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Homework 7

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

what is the prob. that the system functions for atleast 5 months?

$$\int_0^{\infty} cxe^{-x/2}\, \partial x = 1 \qquad \begin{matrix} u = cx & \partial v = e^{-x/2} \\ \partial u = c\partial x & v = -2e^{-x/2} \end{matrix}$$

$$\int_0^{\infty} u\,\partial v = c\left[-2e^{-x/2}x\Big|_0^{\infty} - -2\int_0^{\infty} e^{-x/2}\,\partial x\right]$$

$$= c\left[2\int_0^{\infty} e^{-x/2}\,\partial x\right] = -4c\left(e^{-x/2}\right)\Big|_0^{\infty} = -4c(0-1) \qquad 4c = 1$$

$$c = 1/4$$

prob of atleast 5 months

$$\int_5^{\infty} \frac{1}{4}xe^{-x/2} = \frac{1}{4}\left[-2e^{-x/2}x\Big|_5^{\infty} + 2\int_5^{\infty} e^{-x/2}\,\partial x\right]$$

$$= 1/4\left[10e^{-5/2} + 2\left(-2e^{-x/2}\right)\Big|_5^{\infty}\right]$$

$$= 1/4\left(10e^{-5/2} + 4e^{-5/2}\right) = 7/2\,e^{-5/2}$$

3. Consider the Function $f(x) = \begin{cases} c(2x - x^3) & 0 < x < 5/2 \\ 0 & \text{elsewhere} \end{cases}$

could f be a prob density function? If so, determine c. Repeat if f(x) were given by $f(x) = \begin{cases} c(2x - x^2) & 0 < x < 5/2 \\ 0 & \text{elsewhere} \end{cases}$

part 1) $f(x) = \begin{cases} c(2x - x^3) & 0 < x < 5/2 \\ 0 & \text{elsewhere} \end{cases}$

here, this can be rewritten as $f(x) = cx(2 - x^2)$, this means that $f(x) = 0$ when $x = 0, \pm\sqrt{2}$. Therefore, if $x < \sqrt{2}$, $f(x) > 0$ but if $5/2 > x > \sqrt{2}$ then $f(x)$ is negative. So there is no sign that c can carry to avoid f(x) being negative. Since f can't be negative, it is not a prob. density function.

part 2) $f(x) = cx(2-x) \quad 0 < x < 5/2$

again $f(x) = 0$, when $x = 0, 2$. However since x can be greater than 2, there will be positive and negative values for $f(x)$ no matter what c is. Since f can't be negative, not a prob density function.

7. The density function of x is given by $f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

If $E(x) = 3/5$, find a and b.

$$1 = \int_0^1 a + bx^2 \, dx = ax + \frac{bx^3}{3}\Big|_0^1 = a + \frac{b}{3} = 1$$

$$E(x) = 3/5 = \int_0^1 (ax - bx^3)\, dx = \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1 = \frac{a}{2} + \frac{b}{4} = 3/5$$

$$3a + b = 3 \qquad 5a = 3 \qquad a = 3/5$$

$$10a + 5b = 12 \qquad b = 6/5$$

11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the prob. that the ratio of the shorter to the longer segment is less than $1/4$.

line length L, with a point X selected on interval $[0, L]$

$$P\left\{\frac{\min(X, L-X)}{\max(X, L-X)} < \frac{1}{4}\right\}$$

$$f(x) = \left\{ \frac{1}{L} \quad 0 \le x \le L \right.$$

ratio of shorter to longer $= \min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}$

$$P\left[\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}\right] = 1 - P\left[\min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) > \frac{1}{4}\right]$$

$$= 1 - P\left[\frac{X}{L-X} > \frac{1}{4}, \frac{L-X}{X} > \frac{1}{4}\right]$$

$$= 1 - P\left\{\left(X > \frac{L}{5}\right), \left(X < \frac{4L}{5}\right)\right\} = 1 - P\left\{\frac{L}{5} < X < \frac{4L}{5}\right\}$$

$$1 - \int_{L/5}^{4L/5} \frac{1}{L}\, \partial \quad = 1 - \frac{1}{L}(x)\Big|_{L/5}^{4L/5} \quad = 1 - \frac{1}{L}\left[\frac{4L}{5} - \frac{L}{5}\right]$$

$$= 1 - 3/5 = 2/5$$

14. Let X be a uniform (0,1) RV. Compute $E[X^n]$ by using Proposition 2 and then check the result by using the definition of expectation.

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,\partial x = \int_{-\infty}^{0} x^n(0)\,\partial x + \int_0^1 x^n(1)\,\partial x + \int_1^{\infty} x^n(0)\,\partial x$$

$$= \int_0^1 x^n (1)\, \partial x = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

$Y = X^n$ $\quad P(Y \le a)$ for $0 < a < 1$ $= P(X^n \le a) = P(X \le \sqrt[n]{a})$

$$= \int_0^{\sqrt[n]{a}} 1\, \partial x = \sqrt[n]{a} = F_Y(a)$$

$$f_Y(x) = \frac{x^{\frac{1}{n}-1}}{n} \qquad 0 < x < 1$$

$$E[Y] = \int_{-\infty}^{\infty} x f_Y(x)\, \partial x = \int_{-\infty}^{0} x(0)\,\partial x + \int_0^1 x\left(\frac{x^{1/n-1}}{n}\right) + \int_1^{\infty} x(0)\,\partial x$$

$$= \int_0^1 x\left(\frac{x^{1/n-1}}{n}\right) = \frac{1}{n}\left(\frac{x^{1/n+1}}{1/n+1}\right)\Big|_0^1 = \frac{1}{n+1}$$

16. The annual rainfall in inches in a certain region is normally distributed w/ $\mu = 40$ and $\sigma = 4$. What is the prob that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

$$P(X < 50) = P\left(\frac{X-40}{4} < \frac{50-40}{4}\right) = P(Z < 2.5)$$

$$\Phi(2.5) = \int_0^{2.5} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = .9938$$

$$(P(X<50))^{10} = (.9938)^{10} = .9397$$

You are assuming each year is independent of previous years rainfall

21. Suppose that the heights, in inches, of a 25-year-old man is a normal RV w/ param $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet 2 inches tall (74). What about 6 foot 5 in (77) given they are above 6 feet?

a) $P(X > 74) = P\left(\frac{X-71}{\sqrt{6.25}} > \frac{74-71}{\sqrt{6.25}}\right) = P(Z > 1.2) = 1 - P(Z < 1.2)$

$1 - \Phi(1.2) = .1151$

b) $P(X > 77 \mid X > 72) = \frac{P(X > 77, X > 72)}{P(X > 72)} = \frac{P(X > 77)}{P(X > 72)}$

$$\frac{P\left(\frac{X-72}{\sqrt{6.25}} > \frac{77-71}{\sqrt{6.25}}\right)}{P\left(\frac{X-72}{\sqrt{6.25}} > \frac{77-71}{\sqrt{6.25}}\right)} = \frac{P(Z > 2.4)}{P(Z > .4)} = \frac{1 - \Phi(2.4)}{1 - \Phi(.4)} = .0238$$

Theoretical Exercises

8. Let X be a RV that takes on values between 0 and C. That is $P(0 \le X \le C) = 1$ show that $\mathrm{var}(X) \le C^2/4$

start by arguing that $E(X^2) \le C\,E(X)$

$$E(X^2) = \int_0^C x^2 f(x)\,dx \le C \int_0^C x f(x)\,dx = C\,E(X)$$

because $x \le C$ for all $x \in [0, C]$

$$\mathrm{var}(X) = E(X^2) - E(X)^2 \le C\,E(X) - E(X)^2$$
$$= C^2\left(\frac{E(X)}{C} - \frac{E(X)^2}{C^2}\right)$$

let $a = \frac{E(X)}{C}$ so $= C^2(a - a^2)$ $\quad \mathrm{var}(X) \le C^2(a - a^2) = C^2 a(1-a)$

$$\frac{\partial (C^2(a - a^2))}{\partial a} = C^2(1 - 2a) = 0 \qquad a = 1/2$$

$$\frac{\partial}{\partial a} C^2(1 - 2a) = -2C^2 \text{ negative so } 1/2 \text{ is max}$$

$c^2 \left(\frac{1}{2} - \frac{1}{2^2}\right) = c^2/4$

$var|X| \leq c^2/4$

13. The median of a continuous RV having distribution function F is that value m such that $F(m) = 1/2$. That is a RV is just as likely to be larger than its median as it is to be smaller. Find the median of X if X is

a) uniformly distributed over (a, b)

$F(m) = 1/2 \qquad F(x) = \frac{x-a}{b-a}$

$F(m) = \frac{m-a}{b-a} = 1/2 \qquad 2m - 2a = b - a$

$m = \frac{b+a}{2}$

b) normal with parameters μ, σ^2, we know $f(\mu - x) = f(\mu + x)$
normal distributions are symmetric so $m = \mu = 1/2$

c) exponential with rate λ

$F(x) = \theta e^{-\theta x}$

$F(m) = 1 - e^{-\theta m} = 1/2$

$-\theta m = \ln .5$

$m = \frac{.693147}{\theta}$

15. If X is an exponential RV w/ parameters λ, and $c > 0$, show that cX is exponential with parameters λ/c

$F_X(x) = 1 - e^{-\lambda x} \qquad Y = cX$

$F_Y(x) = P(Y \leq x) = P(cX \leq x) = P(X \leq x/c)$

$= \int_0^{x/c} f(x)\,dx = \int_0^{x/c} \lambda e^{-\lambda x}\,dx = e^{-\lambda x}\Big|_{x/c}^{0}$

$= 1 - e^{-\frac{\lambda x}{c}} = 1 - e^{-(\lambda/c)x}$

so yes cX is exponential w/ param λ/c