\{X \le a, Y \le b\}$$

thus arriving times independent.

$$P\{X \le a, Y \le b\} = P\{X \le a\} \cdot P\{Y \le b\}$$

$$P\{X \le a\} = F_X(a) = \begin{cases} 0 & a \le 15 \\ \frac{a-15}{30} & 15 < a < 45 \\ 1 & a \ge 45 \end{cases}$$

$$P\{Y \le b\} = F_Y(b) = \begin{cases} 0 & b \le 0 \\ b/60 & 0 < b < 60 \\ 1 & b \ge 60 \end{cases}$$

$$\therefore \quad F(a,b) = P\{X \le a\} P\{Y \le b\} = F_X(a) \cdot F_Y(b)$$

$$\therefore \quad F(a,b) = \begin{cases} 0 & (-\infty, 15] \times (-\infty, 0] \\ \frac{a-15}{30} \cdot \frac{b}{60} & (15, 45) \times (0, 60) \\ 1 \cdot \frac{b}{60} & [45, 60) \times (0, 60) \\ 1 & [60, \infty) \times [60, \infty) \end{cases}$$

$\therefore\ f(a,b) = \frac{\partial^2 f}{\partial a \partial b}(a,b) = \begin{cases} 1/1800 & (15,45)\times(0,60) \\ 0 & \text{otherwise.} \end{cases}$

$P\{(X,Y) \mid (X,Y) \in (15,45) \times [X-5,\ X+5]\}$

$= \int_{15}^{45} \int_{X-5}^{X+5} \frac{1}{1800}\, dy\, dX = \frac{1}{6}$

$\therefore\ P\{|X-Y| \le 5\} = \frac{1}{6}$

$P\{X \le Y\} = P\{(X,Y) \mid (15,45) \times (X,60)\}$

$= \int_{15}^{45} \int_{X}^{60} \frac{1}{1800}\, dy\, dX$

$= \frac{1}{30}(45-15) - \frac{1}{1800} \frac{X^2}{2} \Big|_{15}^{45} = 1 - \frac{1}{1800}(30 \cdot 60) \cdot \frac{1}{2}$

$= 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore\ P(X \le Y) = \frac{1}{2}.$

23. The random variables X and Y have joint function $f(x,y) = (2xy(1-x))$ $0<x<1$, $0<y<1$, and 0 otherwise.

a) are X and Y independent?

$P_X(a) = \lim_{b\to\infty} P_{X,Y}(a,b) = \lim_{b\to\infty} \int_0^b \int_0^a 12xy(1-x)\, dx\, dy$

integrating gives us $3a^2 - 2a^3$ $(0<a<1)$

$P_Y(b) = \lim_{a\to\infty} P_{X,Y}(a,b) = \lim_{a\to\infty} \int_0^{\infty} \int_0^b 12xy(1-x)\, dy\, dx$

integrating gives us b^2. $0<b<1$

$P_X(a) = \begin{cases} 0 & a \le 0 \\ 3a^2 - 2a^3 & 0<a<1 \\ 1 & a \ge 1 \end{cases}$ $\quad P_Y(b) = \begin{cases} 0 & b \le 0 \\ b^2 & 0<b<1 \\ 1 & b \ge 1 \end{cases}$

$F_{X,Y}(a,b) = \int_0^b \int_0^a 12xy(1-x)\, dx\, dy$

integrating yields, $= 3a^2b^2 - 2a^3b^3$

$$\therefore P_{X,Y}(a,b) = \begin{cases} 0 & a \le 0 \text{ or } b \le 0 \\ 3a^2 - 2a^3 b^2 & 0 < a < 1,\ 0 < b < 1 \\ 1 & a \ge 1,\ b \ge 1 \end{cases}$$

$P_{XY}(a,b) = P_X(a) \cdot P_Y(b)$

X, Y independent.

b) $E[X]$?

$$P_X(a) = \begin{cases} 0 & a \le 0 \\ 3a^2 - 2a^3 & 0 < a < 1 \\ 1 & a \ge 1 \end{cases} \Rightarrow f_X(a) = \begin{cases} 6a - 6a^2 & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^1 (6a^2 - 6a^3)\, da = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{1}{2}}$$

c) $E[Y] = ?$

$$P_Y(b) = \begin{cases} 0 & b \le 0 \\ b^2 & 0 < b < 1 \\ 1 & b \ge 1 \end{cases} \Rightarrow f_Y(a) = \begin{cases} 2b & 0 < b < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_0^1 b \cdot f_Y(b)\, db = \int_0^1 2b^2\, db = \boxed{\frac{2}{3}}$$

d) $E[X^2] \Rightarrow$ Let $Z = X^2$ $\quad P\{Z \le c\}$

$= P_X\{X^2 < c\}$

$= P_X\{X \le \sqrt{c}\}$

$$= \begin{cases} 3c - 2c\sqrt{c} & 0 < c < 1 \\ 1 & c \ge 1 \end{cases} \Rightarrow f_Z(c) = \begin{cases} 3 - 3c^{1/2} & 0 < c < 1 \\ 0 & c \ge 1 \end{cases}$$

$$E[Z] = \int_0^1 c(3 - 3c^{1/2}) \cdot dc = \frac{3}{10}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$\therefore Var(X) = \boxed{\frac{1}{20}}$$

e) $E[Y^2] \Rightarrow$ Let $Z = Y^2$

$P_Y\{Z \le e\} = P_Y\{Y \le \sqrt{e}\}$

$$= \begin{cases} e & 0 < e < 1 \\ 1 & e \ge 1 \end{cases} \quad \therefore f_Z(e) = \begin{cases} 1 & 0 < e < 1 \\ 0 & e \ge 1 \end{cases}$$

$$E[Y^2] = \int_0^1 e\, de = \frac{1}{2}$$

$$\therefore Var(Y) = E[Y^2] - (E[Y])^2 = \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}$$